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Representations of posets and decompositions of rigid almost completely decomposable groups

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Abstract. Representations of posets in certain modules are used to discuss direct decompositions of almost completely decomposable groups. For almost completely decomposable groups with p-primary regulator quotients direct decompositions are unique up to near–isomorphism. Among the categories of rigid almost completely decomposable groups with p-primary homocyclic regulator quotients we determine those that contain indecomposable groups of any finite rank in which case a complete description is hopeless, and for the remaining cases we completely determine the near–isomorphism classes of indecomposable groups.

1 Introduction

Representations of finite posets S over $\mathbb{Z}_e = \mathbb{Z}/e\mathbb{Z}$ are intimately connected with almost completely decomposable groups, [Mader00, Ch.8]. An ***almost completely decomposable group*** is a torsion-free abelian group of finite rank that contains a completely decomposable subgroup of finite index. An almost completely decomposable group G contains a fully invariant completely decomposable subgroup of finite index, the regulator $R := \mathrm{R}(G)$. The ***regulator quotient*** G/R is a $\mathbb{Z}_e$-module where $e = \exp(G/R)$ denotes the exponent of G/R. The $\mathbb{Z}_e$-module R/eR contains $U_*^G := eG/eR \cong G/R$, and U_*^G together with certain other distinguished submodules form the representation U_G associated with G. The group G is indecomposable if and only if its representation U_G is indecomposable, [Mader00, Corollary 10.1.7].

There is clear evidence that isomorphism is an unworkable equivalence relation for almost completely decomposable groups while near–isomorphism works very well. E.g., [Arnold82] showed that near–isomorphic (torsion-free abelian) groups of finite rank have the same decomposition properties. Also near–isomorphic almost completely decomposable groups have isomorphic regulators and regulator quotients, [Mader00, Theorem 9.2.6]. Two almost completely decomposable groups G and H with regulators isomorphic to R and isomorphic regulator quotients of exponent e can be viewed in the representation setting R/eR differing only by the submodules U_*^G and U_*^H. It turns out that G and H are nearly isomorphic if and only if their representations are isomorphic, [Mader00, Theorem 9.2.4].

A particularly nice subclass of almost completely decomposable groups is the class of almost completely decomposable groups with p-primary regulator quotient. While almost completely decomposable groups in general may have wildly different direct decompositions, for groups with primary regulator quotients the direct decompositions with indecomposable summands are unique up to near–isomorphism, [Faticoni-Schultz96]. This means that a classification of these groups amounts to determining their near–isomorphism classes of indecomposable such groups. Yet, even for these groups the associated representations mostly have ***unbounded representation type***, i.e., there exist indecomposable groups of arbitrarily large finite rank, and these groups are not amenable to classification.

If $R = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ is the decomposition of the completely decomposable group R with homogeneous summands $R_\rho \neq 0$ of type ρ, then $\mathrm{T}_{\mathrm{cr}}(R)$ is the ***critical typeset*** of R. An almost completely decomposable group G is ***rigid*** if $\mathrm{T}_{\mathrm{cr}}(R(G))$ is an antichain. We settle completely the case of rigid almost completely decomposable groups with p-primary ***homocyclic*** regulator quotient, i.e., the regulator quotient is the direct sum of cyclic groups all of the same order p^e for some $e \geq 1$. A type τ is ***p-reduced*** if $pA \neq A$ for any rank-1 group of type τ.

Let e be a positive integer and $S_w = \{\tau_1, \ldots, \tau_w\}$ an anti-chain of p-reduced types and of width w. By $\mathrm{hc}'(S_w, p^e)$ we denote the ***category of rigid homocyclic almost completely***

decomposable groups G with critical typeset $\mathrm{T}_{\mathrm{cr}}(G) = S_w$ and regulator quotient $G/\mathrm{R}(G)$ that is homocyclic of exponent p^e.

The main result of this paper is Theorem 1.1. It will be seen in the Section 3 how almost completely decomposable groups can be described in terms of "representing matrices".

Theorem 1.1.

1. For $w \geq 4$ and any $e \geq 1$, the category $\mathrm{hc}'(S_w, p^e)$ has unbounded representation type.
2. For $w = 3$ and $e \geq 3$, the category $\mathrm{hc}'(S_w, p^e)$ has unbounded representation type.
3. The category $\mathrm{hc}'(S_2, p^e)$ contains up to near–isomorphism exactly one indecomposable group with representing matrix $\begin{bmatrix}1 \| 1\end{bmatrix}$.
4. The category $\mathrm{hc}'(S_3, p)$ contains up to near–isomorphism exactly one indecomposable group with representing matrix $\begin{bmatrix}1 \| 1 \| 1\end{bmatrix}$, and one indecomposable group with representing matrix
$$\begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}.$$
5. The category $\mathrm{hc}'(S_3, p^2)$ contains up to near–isomorphism exactly the fourteen indecomposable groups with representing matrices listed in Theorem 6.3.

Proof. (1) and (2): Corollary 5.9.
(3): Trivial.
(4): Theorem 6.2.
(5): Theorem 6.7. $\square$

Theorem 6.3 overlaps with [Mouser93] and confirms Mouser's assertion that indecomposable groups in $\mathrm{hc}'(S_3, p^2)$ have rank ≤ 9. However, Mouser lists indecomposable representations that do not satisfy the Regulator Criterion, and some indecomposable representations are missing. We therefore produce independent proofs.

The theory of almost completely decomposable groups with two critical types $S = (1, 1)$ is well-known and due to [Arnold73], [Dugas90], [Lewis93]. Indecomposable groups in $\mathrm{hc}'(S_2, p^e)$ have rank 2 and two indecomposable such groups are nearly isomorphic if and only if they have the same critical typeset and isomorphic regulator quotients, [Mader00, Section 12.3].

2 Almost completely decomposable groups

Details on almost completely decomposable groups and representation of posets are found in [Mader00] and [Arnold00]. All "groups" in this paper are torsion-free abelian groups of finite rank.

A completely decomposable subgroup R of an almost completely decomposable group G is a ***regulating subgroup*** of G if $|G/R|$ is minimal. The ***regulator*** $\mathrm{R}(G)$ of G is the intersection of all regulating subgroups of G, a fully invariant, completely decomposable subgroup of finite index in G, [Burkhardt84], [Mader00, Corollary 4.4.5]. Let $R := \mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ be the decomposition of the regulator R with homogeneous summands $R_\rho \neq 0$ of type ρ. Then $\mathrm{T}_{\mathrm{cr}}(R)$ is the critical typeset of R and of G, $\mathrm{T}_{\mathrm{cr}}(G) := \mathrm{T}_{\mathrm{cr}}(R)$. Let (S, p^e) denote the category of almost completely decomposable groups G with $\mathrm{T}_{\mathrm{cr}}(G) \subseteq S$ and regulator quotient $p^e G \subseteq \mathrm{R}(G)$.

Given a prime p, two almost completely decomposable groups G and H are ***isomorphic at*** p if there exist $f : G \to H$, $g : H \to G$, and an integer n prime to p with $fg = n$ and $gf = n$. The groups G and H are ***nearly isomorphic***, $G \cong_{\mathrm{nr}} H$, if they are isomorphic at p for every prime p, [Lady75].

Lemma 2.1. [Arnold00, Lemma 5.4.1] Let S a finite poset of types and assume $G, H \in (S, p^e)$.

1. G and H are nearly isomorphic if and only if G and H are isomorphic at p.
2. G is an indecomposable group if and only if G is isomorphic at p to an indecomposable group.

Another substantial simplification occurs when the almost completely decomposable group contains a unique regulating subgroup which then coincides with the regulator. We then speak of a ***regulating regulator***. This happens when the critical typeset is $\vee$-free or, equivalently, an inverted forest, [Mutzbauer93, 1.4], [Mader00, Proposition 4.5.4], and in particular in the rigid case. In this case the regulator is easily characterized among the completely decomposable subgroups of finite index.

Lemma 2.2. [Mader00, Theorem 4.4.6] Suppose that G is an almost completely decomposable group with regulating regulator. A completely decomposable subgroup A of finite index in the group G is the regulator of G if and only if $\forall\, \tau \in \mathrm{T}_{\mathrm{cr}}(G) : G(\tau) = A(\tau)$.

3 Representations and almost completely decomposable groups

Let G be a group with regulator $R := \mathrm{R}(G) = \bigoplus_{\rho\in\mathrm{T}_{\mathrm{cr}}(R)} R_\rho$ and $p^e G \subseteq R$. Define

$$^{-} : R \to R/p^e R : \overline{x} = x + p^e R, \quad \text{so} \quad \overline{R} = R/p^e R.$$

Set $S := \mathrm{T}_{\mathrm{cr}}(G)$. The ***representation*** U_G ***of*** G is given by

$$U_G = \left(\overline{R}, \overline{R(\sigma)}, \overline{p^e G} : \sigma \in S^{\mathrm{op}}\right),$$

where $U_*^G := \overline{p^e G}$ is the distinguished submodule mentioned earlier.

The construction of U_G makes sense for any completely decomposable subgroup of finite index in G. It is essential to use the regulator because the construction of U_G must be functorial on a suitable category and therefore a canonical choice of the completely decomposable subgroup of finite index is needed. Furthermore, as $R(G)$ is fully invariant in G, we have the well-defined induced map

$$^{-} : \operatorname{End} G \to \operatorname{End} \overline{R} : \overline{f}(\overline{x}) := \overline{f(x)}$$

and the crucial Lemma 3.4 rests on lifting certain maps in $\operatorname{End}\overline{R}$ to $\operatorname{End} G$.

The categories of interest in this paper are $\mathrm{hc}(S_w, p^e)$, a generalization of $\mathrm{hc}'(S_w, p^e)$. We precede the discussion of $\mathrm{hc}(S_w, p^e)$ by establishing properties of groups whose critical typeset is contained in an inverted forest.

Lemma 3.1.

1. Let S be a inverted forest of types and G an almost completely decomposable group with $\mathrm{T}_{\mathrm{cr}}(G) \subseteq S$. Then G has a regulating regulator and $\mathrm{R}(G) = \sum_{\sigma\in S} G(\sigma)$.
2. Let S be an inverted forest and let G and H be almost completely decomposable groups with $\mathrm{T}_{\mathrm{cr}}(G), \mathrm{T}_{\mathrm{cr}}(H) \subseteq S$. Then
   - $\mathrm{T}_{\mathrm{cr}}(G \oplus H)) \subseteq S$, $G \oplus H$ has a regulating regulator, $\mathrm{R}(G \oplus H) = \mathrm{R}(G) \oplus \mathrm{R}(H)$, and
   - $\forall\, \phi \in \operatorname{Hom}(G, H) : \phi(\mathrm{R}(G)) \subseteq \mathrm{R}(H)$.

Proof. (1) Set $R := \mathrm{R}(G)$. By Lemma 2.2,

$$R = \textstyle\sum_{\sigma\in\mathrm{T}_{\mathrm{cr}}(G)} R(\sigma) = \sum_{\sigma\in\mathrm{T}_{\mathrm{cr}}(G)} G(\sigma) \subseteq \sum_{\sigma\in S} G(\sigma).$$

To obtain the claimed equality we need to show that $\forall\, \sigma \in S : G(\sigma) \subseteq R$.

Let $\sigma \in S$. Write $R(\sigma) = \bigoplus_{\sigma\le\rho\in\mathrm{T}_{\mathrm{cr}}(G)} R_\rho$ and suppose that $x \in G(\sigma)$. As $G(\sigma) = R(\sigma)_*$, there exists $0 \neq m \in \mathbb{N}$ such that $mx \in R(\sigma)$. Hence $mx = y_{\tau_1} + \cdots + y_{\tau_k}$ where $0 \neq y_{\tau_i} \in R_{\tau_i}$. It follows that $\forall\, i : \sigma \le \operatorname{tp} x = \operatorname{tp} mx = \bigwedge(\tau_1, \ldots, \tau_k) \le \tau_i$. Using that S is an inverted forest it follows that the τ_i form a chain and without loss of generality $\sigma \le \tau_1 < \cdots < \tau_k$. Hence $\operatorname{tp} x = \operatorname{tp} mx = \tau_1$ and $G(\sigma) = G(\tau_1) = R(\tau_1) \subseteq R$.

(2) We have $\mathrm{T}_{\mathrm{cr}}(G \oplus H) = \mathrm{T}_{\mathrm{cr}}(G) \cup \mathrm{T}_{\mathrm{cr}}(H) \subseteq S$, hence G, H and $G \oplus H$ have regulating regulators. But $\mathrm{R}(G) \oplus \mathrm{R}(H)$ is a regulating subgroup of $G \oplus H$ by the definition of regulating subgroup in terms of Butler complements, [Mader00, Definition 4.1.5], and it follows that $\mathrm{R}(G \oplus H) = \mathrm{R}(G) \oplus \mathrm{R}(H)$. The second claim follows immediately from (1). □

We now formally introduce the category $\mathrm{hc}(S_w, p^e)$ where S_w is an antichain of width w of p-reduced types. The objects of $\mathrm{hc}(S_w, p^e)$ are the almost completely decomposable groups G with $\mathrm{T}_{\mathrm{cr}}(G) \subseteq S_w$ and $p^e G \subseteq \mathrm{R}(G)$, the morphisms of $\mathrm{hc}(S_w, p^e)$ are the ordinary group homomorphisms.

Lemma 3.2.

1. Every $G \in \mathrm{hc}(S_w, p^e)$ has a regulating regulator and $\mathrm{R}(G) = \sum_{\sigma\in S_w} G(\sigma)$.

2. $\mathrm{hc}(S_w, p^e)$ is closed under summands and (finite) direct sums.
3. If $G, H \in \mathrm{hc}(S_w, p^e)$, then $\mathrm{R}(G \oplus H) = \mathrm{R}(G) \oplus \mathrm{R}(H)$.
4. If $G, H \in \mathrm{hc}(S_w, p^e)$, then $\forall\, \phi \in \mathrm{Hom}(G, H) : \phi(\mathrm{R}(G)) \subseteq \mathrm{R}(H)$.

Proof. (1) is a special case of Lemma 3.1(1). (3) and (4) are special cases of Lemma 3.1(2).

(2) $\frac{G \oplus H}{\mathrm{R}(G \oplus H)} \cong \frac{G}{\mathrm{R}(G)} \oplus \frac{H}{\mathrm{R}(H)}$ is homocyclic of exponent dividing p^e, thus $G \oplus H \in \mathrm{hc}(S_w, p^e)$.

Now suppose that $G \in \mathrm{hc}(S_w, p^e)$ and $G = G_1 \oplus G_2$. Then $\mathrm{T}_{\mathrm{cr}}(G_i) \subseteq \mathrm{T}_{\mathrm{cr}}(G) \subseteq S_w$. Hence both G_i have regulating regulators and $\mathrm{R}(G) = \mathrm{R}(G_1) \oplus \mathrm{R}(G_2)$. As before, $\frac{G}{\mathrm{R}(G)} \cong \frac{G_1}{\mathrm{R}(G_1)} \oplus \frac{G_2}{\mathrm{R}(G_2)}$ hence $G_i / \mathrm{R}(G_i)$ is homocyclic of exponent dividing p^e, showing that $G_i \in \mathrm{hc}(S_w, p^e)$. □

When the regulator is regulating, the case of interest here, the regulator can be easily recognized in the representation of G.

Lemma 3.3. [Mader00, Corollary 8.1.12] ***Regulator Criterion.*** Let G be an almost completely decomposable group with regulating regulator. A completely decomposable subgroup R of G with finite index in G and $eG \subseteq R$ is the regulator of G if and only if $\forall\, \tau \in \mathrm{T}_{\mathrm{cr}}(G) : \overline{eG} \cap \overline{R(\tau)} = 0$.

By definition every group $G \in \mathrm{hc}(S_w, p^e)$ has p-reduced critical types, i.e., if $R := \mathrm{R}(G) = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(G)} R_\rho$, then $\forall\, \tau \in \mathrm{T}_{\mathrm{cr}}(G) : pR_\tau \neq R_\tau$. It follows that $\overline{R} = \bigoplus_{\rho \in \mathrm{T}_{\mathrm{cr}}(R)} \overline{R_\rho}$ is a free module over $\mathbb{Z}_{p^e}$ and $\overline{R(\tau)} = \bigoplus_{\rho > \tau} \overline{R_\rho}$ is free as well. Furthermore, $\overline{p^e G} = p^e G / p^e R \cong G/R$ is a free $\mathbb{Z}_{p^e}$-module.

Define $\mathrm{hcRep}(S_w, \mathbb{Z}_{p^e})$ to be the category of representations $U = (U_0, U_\sigma, U_* : \sigma \in S^{\mathrm{op}})$ such that for each $\sigma \in S$, there is a finitely generated free $\mathbb{Z}_{p^e}$-module V_σ with $U_0 = \bigoplus_{\sigma \in S} V_\sigma$, $U_\sigma = \bigoplus_{\sigma \leq \rho \in S} V_\rho$, U_τ a summand of U_σ whenever $\sigma \leq \tau$ in S, U_* a free submodule of U_0, and $U_\sigma \cap U_* = 0$ for each $\sigma \in S$. We set $\mathrm{rk}\, U = \mathrm{rk}\, U_0$.

Morphisms on $U = (U_0, U_\sigma, U_* : \sigma \in S^{\mathrm{op}})$ to $W = (W_0, W_\sigma, W_* : \sigma \in S^{\mathrm{op}})$ are $\mathbb{Z}_{p^e}$-homomorphisms $f : U_0 \to W_0$ with $f(U_\sigma) \subseteq W_\sigma$ for each $\sigma \in S \cup \{*\}$. The category $\mathrm{hcRep}(S_w, \mathbb{Z}_{p^e})$ is additive and has biproducts. A representation U is indecomposable if and only if 0 and 1 are the only idempotents of $\mathrm{End}(U)$, the endomorphism ring of U in $\mathrm{hcRep}(S, \mathbb{Z}_{p^e})$.

Lemma 3.4. Let S be an inverted forest of types.

1. The assignment of near–isomorphism classes of objects of $\mathrm{hc}(S, p^e)$ to isomorphism classes of objects of $\mathrm{hcRep}(S, \mathbb{Z}_{p^e})$ given by
$$[G] \mapsto [U_G] \text{ where } U_G = \left(\overline{\mathrm{R}(G)}, \overline{\mathrm{R}(G)(\sigma)}, \overline{p^e G} : \sigma \in S^{\mathrm{op}}\right),$$
where $^- : \mathrm{R}(G) \to \mathrm{R}(G)/p^e\,\mathrm{R}(G) = \overline{\mathrm{R}(G)}$ is the natural epimorphism, is a bijective correspondence. Note that $\mathrm{rk}\, G = \mathrm{rk}\, U_G$.
2. For $G, H \in \mathrm{hc}(S, p^e)$ and $\phi \in \mathrm{Hom}(G, H)$, there is a natural induced map $\overline{\phi} \in \mathrm{Hom}(U_G, U_H)$ where $\overline{\phi} : \overline{\mathrm{R}(G)} \to \overline{\mathrm{R}(H)} : \forall\, x \in \mathrm{R}(x) : \overline{\phi}(\overline{x}) = \overline{\phi(x)}$.
3. Two groups $G, H \in \mathrm{hc}(S, p^e)$ are nearly isomorphic if and only if U_G is isomorphic to U_H.
4. $G \in \mathrm{hc}(S, p^e)$ is indecomposable if and only if U_G is indecomposable.

Proof. (1) [AMMS09, Lemma 4(1)]. In particular, given a representation $U \in \mathrm{hcRep}(S, \mathbb{Z}_{p^e})$ there exists $G \in \mathrm{hc}(S, p^e)$ such that $U_G \cong U$. The construction of G is described in Remark 3.5(1).

(2) It follows from Lemma 3.2(4) that $\overline{\phi}$ is well-defined.

(3) [Mader00, Theorem 9.2.4] or [AMMS09, Lemma 4].

(4) [AMMS09, Lemma 4]. □

Remark 3.5.

1. Given a representation
$$U = (U_0 = \textstyle\bigoplus_{\rho \in S} V_\rho, U_\sigma, U_* : \sigma \in S^{\mathrm{op}}) \in \mathrm{hcRep}(S, \mathbb{Z}_{p^e}),$$
the construction of a group $G \in \mathrm{hc}(S, p^e)$ such that $U_G \cong U$ is as follows. First choose a completely decomposable group $R = \bigoplus_{\sigma \in S} R_\sigma$ such that $\overline{R_\sigma} = V_\sigma$, choose a divisible hull $\mathbb{Q}R$ of R, let $(^-)^{-1}[U_*] = \{x \in R : \overline{x} \in U_*\}$ (the pre–image of U_* in R) and let $G = p^{-e}(^-)^{-1}[U_*] \leq \mathbb{Q}R$. Then $U_G \cong U$.

2. Let $G \in \mathrm{hc}(S, p^e)$ and suppose that $U_G = U_1 \oplus U_2$ for representations $U_i \in \mathrm{hcRep}(S, \mathbb{Z}_{p^e})$. Then there exist $G_i \in \mathrm{hc}(S, p^e)$ such that $U_{G_i} = U_i$. Then $U_{G_1 \oplus G_2} = U_{G_1} \oplus U_{G_2} = U_1 \oplus U_2 = U_G$. Hence $G \cong_{\mathrm{nr}} G_1 \oplus G_2$ and it follows that G is the direct sum of two groups of rank $\mathrm{rk}\, U_1$ and $\mathrm{rk}\, U_2$.

Let $G \in \mathrm{hc}(S, p^e)$ with representation $U_G = (U_0, U_\sigma, U_*^G : \sigma \in S^{\mathrm{op}})$, $U_0 = \bigoplus_{\sigma \in S} V_\sigma$ and $U_\sigma = \bigoplus_{\sigma \le \rho \in S} V_\rho$. A representing matrix expresses U_*^G in terms of a suitable basis of the free module U_0. Specifically, a $\mathbb{Z}_{p^e}$-matrix M_G is a ***representing matrix of*** G if $B_G = \{B_\sigma : \sigma \in S\}$ is a $\mathbb{Z}_{p^e}$-basis of U_0 with each $B_\sigma = \{x_{\sigma j} : 1 \le j \le n_\sigma\}$ a basis of V_σ, $C_G = \{h_1, \ldots, h_r\}$ a basis of the free $\mathbb{Z}_{p^e}$-module $U_*^G \cong G/\mathrm{R}(G)$,

$$h_i = \sum_{\sigma \in S} \sum_{1 \le j \le n_\sigma} m_{i,\sigma j} x_{\sigma j}, \quad 1 \le i \le r,$$

with $m_{i,\sigma j} \in \mathbb{Z}_{p^e}$ for each $1 \le i \le r$ and $1 \le j \le n_\sigma$, $M_\sigma := \left[m_{i,\sigma j}\right]$, and $M_G := \left[M_\sigma : \sigma \in S\right]_{r \times n}$ with $n = \sum\{n_\sigma : \sigma \in S\}$ and M_σ to the left of $M_{\sigma'}$ if $\sigma < \sigma'$ in S, i.e., the basis elements in B_G are so ordered that the basis elements belonging to larger types are listed to the right of the basis elements belonging to smaller types. We will call a basis of U_0 of this kind a ***proper basis***. Representation matrices are defined only with respect to a proper basis of U_0. The rank of G is equal to the number n of columns of M_G. Looking at U_0 in terms of coordinates, the elements of U_0 are n-tuples and U_*^G is the submodule of U_0 generated by the rows of M_G, in symbols $U_*^G = \vec{\mathbb{Z}} M_G$, where $\vec{\mathbb{Z}}$ is the set of $\mathbb{Z}$-tuples of the appropriate size, here r-tuples, r being the number of rows of M_G.

The representing matrix can be simplified by applying automorphisms of the representation that amount to basis changes. The possible changes (simplifications!) of the representing matrix are effected by "allowed" row and column transformations. We will formulate later (Proposition 6.1) the allowed transformations in the case of rigid almost completely decomposable groups with primary homocyclic regulator quotients. The goal is to simplify the representing matrix to the degree that either no further simplifications are possible and the representation can be shown to be indecomposable or the matrix is "decomposed" and shows that the representation is decomposed. These representations are the representations of groups that then are indecomposable or decomposed as are their representations. If the group is decomposed, by Remark 3.5, the ranks of the summands are equal to ranks the representation summands and these in turn are equal the number of columns in their representing matrix.

A representing matrix is ***decomposed*** if it can be put in block diagonal form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ by rearranging rows and columns. We will not formulate the technical details but the proof of Theorem 6.7 contains numerous illustrations that are easily understood.

4 General Results

We first recall a form of the Regulator Criterion that is suited to the representation approach that we use. [AMMS11, Lemma 13] deals with the general case of a regulating regulator and primary regulator quotient. We will state the result for the rigid homocyclic case that we need. It should be noted that in this case the representing matrix is just the "coordinate matrix" of [AMMS11, Lemma 13] considered modulo the exponent of the regulator quotient.

Lemma 4.1. ([AMMS11, Lemma 13]) ***Matrix Regulator Criterion.*** Let $G \in \mathrm{hc}(S_w, p^e)$ and let $M = \left[M_1 \| M_2 \| \cdots \| M_w\right]$ be a representing matrix of G having r rows. Then

$$\forall\, i \in \{1, \ldots, w\} : \begin{bmatrix} M_1 & \cdots & M_{i-1} & M_{i+1} & \cdots & M_w \end{bmatrix} \text{ has rank } r \text{ modulo } p.$$

We first establish a criterion that is very efficient and turns indecomposability proofs in most cases into a routine exercise. The hypothesis that a representing matrix M has a right inverse M^* is in all our applications an evident matter. But it is a fact that M^* exists if and only if $G/\mathrm{R}(G)$ is homocyclic.

Remark 4.2. A representing matrix M_G of a rigid homocyclic group has a right inverse. This follows from the Matrix Regulator Criterion. On the other hand if a representing matrix M of G has a right inverse M^*, then it follows from $MM^* = I$ that no row of M is a p-fold of some row vector which means that each row has maximal order $p^e = \exp(G/\mathrm{R}(G))$. In addition the rows of M are linearly independent by definition of the representing matrix, so $G/\mathrm{R}(G) \cong p^e G / p^e \mathrm{R}(G) = \overline{p^e G}$ is a free $\mathbb{Z}_{p^e}$-module, i.e., a homocyclic group of exponent p^e.

A representation morphism $f \in \operatorname{End} U_G$ is a module homomorphism $f : \overline{R} \to \overline{R}$ such that $\forall\, \tau \in \mathrm{T}_{\mathrm{cr}}(G) : f(\overline{R(\tau)}) \subseteq \overline{R(\tau)}$ and $f(\overline{p^e G}) \subseteq \overline{p^e G}$. We will use weaker concept called ***type endomorphism*** of $\overline{R}$. A type endomorphism is a module homomorphism $g : \overline{R} \to \overline{R}$ such that $\forall\, \tau \in \mathrm{T}_{\mathrm{cr}}(R) : f(\overline{R(\tau)}) \subseteq \overline{R(\tau)}$. So type endomorphism are just representation morphisms of U_R.

Theorem 4.3. Let G be an almost completely decomposable group, $R = \mathrm{R}(G)$ its regulator, let U_G be the representation of G, and let $M := M_G$ be a representing matrix of G. Assume that M^* is a right inverse of M. Let f be a type endomorphism of G. Then $f \in \operatorname{End} U_G$ if and only if it satisfies the matrix equation

$$Mf = MfM^*M.$$

Proof. We have $U_G = \big(U_0, U_\rho, U_*^G : \rho \in \mathrm{T}_{\mathrm{cr}}(G)^{\mathrm{op}}\big)$ where $U_*^G = \overrightarrow{\mathbb{Z}}M$. Suppose that $f \in \operatorname{End} U_G$ and that M has t rows. Then for every $\overrightarrow{x} \in \overrightarrow{\mathbb{Z}} = \mathbb{Z}^r$ there is $\overrightarrow{u} \in \overrightarrow{\mathbb{Z}}$ such that $\overrightarrow{x}Mf = \overrightarrow{u}M$. Hence $\overrightarrow{u} = \overrightarrow{x}MfM^*$. Substituting this expression for $\overrightarrow{u}$ in $\overrightarrow{x}Mf = \overrightarrow{u}M$ and dropping the argument $\overrightarrow{x}$ results in the claimed matrix identity.

Conversely, assume that f is a type endomorphism of U_0 and $Mf = MfM^*M$. Then take $\overrightarrow{u} = \overrightarrow{x}MfM^*$, so that $\overrightarrow{u}M = \overrightarrow{x}Mf$ showing that $f(U_*^G) \subseteq U_*^G$ and $f \in \operatorname{End} U_G$. □

Corollary 4.4. Let G be an almost completely decomposable group, $R = \mathrm{R}(G)$ its regulator, let U_G be the representation of G, and let $M := M_G$ be a representing matrix of G. Assume that M^* is a right inverse of M. Let $f^2 = f \in \operatorname{End} U_G$. Then G is indecomposable if and only if the matrix equation

$$Mf = MfM^*M$$

is satisfied only by the trivial solutions $f = 0$ and $f = 1$.

Lemma 4.5 relates indecomposable groups in (S, p^k) to indecomposable groups in (S, p^e) for $1 \le k \le e$ in the representation setting.

Lemma 4.5. Assume S is an inverted forest of p-reduced types and $1 \le k \le e$.

1. If $G \in \mathrm{hc}(S, p^k)$ is indecomposable, then there is an indecomposable group $H \in \mathrm{hc}(S, p^e)$ with $\mathrm{rk}(H) = \mathrm{rk}(G)$.
2. If $\mathrm{hc}(S, p^k)$ has unbounded representation type, then $\mathrm{hc}(S, p^e)$ has unbounded representation type.
3. If $\mathrm{hc}(S, p^e)$ has bounded representation type, then $\mathrm{hc}(S, p^k)$ has bounded representation type.

Proof. (1) We identify $\mathbb{Z}_{p^k}$ with $\mathbb{Z}_{p^e}/p^k\mathbb{Z}_{p^e}$. Given a free $\mathbb{Z}_{p^k}$-module F, let F^* be a free $\mathbb{Z}_{p^e}$-module with $F = F^*/p^kF^*$. If $M = \Big[m_{rs} + p^k\mathbb{Z}_{p^e}\Big]$ is a $\mathbb{Z}_{p^k}$-matrix, then $M^* = \Big[m_{rs}\Big]$ is a $\mathbb{Z}_{p^e}$-matrix with $M^* \equiv M \mod p^k$.

Let $U_G = (U_0, U_\sigma, U_*^G : \sigma \in S^{\mathrm{op}})$ be the representation of the indecomposable group $G \in \mathrm{hc}(S, p^k)$ with M_G a representing matrix of G. Define $U_G^* = (U_0^*, U_\sigma^*, U_*^* : \sigma \in S^{\mathrm{op}})$ with $U_*^* = \overrightarrow{\mathbb{Z}}M_G^*$. Then $U_\sigma^* \cap U_*^* = 0$, because $U_\sigma \cap U_*^G = 0$, and so U_*^G is in $\mathrm{cdRep}(S, \mathbb{Z}_{p^e})$.

Let f be an idempotent endomorphism of U_G^*, i.e., $f^2 = f : U_0^* \to U_0^*$ is an idempotent $\mathbb{Z}_{p^e}$-endomorphism with $f(U_\sigma^*) \subseteq U_\sigma^*$ and $f(U_*^*) \subseteq U_*^*$. Then f induces an idempotent $\mathbb{Z}_{p^k}$-endomorphism $g : U_0 \to U_0$ since $U_0 = U_0^*/p^kU_0^*$. Furthermore, $g(U_\sigma) \subseteq U_\sigma$ and $g(U_*) \subseteq U_*$ because

$$U_G = (U_0, U_\sigma, U_*^G : \sigma \in S) = \left(\frac{U_0^*}{p^kU_0^*}, \frac{U_\sigma^* + p^kU_0^*)}{p^kU_0^*}, \frac{U_*^* + p^kU_0^*}{p^kU_0^*} : \sigma \in S\right).$$

Hence, g is an idempotent endomorphism of U_G. As U_G is indecomposable, $g = 0$ or $g = 1$. If $g = 0$, then $f = pf'$ is an idempotent nilpotent and $f = 0$. If $g = 1$, then $f = 1 + pf'$ is an idempotent unit and $f = 1$. This shows that U_G^* is indecomposable.

By Lemma 3.4, $U_*^G \cong U_H$ for some indecomposable $\mathrm{hc}(S, p^e)$-group H. Then $\mathrm{rk}(G) = \mathrm{rk}(U_0) = \mathrm{rk}(U_0^*) = \mathrm{rk}(H)$.

(2) and (3) follow from (1). □

Example 4.6. The group G_{21} with $M_{G_{21}} = \begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}$ is indecomposable in $\mathrm{hc}(S_3, p^2)$. Hence there is an indecomposable group H in $\mathrm{hc}(S_3, p^3)$ with $M_H = M_{G_{21}}$.

Lemma 4.7 shows that the abundance of indecomposable groups increases if to the poset S of critical types a further type σ is added that is incomparable with any of the types in S. In particular, if hcRep(S, p^e) has unbounded representation type, then so has hcRep$(S \cup \{\sigma\}, p^e)$.

Lemma 4.7. Let G be a p-reduced almost completely decomposable group with regulating regulator $R = \mathrm{R}(G)$ and p-primary regulator quotient, let U_G be the representation of G, and let $M := M_G$ be a representing matrix of G. Assume that M^* is a right inverse of M. Then, for any non-zero column vector N with entries in $\mathbb{Z}_{p^e}$, the matrix $M' = \begin{bmatrix} M \| N \end{bmatrix}$, is the representing matrix of an indecomposable H with $\mathrm{T_{cr}}(H) = \mathrm{T_{cr}}(G) \cup \{\sigma\}$ where σ is p-reduced and incomparable with any type in $\mathrm{T_{cr}}(G)$.

Proof. We have $U_G = \left(U_0, U_\rho, U_*^G : \rho \in \mathrm{T_{cr}}(G)^{\mathrm{op}}\right)$ where $U_*^G = \overrightarrow{\mathbb{Z}} M$ is the row space of M. The matrix $M' = \begin{bmatrix} M \| N \end{bmatrix}$ satisfies the Regulator Criterion because M does, where N belongs to the additional type σ. (This is certainly easy in the rigid case using Lemma 4.1, but it is also true in the general case using [AMMS11, Lemma 13].) There exists an almost completely decomposable H having the representing matrix M' and the representation $U_H = (U_0 \oplus V, U_\rho, V, U_* : \rho \in \mathrm{T_{cr}}(G))$ where $U_* = \overrightarrow{\mathbb{Z}} M'$. The entries of M' are in $\mathbb{Z}_{p^e}$ so U_* is p-primary as a group.

Suppose that $f'^2 = f' \in \operatorname{End} U_H$. Then $f' = \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}$ where $f \in \operatorname{End} U_G$ and g is a scalar. Then $M'^* = \begin{bmatrix} M^* \\ 0 \end{bmatrix}$ is a right inverse of M'. Furthermore $M'f' = \begin{bmatrix} Mf & Ng \end{bmatrix}$, $M'^*M' = \begin{bmatrix} M^*M & M^*N \\ 0 & 0 \end{bmatrix}$, and $M'f'M'^*M' = \begin{bmatrix} MfM^*M & MfM^*N \end{bmatrix}$. As $f' \in \operatorname{End} U_H$ it follows that $M'f' = M'f'M'^*M'$ and so $Mf = MfM^*M$ and $Ng = MfM^*N$.

By hypothesis the first condition implies that $f = 0$ or $f = 1$. Suppose that $f = 0$. Then $Ng = 0$, so $g = 0$ and $f' = 0$. Suppose that $f = 1$. Then $Ng = N$, $g = 1$ and $f' = 1$. □

Example 4.8. The group G_{21} with $M_{G_{21}} = \begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}$ is indecomposable hc(S_3, p^2). Hence there are there are indecomposable groups in hc(S_4, p^2) with representing matrices

$$\begin{bmatrix} 1 & \| & 0 & \| & 1 & \| & 1 \\ 0 & \| & 1 & \| & 1 & \| & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & \| & 0 & \| & 1 & \| & 0 \\ 0 & \| & 1 & \| & 1 & \| & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & \| & 0 & \| & 1 & \| & 1 \\ 0 & \| & 1 & \| & 1 & \| & 1 \end{bmatrix}.$$

It readily follows from Remark 4.2 that the new group H of Lemma 4.7 is homocyclic, and if G was rigid, so is H. It is also clear that H has a regulating regulator if $\mathrm{T_{cr}}(G)$ is an inverted forest, in particular if $\mathrm{T_{cr}}(G)$ is an antichain.

It is not clear whether the new groups H for different choices of N are nearly isomorphic or not. This however is not needed to infer Corollary 4.9 from Corollary 4.7 as rank considerations suffice.

Corollary 4.9. If the category hc(S_3, p^k) has unbounded representation type and $w \geq 3$, then hc(S_w, p^k) has unbounded representation type.

There is a general criterion in terms of representing matrices to decide whether two groups with the same regulator and isomorphic regulator quotients are nearly isomorphic. It is a fact that nearly isomorphic almost completely decomposable groups have isomorphic regulators and regulator quotients, [Mader00, Lemma 9.1.10.4].

Theorem 4.10. Let G and H be almost completely decomposable groups with $R := \mathrm{R}(G) = \mathrm{R}(H)$ and $G/R \cong H/R$. Set $\exp(G/R) = e$. Then the representations of the two groups are the same except for the terms $U_*^G = eG/eR$ and $U_*^H = eH/eR$. Let M_G and M_H be representing matrices of G and H, respectively, using the same basis. Suppose that M_H has a right inverse M_H^*. Then G and H are nearly isomorphic if and only if there is a type automorphism α of R/eR such that $M_G\alpha = M_G\alpha M_H^* M_H$.

Proof. By [Mader00, Theorem 9.2.4] $G \cong_{\mathrm{nr}} H$ if and only if there is a type isomorphism (= bijective type endomorphism) α of R/eR such that $U_*^G\alpha = U_*^H$. This is the case if and only if

$$\forall \vec{x} \exists \vec{u} : \vec{x} M_G \alpha = \vec{u} M_H.$$

If so $\vec{u} = \vec{x} M_G \alpha M_H^*$. Substituting and omitting the variable $\vec{x}$ we obtain $M_G \alpha = M_G \alpha M_H^* M_H$. The converse is clear. □

We note a special case relevant to Lemma 4.7.

Corollary 4.11. Suppose that X is an almost completely decomposable group with regulating regulator R and representing matrix M that has a right inverse M^*. Let N_G and N_H be column matrices and let G and H be groups with representing matrices $M_G = \begin{bmatrix} M \| N_G \end{bmatrix}$ and $M_H = \begin{bmatrix} M \| N_H \end{bmatrix}$ respectively. Then $G \cong_{\text{nr}} H$ if and only if there is $\alpha \in \operatorname{Aut} U_R$ and a unit a such that $M\alpha M^* N_H = N_G a$.

Proof. The new groups G and H have regulator $R' = R \oplus R_\sigma$ where R_σ is a rank one group of type σ. Let f be a type automorphism of $U_{R'}$. Then

$$f = \begin{bmatrix} \alpha & 0 \\ 0 & a \end{bmatrix},\ \alpha, a \text{ invertible, and } M_H^* = \begin{bmatrix} M^* \\ 0 \end{bmatrix} \text{ is a right inverse of } M_H.$$

Easy computations result in

$$M_G f = \begin{bmatrix} M\alpha & N_G a \end{bmatrix}, \quad M_G f M_H^* M_H = \begin{bmatrix} M\alpha & M\alpha M^* N_H \end{bmatrix}.$$

Hence $M_G f = M_G f M_H^* M_H$ if and only if $N_G a = M\alpha M^* N_H$. □

Corollary 4.11 can settle concrete cases.

Example 4.12. The three constructs in Example 4.8 belong to different near–isomorphism classes. On the other hand

$$\begin{bmatrix} 1 & \| & 0 & \| & 1 & \| & 1 \\ 0 & \| & 1 & \| & 1 & \| & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \| & 0 & \| & 1 & \| & u \\ 0 & \| & 1 & \| & 1 & \| & v \end{bmatrix} \text{ with } u \neq 0 \neq v$$

belong to nearly isomorphic groups.

Proof. Here $M^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\alpha = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ and $M\alpha M^* = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. Consider the general case $N_G = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ and $N_H = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$. Then $N_G a = M\alpha M^* N_H$ if and only if

$$\begin{bmatrix} a_1 n_1 \\ a_2 n_2 \end{bmatrix} = \begin{bmatrix} a m_1 \\ a m_2 \end{bmatrix}.$$

Here a_1, a_2, a are non-zero elements in $\mathbb{Z}_p$, hence units. It is easily seen that no two of the three groups of Example 4.8 are nearly isomorphic; for example $\begin{bmatrix} a_1 \cdot 1 \\ a_2 \cdot 1 \end{bmatrix} = \begin{bmatrix} a \cdot 1 \\ a \cdot 0 \end{bmatrix}$ results in the contradiction $a_2 = 0$. On the other hand

$$\begin{bmatrix} a_1 \cdot 1 \\ a_2 \cdot 1 \end{bmatrix} = \begin{bmatrix} au \\ av \end{bmatrix} \text{ has the solution } a = 1, \alpha = \begin{bmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \square$$

Proposition 4.13 notes consequences of the symmetry of antichains.

Proposition 4.13. Suppose that the representing matrix $M = \begin{bmatrix} M_1 \| \cdots \| M_w \end{bmatrix}$ defines a group $G \in \operatorname{hc}(S_w, p^e)$ that is indecomposable.

Let π be a permutation of $\{1, 2, \ldots, w\}$. Then $M' = \begin{bmatrix} M_{\pi(1)} \| \cdots \| M_{\pi(w)} \end{bmatrix}$ defines a group $H \in \operatorname{hc}(S_w, p^e)$ that is indecomposable.

Proof. This is due to the symmetry in the critical types. In fact, the indecomposability criterion Corollary 4.4 depends only on the poset structure of the critical typeset and the critical types can be assigned to the column blocks of M in any desired way. Then the types can be listed as desired without changing the group resulting in a permutation of the column blocks of M. □

5 Unbounded representation type

We demonstrate the use of the Indecomposability Criterion in a particularly simple case. We will make use of the fact that there exist $n \times n$ integral matrices A with the property that an $n \times n$ matrix X must equal 0 or 1 modulo p if $AX \equiv XA \mod p$. This is the case if the minimal polynomial of A is the power of an irreducible polynomial. Making an n-dimensional $\mathbb{Z}_p$-vector space V into a $\mathbb{Z}_p[x]$-module using A (via $xv = Av$, $v \in V$), the matrix A has the desired property if the $\mathbb{Z}_p[x]$-module V turns out to be indecomposable.

Theorem 5.1. The category $\mathrm{hc}(S_4, p)$ has unbounded representation type.

Proof. Let A be an $n \times n$ matrix with entries in $\mathbb{Z}_p$ such that $AX = XA$ implies that $X \in \{0, 1\}$, and let G be a group in $\mathrm{hc}(S_4, p)$ with representing matrix

$$M = \begin{bmatrix} I_n & \| & 0 & \| & I_n & \| & I_n \\ 0 & \| & I_n & \| & I_n & \| & A \end{bmatrix}$$

Clearly $M^* = \begin{bmatrix} I_n & 0 \\ 0 & I_n \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a right inverse of M. By an easy computation

$$M^*M = \begin{bmatrix} I_n & 0 & I_n & I_n \\ 0 & I_n & I_n & A \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $f^2 = f$ be a type idempotent in U_G. Then, for $n \times n$ idempotent matrices a, b, c, d,

$$f = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \quad Mf = \begin{bmatrix} a & 0 & c & d \\ 0 & b & c & Ad \end{bmatrix}, \quad MfM^*M = \begin{bmatrix} a & 0 & a & a \\ 0 & b & b & bA \end{bmatrix}.$$

Now assume that f is a representation idempotent. Then $Mf = MfM^*M$ implies that $c = a = d = b$ and $Ad = bA$, hence $Ab = bA$ and it follows that $b \in \{0, 1\}$ and that $f \in \{0, 1\}$. □

Theorem 5.2 is stated in [Arnold-Dugas98] with an equivalent matrix but without proof. The proof is delicate and we include it in detail.

Theorem 5.2. The category $\mathrm{hc}(S_3, p^3)$ has unbounded representation type.

Proof. Let $G \in \mathrm{hc}(S_3, p^3)$-group with representing matrix

$$M = \begin{bmatrix} I_n & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 & \| & 0 & I_n & pI_n & 0 \\ 0 & I_n & 0 & 0 & \| & 0 & 0 & 0 & p^2I_n & \| & I_n & I_n & 0 & p^2I_n \\ 0 & 0 & I_n & 0 & \| & 0 & 0 & I_n & 0 & \| & 0 & 0 & A & pI_n \\ 0 & 0 & 0 & p^2I_n & \| & 0 & I_n & 0 & 0 & \| & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \| & I_n & 0 & 0 & 0 & \| & I_n & 0 & 0 & 0 \end{bmatrix}$$

where A is a square matrix such that $AX \equiv XA \mod p$ implies that $X \in \{0, 1\}$.

Clearly $M^* = \begin{bmatrix} I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{tr}}$ is a right inverse of M.

Let $f^2 = f$ be an idempotent type endomorphism of U_G. Then $f = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$,

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}, \quad c = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{bmatrix}$$

where the a_{ij}, b_{ij}, c_{ij} are $n \times n$ matrices.

Suppose now that f is a representation morphism. Then $Mf = MfM^*M$ implies the matrix equations

- $a_{14} = 0 \quad a_{13} = 0 \quad p^2a_{12} = 0 \quad a_{24} = 0 \quad a_{23} = p^2b_{43} \quad a_{34} = p^2b_{32}$
- $b_{34} = p^2a_{32} \quad b_{23} = p^2a_{43} \quad b_{24} = 0 \quad p^2b_{12} = 0 \quad b_{13} = 0 \quad b_{14} = 0$
- $c_{13} = 0 \quad c_{14} = 0 \quad c_{24} = 0$
- $p^2b_{44} = p^2a_{22} \quad b_{33} = a_{33} \quad p^2a_{44} = p^2b_{22} \quad c_{11} = b_{11} \quad c_{22} = b_{22} + p^2a_{41} + p^2a_{42}$
- $c_{21} + pc_{31} = a_{12}$
- $c_{22} + pc_{32} = a_{12} + a_{11}$
- $c_{11} + c_{21} + p^2c_{41} = a_{22} + p^2b_{41}$
- $c_{12} + c_{22} + p^2c_{42} = a_{22} + a_{21} + p^2b_{42}$
- $c_{21} = p^2a_{42} + b_{21}$
- $c_{12} = b_{12}$
- $c_{23} + pc_{33} = a_{13}A + pa_{11}$
- $c_{24} + pc_{34} = pa_{13} + p^2a_{12}$
- $p^2c_{43} + c_{23} + c_{13} = pa_{21} + a_{23}A$
- $c_{14} + c_{24} + p^2c_{44} = pa_{23} + p^2a_{22}$
- $Ac_{33} + pc_{43} = a_{33}A + pa_{31}$
- $Ac_{34} + pc_{44} = pa_{33} + p^2a_{32}$
- $c_{23} = p^2a_{43}A$

We interpret these equations modulo p, but retain stronger information as needed.

$$a_{14} = 0 \quad a_{13} = 0 \quad a_{12} \equiv 0 \quad a_{24} = 0 \quad pa_{23} = 0 \quad a_{34} \equiv 0 \tag{5.3}$$

$$b_{34} \equiv 0 \quad b_{23} \equiv 0 \quad b_{24} = 0 \quad b_{12} \equiv 0 \quad b_{13} = 0 \quad b_{14} = 0 \tag{5.4}$$

$$c_{13} = 0 \quad c_{14} = 0 \quad c_{24} = 0 \tag{5.5}$$

$$b_{44} \equiv a_{22} \quad b_{33} = a_{33} \quad a_{44} \equiv b_{22} \quad c_{11} = b_{11} \quad c_{22} \equiv b_{22}$$

$$c_{21} \equiv a_{12} \text{ so } c_{21} \equiv 0 \text{ by (5.3)} \tag{5.6}$$

$$c_{22} \equiv a_{12} + a_{11} \text{ so } c_{22} \equiv a_{11} \text{ by (5.3)}$$

$$c_{11} + c_{21} \equiv a_{22} \text{ so } c_{11} \equiv a_{22} \text{ by (5.6)}$$

$$c_{12} + c_{22} \equiv a_{22} + a_{21} \text{ so } c_{22} \equiv a_{22} \text{ by (5.7), (5.8)}$$

$$c_{21} \equiv b_{21} \text{ so } b_{21} \equiv 0 \text{ by (5.6)}$$

$$c_{12} = b_{12} \text{ so } c_{12} \equiv 0 \text{ by (5.4)}$$

$$c_{23} \equiv a_{13}A \text{ so } c_{23} \equiv 0 \text{ by (5.3)} \tag{5.7}$$

$$\begin{aligned} &c_{24} + pc_{34} = pa_{13} + p^2a_{12} \text{ so } pc_{34} = pa_{13} \text{ by (5.5), (5.3)} \\ &\quad \text{so } pc_{34} = 0 \text{ by (5.3)} \end{aligned}$$

$$\begin{aligned} &p^2c_{43} + c_{23} + c_{13} = pa_{21} + a_{23}A \text{ so } pc_{23} = p^2a_{21} + pa_{23}A \\ &\quad \text{using (5.5) so } p^2a_{21} = 0 \text{ by (5.3), (5.9), so } a_{21} \equiv 0 \end{aligned}$$

$$\begin{aligned} &c_{14} + c_{24} + p^2c_{44} = pa_{23} + p^2a_{22} \\ &\quad \text{so } p^2c_{44} = p^2a_{22} \text{ by (5.5), (5.3) so } c_{44} \equiv a_{22} \end{aligned}$$

$$Ac_{33} \equiv a_{33}A \tag{5.8}$$

$$\begin{aligned} &Ac_{34} + pc_{44} = pa_{33} + p^2a_{32} \text{ so} \\ &Apc_{34} + p^2c_{44} = p^2a_{33} \text{ so } p^2c_{44} = p^2a_{33} \text{ by (5.8) so } c_{44} \equiv a_{33} \end{aligned}$$

$$c_{23} = p^2a_{43}A \text{ so } pc_{23} = 0$$

Combining the information above we now have that, modulo p,

$$\alpha := a_{11} \equiv a_{22} \equiv a_{33} \equiv a_{44} \equiv b_{11} \equiv b_{22} \equiv b_{33} \equiv b_{44} \equiv c_{11} \equiv c_{22} \equiv c_{33} \equiv c_{44}$$

and the idempotent matrices a, b, c are lower triangular with the one value α as diagonal blocks. By (5.8) we have $A\alpha = \alpha A$ and by choice of the matrix A, either $\alpha \equiv 0 \mod p$ or $\alpha \equiv 1 \mod p$. By then α itself must be 0 or 1, therefore the lower triangular matrices a, b, c must be 0 or 1 and finally $f \in \{0, 1\}$. □

Corollary 5.9. The category $\mathrm{hc}(S_w, p^m)$ has unbounded representation type if

1. $w \geq 4$, $m \geq 1$,
2. $w = 3$ and $m \geq 3$.

Proof. (1) follows from Theorem 5.1 and Lemma 4.7.
(2) follows from Theorem 5.2 and Lemma 4.5. □

6 Indecomposable groups in $\mathrm{hc}(S_3, p^e)$, $e \leq 3$

The following proposition, expressing a change of bases B_G and C_G of a representing matrix in terms of row and column operations, is analogous to [AMMS12] for $(1, n)$-groups and to [AMMS11, Lemmas 14 and 24] and [Mader00, Proposition 12.4.1] with different notation.

Lemma 6.1. Let $G \in \mathrm{hc}(S_w, p^e)$ with representing matrix

$$M_G = \left[M_1 \| M_2 \| \cdots \| M_w \right].$$

If M_G can be transformed into a matrix M by a sequence of the six types of transformations below, then $M = M_H$ is the representing matrix of a group $H \in \mathrm{hc}(S_w, p^e)$ that is nearly isomorphic with G.

1. Add a $\mathbb{Z}_{p^e}$-multiple of a row of M_G to any other row.
2. Multiply a row of M_G by a unit of $\mathbb{Z}_{p^e}$.
3. Interchange any two rows of M_G.
4. Add a $\mathbb{Z}_{p^e}$-multiple of a column of M_j to another column of M_j.
5. Multiply a column of M_G by a unit of $\mathbb{Z}_{p^e}$.
6. Interchange any two columns of M_j.

Proof. Let $U_G = (U_0, U_\sigma, U_* \mid \sigma \in S_w)$ be the representation of G where $U_0 = \bigoplus_{\sigma\in S_w} V_\sigma$. The representing matrix is obtained by choosing a proper basis $B = \bigcup_{\sigma\in S_w} B_\sigma$ of U_0 where B_σ is a basis of the free module V_σ and choosing a basis C of $U_* \cong G/\mathrm{R}(G)$. In terms of coordinates with respect to these bases U_* is just the row space of the representing matrix M_G.

Elementary row transformations do not change the row space and hence the corresponding group is still G. We have justified (1)–(3).

Elementary column transformations in the blocks M_i correspond to basis changes in the V_σ and hence to an automorphism ξ of U_0 with $\xi(V_\sigma) = V_\sigma$. There is a group $H \in \mathrm{hc}(S_w, p^e)$ with representation $U_H = (U_0, U_\sigma, \xi(U_*) \mid \sigma \in S_w)$. Then $H \cong_{\mathrm{nr}} G$ and the representing matrix of H with respect to the new bases of the U_σ is just the transform of M_G. This justifies (4)–(6). □

The following proofs start with an arbitrary representing matrix of an indecomposable group and then simplify the matrix using allowed transformation until direct summands appear that must coincide with the group. Recall that $\mathrm{hc}'(S, p^e)$ contains those groups $G \in \mathrm{hc}(S, p^e)$ with $\exp(G/\mathrm{R}(G)) = p^e$.

Theorem 6.2. Let $\{\tau_1, \tau_2, \tau_3\}$ be a given antichain of types. Then there are in $\mathrm{hc}'(S_3, p)$ up to near–isomorphism exactly two indecomposable groups with critical typeset $\{\tau_1, \tau_2, \tau_3\}$ and representing matrices

$$\begin{bmatrix}1 \| 1 \| 1\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}1 & \| & 0 & \| & 1\\ 0 & \| & 1 & \| & 1\end{bmatrix}.$$

Proof. Let G be an indecomposable group in $\mathrm{hc}'(S_3, p)$. Then $\mathrm{rk}(G) \geq 3$. Let $M_G = \begin{bmatrix}M_1 \| M_2 \| M_3\end{bmatrix}$ be a representing matrix for G. Each of the submatrices M_i must have at least one column. The entries of M_G are either 0 or units. The matrix M_G cannot have 0-columns and any row transformation on the whole matrix and any column transformation within blocks is allowed corresponding to basis changes or switches to a nearly isomorphic group, Lemma 6.1.

If M_G has just one row it is easy to see that the representing matrix has to be $\begin{bmatrix}1 \| 1 \| 1\end{bmatrix}$ and there is only one group of this kind up to near–isomorphism.

Thus let M_G have $r \geq 2$ rows. It may be assumed that $M_1 = \begin{bmatrix}I_a\\ 0\end{bmatrix}$ where I_a is an identity matrix of size $a \times a$ and 0 is a zero matrix of size $(r-a) \times a$. As M_1 must have columns, $a \geq 1$. The Regulator Criterion requires that M_2 contains units in the last $r - a$ rows. These can be used to annihilate upward and to the right. Let J_s denote the $s \times s$-matrix with entries 1 on the co-diagonal and and 0 entries otherwise, e.g. $J = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. With this notation we get

$$M_G = \begin{bmatrix}I_a & \| & 0 & X & \| & Y\\ 0 & \| & J_{r-a} & 0 & \| & Z\end{bmatrix}$$

The submatrix X may be transformed to (reversed) Smith Normal Form and cannot contain a 0-column. Thus we get the form

$$M_G = \begin{bmatrix}I_b & 0 & \| & 0 & 0 & \| & X\\ 0 & I_{a-b} & \| & 0 & J_{a-b} & \| & Y\\ 0 & 0 & \| & J_{r-a} & 0 & \| & Z\end{bmatrix}, \quad a \geq b \geq 0.$$

The submatrix $\begin{bmatrix}X\\ Y\\ Z\end{bmatrix}$ may be assumed to be in column echelon form and the first b rows must have full rank. With some annihilations we have

$$M_G = \begin{bmatrix} I_b & 0 & \| & 0 & 0 & \| & I_b & 0 \\ 0 & I_{a-b} & \| & 0 & J_{a-b} & \| & 0 & Y_1 \\ 0 & 0 & \| & J_{r-a} & 0 & \| & Z_1 & Z_2 \end{bmatrix}.$$

If $Z_2 \neq 0$, we can and annihilate in Z_1 and get a rank-2 summand. On the other hand if $Z_2 = 0$, then there is a rank-2 or rank-3 summand. Hence the last column must be absent and we must have $a = b$ resulting in

$$M_G = \begin{bmatrix} I_a & \| & 0 & \| & I_a \\ 0 & \| & J_{r-a} & \| & Z_1 \end{bmatrix}.$$

Without loss of generality Z_1 is in Smith Normal Form and Z_1 cannot have a 0-row. Hence $Z_1 = \begin{bmatrix} I_{r-a} & 0 \end{bmatrix}$ and if the 0 entry actually appears, then we have a forbidden rank-2 summand. So $Z_1 = \begin{bmatrix} I_b \end{bmatrix}$ and to be indecomposable we must have $a = 1$, $r = 2$ and obtain the indecomposable matrix

$$M_G = \begin{bmatrix} M_1 \| M_2 \| M_3 \end{bmatrix} = \begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}.$$

Any permutation of the submatrices M_i can be changed by allowed transformations to the form M_G showing there is only one such group up to near–isomorphism. □

Theorem 6.3. Groups in $\mathrm{hc}'(S_3, p^2)$ are indecomposable if they have one of the following representing matrices $\begin{bmatrix} M_1 \| M_2 \| M_3 \end{bmatrix}$.

1. Groups with cyclic regulator quotient:
   a. $M_{G_{31}} = \begin{bmatrix} 1 \| 1 \| 1 \end{bmatrix}$, $\mathrm{rk}(G_{31}) = 3$,
   b. $M_{G_{32}} = \begin{bmatrix} 1 \| 1 \| p \end{bmatrix}$, $\mathrm{rk}(G_{32}) = 3$, with two permutations $\begin{bmatrix} 1 \| p \| 1 \end{bmatrix}$ and $\begin{bmatrix} p \| 1 \| 1 \end{bmatrix}$,
2. Groups with 2-generated regulator quotient:
   a. $M_{G_{21}} = \begin{bmatrix} 1 & \| & 0 & \| & 1 \\ 0 & \| & 1 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{21}) = 3$,
   b. $M_{G_{22}} = \begin{bmatrix} 1 & 0 & \| & 0 & \| & 1 \\ 0 & p & \| & 1 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{22}) = 4$, with two permutations $\begin{bmatrix} 0 & \| & 1 & 0 & \| & 1 \\ 1 & \| & 0 & p & \| & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & \| & 1 & \| & 1 & 0 \\ 1 & \| & 1 & \| & 0 & p \end{bmatrix}$,
   c. $M_{G_{23}} = \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix}$, $\mathrm{rk}(G_{23}) = 5$, with two permutations $\begin{bmatrix} 1 & 0 & \| & 1 & \| & 0 & p \\ 0 & p & \| & 1 & \| & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & \| & 1 & 0 & \| & 0 & p \\ 1 & \| & 0 & p & \| & 1 & 0 \end{bmatrix}$,
   d. $M_{G_{24}} = \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 & 0 \\ 0 & p & \| & 1 & 0 & \| & 1 & p \end{bmatrix}$, $\mathrm{rk}(G_{24}) = 6$,
3. Groups with 3-generated regulator quotient:
   $$M_{G_{31}} = \begin{bmatrix} 1 & 0 & \| & 0 & 0 & \| & 1 & 0 \\ 0 & 1 & \| & 0 & 1 & \| & 0 & 1 \\ 0 & 0 & \| & 1 & 0 & \| & 1 & p \end{bmatrix}, \ \mathrm{rk}(G_{31}) = 6,$$
4. Groups with 4-generated regulator quotient:
   $$M_{G_{41}} = \begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 0 & 1 & p \end{bmatrix}, \ \mathrm{rk}(G_{41}) = 9.$$

Proof. We only show that G_{41} is indecomposable. The other proofs are similar and simpler. We also demonstrate in one case why some permutations as in Proposition 4.13 do not result in new near–isomorphism classes of indecomposable groups.

The matrix $M^* = \begin{bmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0&0 \end{bmatrix}^{\mathrm{tr}}$ is a right inverse of the representing matrix $M_{G_{41}}$. A type endomorphism of G_{41} is $f = \operatorname{diag}(a, b, c)$ where

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad b = \begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \quad c = \begin{bmatrix} c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33} \end{bmatrix}.$$

Then $Mf = MfM^*M$ if and only if

$$\begin{array}{ccccc} a_{13} = 0 & 0 = pa_{12} & c_{11} = a_{11} & c_{12} = a_{12} & c_{13} = 0\\ a_{23} = 0 & pb_{33} = pa_{22} & c_{21} = a_{21} + pb_{32} & c_{22} = a_{22} + pb_{31} & c_{23} = 0\\ pa_{33} = pb_{22} & b_{23} = 0 & c_{11} + pc_{31} = b_{22} + pa_{31} & c_{12} + pc_{32} = b_{21} + pa_{32} & c_{13} + pc_{33} = pb_{21} + pb_{22}\\ 0 = pb_{12} & b_{13} = 0 & c_{21} + pc_{31} = b_{12} & c_{22} + pc_{32} = b_{11} & c_{23} + pc_{33} = pb_{11} + pb_{12} \end{array}$$

From these we immediately get with $\alpha := a_{11}$ and $\beta := a_{22}$ that, modulo p,

$$a \equiv \begin{bmatrix} \alpha & 0 & 0\\ a_{21} & \beta & 0\\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad b \equiv \begin{bmatrix} \beta & 0 & 0\\ 0 & \alpha & 0\\ b_{31} & b_{32} & \beta \end{bmatrix}, \quad c \equiv \begin{bmatrix} \alpha & 0 & 0\\ c_{21} & \beta & 0\\ c_{31} & c_{32} & \beta \end{bmatrix}.$$

Using a combination of equalities we further get $a_{33} \equiv b_{22} \equiv \beta$ and $\beta \equiv c_{33} \equiv b_{22} \equiv \alpha$, hence $\alpha \equiv \beta$. The equalities $c_{21} = a_{21} + pb_{32}$ and $c_{21} + pc_{31} = b_{12}$ were not needed. Now assume that $f^2 = f$. Then $a^2 = a$, $b^2 = b$ and $c^2 = c$. Modulo p these are lower triangular matrices with α on all diagonals. So $a \equiv b \equiv c \in \{0, 1\}$, hence $a = b = c \in \{0, 1\}$ and finally $f \in \{0, 1\}$.

According to Proposition 4.13 the group H with representing matrix

$$M = \left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&p\\ 1&0&p&0&0&p&0&1&0\\ 0&1&p&0&0&0&1&0&0 \end{array}\right]$$

is indecomposable. To show that $H \cong_{\mathrm{nr}} G_{41}$ we will transform M. The steps mirror the construction of the normal form (6.6): First obtain the Smith Normal Form of the first block, then the reverse Smith Normal Form of the second block and finally simplify the third block using column transformations and some row transformations that are still possible without destroying the special form of the first two blocks.

To wit, we first subtract row 1 from row 3 and row 2 from row 4 to get

$$\left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&p\\ 0&0&p&-1&0&p&0&1&0\\ 0&0&p&0&-1&0&1&0&-p \end{array}\right]$$

Next we subtract row 3 from row 4 and get

$$\left[\begin{array}{ccc|ccc|ccc} 1&0&0&1&0&0&0&0&0\\ 0&1&0&0&1&0&0&0&p\\ 0&0&p&-1&0&p&0&1&0\\ 0&0&0&1&-1&-p&1&-1&-p \end{array}\right]$$

Next we clear entries above 1 in block 2 and get

$$\left[\begin{array}{ccc|ccc|ccc} 1&0&0&0&1&p&-1&1&p\\ 0&1&0&0&1&0&0&0&p\\ 0&0&p&0&-1&0&1&0&-p\\ 0&0&0&1&-1&-p&1&-1&-p \end{array}\right]$$

We get after several column and row transformations

$$\begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 0 & p & \| & 0 & -1 & 0 & \| & 1 & 0 & -p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 1 & -1 & -p \end{bmatrix}$$

Exchanging the first two rows and making adjustments we get

$$\begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & -p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 1 & -1 & -p \end{bmatrix}$$

Finally, column transformations in the third block and clearing fill-ins results in

$$\begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 0 & 1 & p \end{bmatrix} = M_{G_{41}}.$$

On the other hand, the representing matrices

$$M_{G_{23}} = \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix}, \quad \text{and} \quad M := \begin{bmatrix} 1 & 0 & \| & 1 & \| & 0 & p \\ 0 & p & \| & 1 & \| & 1 & 0 \end{bmatrix}$$

cannot belong to nearly isomorphic groups because the ranks of the homogeneous components of the regulator are different. □

We next establish a sort of "normal form" of the representing matrices that we have to deal with. Let J_a denote the $a \times a$ matrix with entries 1 on the co-diagonal and vanishing entries elsewhere.

Lemma 6.4. Let $G \in \mathrm{hc}'(S_3, p^2)$ be indecomposable. Then there is a $H \in \mathrm{hc}'(S_3, p^2)$ nearly isomorphic to G with representing matrix

$$M = \begin{bmatrix} M_1 \| M_2 \| M_3 \end{bmatrix} =$$

$$\begin{bmatrix} I_{a_1} & 0 & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 & 0 & \| & I_{a_1} & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 & pJ_{a_2} & \| & 0 & I_{a_2} & 0 & 0 \\ 0 & 0 & I_{a_{31}} & 0 & 0 & \| & 0 & 0 & 0 & J_{a_{31}} & 0 & \| & 0 & 0 & I_{a_{31}} & 0 \\ 0 & 0 & 0 & I_{a_{32}} & 0 & \| & 0 & 0 & J_{a_{32}} & 0 & 0 & \| & 0 & 0 & 0 & pX_1 \\ 0 & 0 & 0 & 0 & pI_b & \| & 0 & J_b & 0 & 0 & 0 & \| & X_2 & X_3 & 0 & pX_4 \\ 0 & 0 & 0 & 0 & 0 & \| & J_c & 0 & 0 & 0 & 0 & \| & X_5 & X_6 & pX_7 & pX_8 \end{bmatrix}$$

where one or more of the rows of M and columns of the M_i may be absent.

Let

- r denote the number of rows of M,
- n_i the number of columns of M_i, and
- d the number of columns of pX_1.

Then we may further assume that $n_1 \geq n_2 \geq n_3$ and if $n_1 = n_2$, then $\mathrm{rk}_p\, M_1 \geq \mathrm{rk}_p\, M_2$, hence

$$a_1 \geq c, \quad d \leq (b+c) + a_{32} - a_1, \quad b + c \leq a_1 + a_2, \quad 1 \leq b + c \leq \frac{r}{2}.$$

In particular, $a_1 \leq (b+c) + a_{32}$. If $n_1 = n_2$, then $a_1 = c$, and $b \leq a_2$.

Proof. Let G be indecomposable in $\mathrm{hc}'(S_3, p^2)$. Then $\mathrm{rk}(G) \geq 3$. The case $\mathrm{rk}(G) = 3$ being simple we assume that $\mathrm{rk}(G) \geq 4$. Hence if a summand of rank ≤ 4 shows up, it must be omitted.

By symmetry in the poset of types we may rearrange the M_i and hence can assume that $n_1 \geq n_2 \geq n_3$. Furthermore, if $n_1 = n_2$, then we may assume that the rank of M_1 modulo p is $\geq$ to the rank of M_2 modulo p.

Arbitrary row transformations on M and arbitrary column transformations within each M_i will produce representing matrices of groups nearly isomorphic to G and the Smith Normal Form of M_1 has no zero column because G is indecomposable.

$$M_1 = \begin{bmatrix} I_a & 0 \\ 0 & pI_b \\ 0 & 0 \end{bmatrix}.$$

We can establish the column echelon form of M_2 from the bottom up, and using that, due to the Regulator Criterion, there must be units in all rows of M_2 except possibly in the first a rows, we obtain the form

$$M_2 = \begin{bmatrix} * & * & X \\ 0 & J_b & 0 \\ J_c & 0 & 0 \end{bmatrix}.$$

Row transformations upward in M_2 are allowed because the changes so produced in M_1 can be undone by column transformations. Therefore we can assume that

$$\Big[M_1 \| M_2\Big] = \begin{bmatrix} I_a & 0 & \| & 0 & 0 & X \\ 0 & pI_b & \| & 0 & J_b & 0 \\ 0 & 0 & \| & J_c & 0 & 0 \end{bmatrix}.$$

Any row and column transformations can be used on X because whatever change occurs in I_a can be undone by column transformations. Thus we can achieve the reverse Smith Normal Form of X and, as no zero columns can appear, we have

$$\Big[M_1 \| M_2\Big] = \begin{bmatrix} I_{a_1} & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & \| & 0 & 0 & 0 & pJ_{a_2} \\ 0 & 0 & I_{a_3} & 0 & \| & 0 & 0 & J_{a_3} & 0 \\ 0 & 0 & 0 & pI_b & \| & 0 & J_b & 0 & 0 \\ 0 & 0 & 0 & 0 & \| & J_c & 0 & 0 & 0 \end{bmatrix}. \tag{6.5}$$

We now look at M_3 assuming that $\Big[M_1 \| M_2\Big]$ are in the form (6.5). First of all we may assume that M_3 is in reduced column echelon form and the first $a_1 + a_2$ rows must contain units to satisfy the Regulator Criterion, Lemma 4.1. These units may be used to annihilate below and we get

$$M = \begin{bmatrix} I_{a_1} & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 & \| & I_{a_1} & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & \| & 0 & 0 & 0 & pJ_{a_2} & \| & \mathbf{0} & I_{a_2} & 0 \\ 0 & 0 & I_{a_3} & 0 & \| & 0 & 0 & J_{a_3} & 0 & \| & \mathbf{0} & \mathbf{0} & X_1 \\ 0 & 0 & 0 & pI_b & \| & 0 & J_b & 0 & 0 & \| & X_2 & X_3 & X_4 \\ 0 & 0 & 0 & 0 & \| & J_c & 0 & 0 & 0 & \| & X_5 & X_6 & X_7 \end{bmatrix},$$

where the boldface zeros in M_3 indicate that they can always be restored if filled in by column transformations.

If X_7 contains a unit, it can be used to annihilate up, down, right and left and we get a rank-3 summand. So $X_7 = pX_7'$. Now if X_4 contains a unit, it can be used to annihilate up, down (because $X_7 = pX_7'$ and $p^2 = 0$), left, right and get a rank-3 summand. So $X_4 = pX_4'$. We now have

$$M = \begin{bmatrix} I_{a_1} & 0 & 0 & 0 & \| & 0 & 0 & 0 & 0 & \| & I_{a_1} & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & \| & 0 & 0 & 0 & pJ_{a_2} & \| & \mathbf{0} & I_{a_2} & 0 \\ 0 & 0 & I_{a_3} & 0 & \| & 0 & 0 & J_{a_3} & 0 & \| & \mathbf{0} & \mathbf{0} & X_1 \\ 0 & 0 & 0 & pI_b & \| & 0 & J_b & 0 & 0 & \| & X_2 & X_3 & pX_4' \\ 0 & 0 & 0 & 0 & \| & J_c & 0 & 0 & 0 & \| & X_5 & X_6 & pX_7' \end{bmatrix},$$

Arbitrary row and column transformations may be used on X_1 and we therefore may assume that X_1 is in Smith Normal Form which we can compress to $X_1 = \begin{bmatrix} I_{a_{31}} & 0 \\ 0 & pX_1' \end{bmatrix}$. With $I_{a_{31}}$ we can annihilate in pX_4' to get $M=$

$$\left[\begin{array}{ccccc|ccccc|cccc} I_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{a_1} & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pJ_{a_2} & \mathbf{0} & I_{a_2} & 0 & 0 \\ 0 & 0 & I_{a_{31}} & 0 & 0 & 0 & 0 & 0 & J_{a_{31}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ 0 & 0 & 0 & I_{a_{32}} & 0 & 0 & 0 & J_{a_{32}} & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & pX_1' \\ 0 & 0 & 0 & 0 & pI_b & 0 & J_b & 0 & 0 & 0 & X_2 & X_3 & \mathbf{0} & pX_4'' \\ 0 & 0 & 0 & 0 & 0 & J_c & 0 & 0 & 0 & 0 & X_5 & X_6 & pX_7'' & pX_7''' \end{array}\right]$$

and we have established the claimed form after adjusting notation.

With this form of the representing matrix we see that

1. $n_1 = a_1 + a_2 + a_{31} + a_{32} + b = \mathrm{rk}_p(M_1) + b$,
2. $n_2 = a_2 + a_{31} + a_{32} + b + c = \mathrm{rk}_p(M_2) + a_2$,
3. $n_3 = a_1 + a_2 + a_{31} + d$.

The assumption $n_1 \geq n_2 \geq n_3$ implies that $a_1 \geq c$ and $a_{32} + b + c \geq a_1 + d$. Suppose that $n_1 = n_2$, i.e., $a_1 = c$. Then $\mathrm{rk}_p(M_1) \geq \mathrm{rk}_p(M_2)$ if and only if $n_1 - b \geq n_1 - a_2$ if and only if $a_2 \geq b$. It should be noted here that the established form of $[M_1 \| M_2]$ can be reestablished in $[M_2 \| M_1]$ by reversing the order of the rows.

Finally, the Regulator Criterion requires that the submatrix

$$[M_1\ M_3] = \begin{bmatrix} I_{a_1} & 0 & 0 & 0 & 0 & I_{a_1} & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & 0 & 0 & I_{a_2} & 0 & 0 \\ 0 & 0 & I_{a_{31}} & 0 & 0 & 0 & 0 & I_{a_{31}} & 0 \\ 0 & 0 & 0 & I_{a_{32}} & 0 & 0 & 0 & 0 & pX_1' \\ 0 & 0 & 0 & 0 & pI_b & X_2 & X_3 & 0 & pX_4'' \\ 0 & 0 & 0 & 0 & 0 & X_5 & X_6 & pX_7'' & pX_7''' \end{bmatrix}$$

has rank r modulo p. This means that the rank modulo p of $\begin{bmatrix} X_2 & X_3 \\ X_5 & X_6 \end{bmatrix}$ must be $b+c$ which equals its number of rows and this must be less than or equal to its number of columns $a_1 + a_2$. □

In order to keep the matrices from growing too large, we often use the abbreviated "normal" form

$$M = \left[\ M_1 \,\|\, M_2 \,\|\, M_3\ \right] = \left[\begin{array}{c|c|cccc} I_{a_1} & 0 & I_{a_1} & 0 & 0 & 0 \\ I_{a_2} & pJ_{a_2} & \mathbf{0} & I_{a_2} & 0 & 0 \\ I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ I_{a_{32}} & J_{a_{32}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & pX_1 \\ pI_b & J_b & X_2 & X_3 & 0 & pX_4 \\ 0 & J_c & X_5 & X_6 & pX_7 & pX_8 \end{array}\right]. \qquad (6.6)$$

The boldface **0**'s signal that these zeros can always be reestablished from above if they are "filled in" by some column transformations in M_3.

The case $\mathrm{rk}\, G = 3$ being easy, we assume that $\mathrm{rk}\, G > 3$, i.e., if M has a non-zero summand with up to 3 columns, then we have a contradiction and can exclude whatever conditions produce this outcome. In general, if $G = G_1 \oplus G_2$, then either G_1 is indecomposable and $G = G_1$ or $G_1 = 0$ and $G = G_2$.

Theorem 6.7. The list of fourteen near–isomorphism classes of indecomposable groups in $\mathrm{hc}'(S_3, p^2)$ given in Theorem 6.3 is complete.

Proof. We will use certain **language** in this proof that we wish to clarify first.

- The term ***line*** means a row or a column. A 0-***line*** is a line of zeros and a p-***line*** is a line all of whose entries are in $p\mathbb{Z}_{p^2}$.
- A ***cross at*** (i,j) in a matrix A occurs if all entries in row i and column j are zero except possibly the entry at location (i,j), the ***pivot*** of the cross.
- By *"$x \in A$ leads to a cross"* we mean that this entry x can be used to annihilate in its row and its column to produce a cross.

- We apply transformations to annihilate entries. While doing this, some entries that were originally zero may change to nonzero entries; these entries are called *fill-ins*. The phrase *"we annihilate"* tacitly includes that the fill-ins caused in the process can be removed by subsequent transformations without destroying previously established desired forms, mostly blocks p^iI for $i \geq 0$.
- The phrase *"we form the Smith Normal Form of A"* implies that the Smith Normal Form can be established by allowed line-transformations. It is tacitly ascertained that destroyed p^iI-blocks can be reestablished.

Let $M = \begin{bmatrix} M_1 \| M_2 \| M_3 \end{bmatrix}$ be the representing matrix of an indecomposable group $G \in \mathrm{hc}'(S_3, p^2)$ of rank ≥ 4. Then without loss of generality

$$M = \left[\begin{array}{ccccc|ccccc|cccc} I_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{a_1} & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pJ_{a_2} & \mathbf{0} & I_{a_2} & 0 & 0 \\ 0 & 0 & I_{a_{31}} & 0 & 0 & 0 & 0 & 0 & J_{a_{31}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ 0 & 0 & 0 & I_{a_{32}} & 0 & 0 & 0 & J_{a_{32}} & 0 & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & pX_1 \\ 0 & 0 & 0 & 0 & pI_b & 0 & J_b & 0 & 0 & 0 & X_2 & X_3 & 0 & pX_4 \\ 0 & 0 & 0 & 0 & 0 & J_c & 0 & 0 & 0 & 0 & X_5 & X_6 & pX_7 & pX_8 \end{array}\right]$$

(a) pX_1 is not present.

The submatrix $\begin{bmatrix} pX_1 \\ pX_4 \\ pX_8 \end{bmatrix}$ allows for row transformations upward and arbitrary column transformations. Thus it can be transformed to reduced column echelon form, implying that there is at most one nonzero entry in a row, i.e., a row is either a 0-row or it contains the pivot of a cross. So the row block containing pX_1 is not present, hence

$$M = \left[\begin{array}{cccc|cccc|cccc} I_{a_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{a_1} & 0 & 0 & 0 \\ 0 & I_{a_2} & 0 & 0 & 0 & 0 & 0 & pJ_{a_2} & \mathbf{0} & I_{a_2} & 0 & 0 \\ 0 & 0 & I_{a_{31}} & 0 & 0 & 0 & J_{a_{31}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ 0 & 0 & 0 & pI_b & 0 & J_b & 0 & 0 & X_2 & X_3 & 0 & pX_4 \\ 0 & 0 & 0 & 0 & J_c & 0 & 0 & 0 & X_5 & X_6 & pX_7 & pX_8 \end{array}\right]$$

(b) Simplified Form of $\begin{bmatrix} X_2 & X_3 \\ X_5 & X_6 \end{bmatrix}$.

We can obtain the partial Smith Normal Form of $X_6 = \begin{bmatrix} I_{c_1} & 0 \\ 0 & pX \end{bmatrix}$ and get

$$M = \left[\begin{array}{c|c|ccccc} I_{a_1} & 0 & I_{a_1} & 0 & 0 & 0 & 0 \\ I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\ I_{a_{22}} & pJ_{a_{22}} & \mathbf{0} & \mathbf{0} & I_{a_{22}} & 0 & 0 \\ I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ pI_b & J_b & X_2 & X_{31} & X_{32} & 0 & pX_4 \\ 0 & J_{c_1} & X_{51} & I_{c_1} & 0 & pX_{71} & pX_{81} \\ 0 & J_{c_2} & X_{52} & 0 & pX & pX_{72} & pX_{82} \end{array}\right]$$

With I_{c_1} we can annihilate X_{31}, X_{51} and also pX_{71} to get

$$M = \left[\begin{array}{c|c|ccccc} I_{a_1} & 0 & I_{a_1} & 0 & 0 & 0 & 0 \\ I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\ I_{a_{22}} & pJ_{a_{22}} & \mathbf{0} & \mathbf{0} & I_{a_{22}} & 0 & 0 \\ I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ pI_b & J_b & X_2 & 0 & X_{32} & 0 & pX_4 \\ 0 & J_{c_1} & 0 & I_{c_1} & 0 & 0 & pX_{81} \\ 0 & J_{c_2} & X_{52} & 0 & pX & pX_{72} & pX_{82} \end{array}\right]$$

The Smith Normal Form of X_{52} cannot contain a p-row, so without loss of generality $X_{52} = \begin{bmatrix} I_{c_2} & 0 \end{bmatrix}$ and we get

$$M = \left[\begin{array}{c|c|cccccc} I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 \\ I_{a_{12}} & 0 & \mathbf{0} & I_{a_{12}} & 0 & 0 & 0 & 0 \\ I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\ I_{a_{22}} & pJ_{a_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{22}} & 0 & 0 \\ I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\ pI_b & J_b & X_{21} & X_{22} & 0 & X_{32} & 0 & pX_4 \\ 0 & J_{c_1} & 0 & 0 & I_{c_1} & 0 & 0 & pX_{81} \\ 0 & J_{c_2} & I_{c_2} & 0 & 0 & pX & pX_{72} & pX_{82} \end{array}\right]$$

With I_{c_2} we can now annihilate X_{21} but also pX because the apparent fill-in above $I_{a_{22}}$ is absent as it contains a factor p^2 and $p^2 = 0$. We now have

$$M = \left[\begin{array}{c||c||cccccc}
I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{12}} & 0 & \mathbf{0} & I_{a_{12}} & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\
I_{a_{22}} & pJ_{a_{22}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{22}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_b & J_b & 0 & X_{22} & 0 & X_{32} & 0 & pX_4 \\
0 & J_{c_1} & 0 & 0 & I_{c_1} & 0 & 0 & pX_{81} \\
0 & J_{c_2} & I_{c_2} & 0 & 0 & 0 & pX_{72} & pX_{82}
\end{array}\right]$$

We form next the partial Smith Normal Form of X_{32} that is $\begin{bmatrix} I_{b_1} & 0 \\ 0 & pY \end{bmatrix}$, and we get

$$M = \left[\begin{array}{c||c||ccccccc}
I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{12}} & 0 & \mathbf{0} & I_{a_{12}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\
I_{a_{222}} & pJ_{a_{222}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{222}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & X_{221} & 0 & I_{b_1} & 0 & 0 & pX_{41} \\
pI_{b_2} & J_{b_2} & 0 & X_{222} & 0 & 0 & pY & 0 & pX_{42} \\
0 & J_{c_1} & 0 & 0 & I_{c_1} & 0 & 0 & 0 & pX_{81} \\
0 & J_{c_2} & I_{c_2} & 0 & 0 & 0 & 0 & pX_{72} & pX_{82}
\end{array}\right]$$

We annihilate X_{221} with I_{b_1} and introduce the Smith Normal Form of X_{222} that cannot contain a p-row and get

$$M = \left[\begin{array}{c||c||cccccccc}
I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & \mathbf{0} & I_{a_{121}} & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{122}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{122}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\
I_{a_{222}} & pJ_{a_{222}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{222}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & 0 & I_{b_1} & 0 & 0 & pX_{41} \\
pI_{b_2} & J_{b_2} & 0 & I_{b_2} & 0 & 0 & 0 & pY & 0 & pX_{42} \\
0 & J_{c_1} & 0 & 0 & 0 & I_{c_1} & 0 & 0 & 0 & pX_{81} \\
0 & J_{c_2} & I_{c_2} & 0 & 0 & 0 & 0 & 0 & pX_{72} & pX_{82}
\end{array}\right]$$

Next we see that pY can be annihilated, the fill-in above can be cleared from below using that $p^2 = 0$ resulting in

$$M = \left[\begin{array}{c||c||cccccccc}
I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & \mathbf{0} & I_{a_{121}} & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{122}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{122}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\
I_{a_{222}} & pJ_{a_{222}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{222}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & 0 & I_{b_1} & 0 & 0 & pX_{41} \\
pI_{b_2} & J_{b_2} & 0 & I_{b_2} & 0 & 0 & 0 & 0 & 0 & pX_{42} \\
0 & J_{c_1} & 0 & 0 & 0 & I_{c_1} & 0 & 0 & 0 & pX_{81} \\
0 & J_{c_2} & I_{c_2} & 0 & 0 & 0 & 0 & 0 & pX_{72} & pX_{82}
\end{array}\right]$$

We have low rank summands unless $a_{122} = 0$ and $a_{222} = 0$, so without loss of generality

$$M = \left[\begin{array}{c||c||cccccc}
I_{a_{11}} & 0 & I_{a_{11}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & 0 & I_{a_{121}} & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & I_{b_1} & 0 & pX_{41} \\
pI_{b_2} & J_{b_2} & 0 & I_{b_2} & 0 & 0 & 0 & pX_{42} \\
0 & J_{c_1} & 0 & 0 & I_{c_1} & 0 & 0 & pX_{81} \\
0 & J_{c_2} & I_{c_2} & 0 & 0 & 0 & pX_{72} & pX_{82}
\end{array}\right]$$

(c) The remaining groups.
We can produced the Smith Normal Form of pX_{82} and get without loss of generality

$$M = \left[\begin{array}{c||c||cccccccc}
I_{a_{111}} & 0 & I_{a_{111}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{112}} & 0 & \mathbf{0} & I_{a_{112}} & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{121}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & 0 & I_{b_1} & 0 & pX_{411} & pX_{412} \\
pI_{b_2} & J_{b_2} & 0 & 0 & I_{b_2} & 0 & 0 & 0 & pX_{421} & pX_{422} \\
0 & J_{c_1} & 0 & 0 & 0 & I_{c_1} & 0 & 0 & pX_{811} & pX_{812} \\
0 & J_{c_{21}} & I_{c_{21}} & 0 & 0 & 0 & 0 & pX_{721} & pI_{c_{21}} & 0 \\
0 & J_{c_{22}} & 0 & I_{c_{22}} & 0 & 0 & 0 & pX_{722} & 0 & 0
\end{array}\right]$$

With $pI_{c_{21}}$ we annihilate upward to get

$$M = \left[\begin{array}{c||c||cccccccc}
I_{a_{111}} & 0 & I_{a_{111}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{112}} & 0 & \mathbf{0} & I_{a_{112}} & 0 & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & \mathbf{0} & \mathbf{0} & I_{a_{121}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & 0 & I_{b_1} & 0 & 0 & pX_{412} \\
pI_{b_2} & J_{b_2} & 0 & 0 & I_{b_2} & 0 & 0 & 0 & 0 & pX_{422} \\
0 & J_{c_1} & 0 & 0 & 0 & I_{c_1} & 0 & 0 & 0 & pX_{812} \\
0 & J_{c_{21}} & I_{c_{21}} & 0 & 0 & 0 & 0 & 0 & pI_{c_{21}} & 0 \\
0 & J_{c_{22}} & 0 & I_{c_{22}} & 0 & 0 & 0 & pX_{722} & 0 & 0
\end{array}\right]$$

Now if $c_{21} \neq 0$, then we get

$$\begin{aligned}
M &= \left[\begin{array}{c||c||cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & p \end{array}\right] \rightsquigarrow \left[\begin{array}{c||c||cc} 0 & 1 & 1 & p \\ 1 & 0 & 1 & 0 \end{array}\right] \\
&\rightsquigarrow \left[\begin{array}{c||c||cc} 0 & 1 & 1 & p \\ 1 & 1 & 0 & p \end{array}\right] \rightsquigarrow \left[\begin{array}{c||c||cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & p \end{array}\right], \quad \text{a permutation of } M_{G_{22}}.
\end{aligned}$$

We now assume without loss of generality that $c_{21} = 0$ and we have

$$M = \left[\begin{array}{c||c||cccccc}
I_{a_{112}} & 0 & I_{a_{112}} & 0 & 0 & 0 & 0 & 0 \\
I_{a_{121}} & 0 & \mathbf{0} & I_{a_{121}} & 0 & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & \mathbf{0} & I_{a_{21}} & 0 & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{31}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & 0 & I_{b_1} & 0 & pX_{412} \\
pI_{b_2} & J_{b_2} & 0 & I_{b_2} & 0 & 0 & 0 & pX_{422} \\
0 & J_{c_1} & 0 & 0 & I_{c_1} & 0 & 0 & pX_{812} \\
0 & J_{c_{22}} & I_{c_{22}} & 0 & 0 & 0 & pX_{722} & 0
\end{array}\right]$$

There are now two possibilities:

$$M_1 = \left[\begin{array}{c||c||cc}
I_{a_{112}} & 0 & I_{a_{112}} & 0 \\
I_{a_{31}} & J_{a_{31}} & \mathbf{0} & I_{a_{31}} \\
0 & J_{c_{22}} & I_{c_{22}} & pX_{722}
\end{array}\right]$$

$$M_2 = \left[\begin{array}{c||c||cccc}
I_{a_{121}} & 0 & I_{a_{121}} & 0 & 0 & 0 \\
I_{a_{21}} & pJ_{a_{21}} & \mathbf{0} & I_{a_{21}} & 0 & 0 \\
I_{a_{221}} & pJ_{a_{221}} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 \\
pI_{b_1} & J_{b_1} & 0 & 0 & I_{b_1} & pX_{412} \\
pI_{b_2} & J_{b_2} & I_{b_2} & 0 & 0 & pX_{422} \\
0 & J_{c_1} & 0 & I_{c_1} & 0 & pX_{812}
\end{array}\right]$$

In M_1 we establish the Smith Normal Form of pX_{722} and get

$$M_1 = \left[\begin{array}{c||c||cccc}
I_{a_{1121}} & 0 & I_{a_{112}} & 0 & 0 & 0 \\
I_{a_{1122}} & 0 & 0 & I_{a_{112}} & 0 & 0 \\
I_{a_{311}} & J_{a_{311}} & \mathbf{0} & \mathbf{0} & I_{a_{311}} & 0 \\
I_{a_{312}} & J_{a_{312}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{312}} \\
0 & J_{c_{221}} & I_{c_{221}} & 0 & pI_{221} & 0 \\
0 & J_{c_{222}} & 0 & I_{c_{222}} & 0 & 0
\end{array}\right]$$

Rank-3 summands being excluded we are left the possibility

$$\left[\begin{array}{cc||cc||cc}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & p
\end{array}\right] = M_{G_{31}}.$$

In M_2 we establish the Smith Normal Form of pX_{812} to get

$$M_2 = \begin{bmatrix} I_{a_{121}} & \| & 0 & \| & I_{a_{121}} & 0 & 0 & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & I_{a_{211}} & 0 & 0 & 0 & 0 \\ I_{a_{212}} & \| & pJ_{a_{212}} & \| & \mathbf{0} & \mathbf{0} & I_{a_{212}} & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & 0 & 0 & I_{b_1} & pX_{4121} & pX_{4122} \\ pI_{b_2} & \| & J_{b_2} & \| & I_{b_2} & 0 & 0 & 0 & pX_{4221} & pX_{4222} \\ 0 & \| & J_{c_{11}} & \| & 0 & I_{c_{11}} & 0 & 0 & pI_{c_{11}} & 0 \\ 0 & \| & J_{c_{12}} & \| & 0 & 0 & I_{c_{12}} & 0 & 0 & 0 \end{bmatrix}$$

Dismissing the possible summand G_{21} and annihilating pX_{4121}, we are left with

$$M_2 = \begin{bmatrix} I_{a_{121}} & \| & 0 & \| & I_{a_{121}} & 0 & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & I_{a_{211}} & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & 0 & I_{b_1} & 0 & pX_{4122} \\ pI_{b_2} & \| & J_{b_2} & \| & I_{b_2} & 0 & 0 & pX_{4221} & pX_{4222} \\ 0 & \| & J_{c_{11}} & \| & 0 & I_{c_{11}} & 0 & pI_{c_{11}} & 0 \end{bmatrix}$$

Create the Smith Normal Form of pX_{4222}, $\begin{bmatrix} pI_{b_{21}} & 0 \\ 0 & 0 \end{bmatrix}$, and get

$$M_2 = \begin{bmatrix} I_{a_{1211}} & \| & 0 & \| & I_{a_{1211}} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_{a_{1212}} & \| & 0 & \| & 0 & I_{a_{1212}} & 0 & 0 & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & \mathbf{0} & I_{a_{211}} & 0 & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & 0 & 0 & I_{b_1} & 0 & pX_{41221} & pX_{41222} \\ pI_{b_{21}} & \| & J_{b_{21}} & \| & I_{b_{21}} & 0 & 0 & 0 & pX_{42211} & pI_{b_{21}} & 0 \\ pI_{b_{22}} & \| & J_{b_{22}} & \| & 0 & I_{b_{22}} & 0 & 0 & pX_{42212} & 0 & 0 \\ 0 & \| & J_{c_{11}} & \| & 0 & 0 & I_{c_{11}} & 0 & pI_{c_{11}} & 0 & 0 \end{bmatrix}$$

Now annihilate left with $pI_{b_{21}}$ and clear the fill-in with $pI_{c_{11}}$ to get

$$M_2 = \begin{bmatrix} I_{a_{1211}} & \| & 0 & \| & I_{a_{1211}} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_{a_{1212}} & \| & 0 & \| & 0 & I_{a_{1212}} & 0 & 0 & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & \mathbf{0} & I_{a_{211}} & 0 & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & \mathbf{0} & \mathbf{0} & I_{a_{221}} & 0 & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & 0 & 0 & I_{b_1} & 0 & pX_{41221} & pX_{41222} \\ pI_{b_{21}} & \| & J_{b_{21}} & \| & I_{b_{21}} & 0 & 0 & 0 & 0 & pI_{b_{21}} & 0 \\ pI_{b_{22}} & \| & J_{b_{22}} & \| & 0 & I_{b_{22}} & 0 & 0 & pX_{42212} & 0 & 0 \\ 0 & \| & J_{c_{11}} & \| & 0 & 0 & I_{c_{11}} & 0 & pI_{c_{11}} & 0 & 0 \end{bmatrix}$$

There are two summands now

$$M_{21} = \begin{bmatrix} I_{a_{1212}} & \| & 0 & \| & I_{a_{1212}} & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & I_{a_{211}} & 0 \\ pI_{b_{22}} & \| & J_{b_{22}} & \| & I_{b_{22}} & 0 & pX_{42212} \\ 0 & \| & J_{c_{11}} & \| & 0 & I_{c_{11}} & pI_{c_{11}} \end{bmatrix}$$

and

$$M_{22} = \begin{bmatrix} I_{a_{1211}} & \| & 0 & \| & I_{a_{1211}} & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & I_{a_{221}} & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & I_{b_1} & pX_{41221} & pX_{41222} \\ pI_{b_{21}} & \| & J_{b_{21}} & \| & I_{b_{21}} & 0 & pI_{b_{21}} & 0 \end{bmatrix}$$

Case $M = M_{21}$**.** We introduce the Smith Normal Form of pX_{42212} and get

$$M = \begin{bmatrix} I_{a_{12121}} & \| & 0 & \| & I_{a_{12121}} & 0 & 0 & 0 & 0 & 0 \\ I_{a_{12122}} & \| & 0 & \| & 0 & I_{a_{12122}} & 0 & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & \mathbf{0} & I_{a_{211}} & 0 & 0 & 0 \\ I_{a_{211}} & \| & pJ_{a_{211}} & \| & \mathbf{0} & \mathbf{0} & 0 & I_{a_{211}} & 0 & 0 \\ pI_{b_{2211}} & \| & J_{b_{2211}} & \| & I_{b_{2211}} & 0 & 0 & 0 & pI_{b_{2211}} & 0 \\ pI_{b_{2222}} & \| & J_{b_{2222}} & \| & 0 & I_{b_{2222}} & 0 & 0 & 0 & 0 \\ 0 & \| & J_{c_{11}} & \| & 0 & 0 & I_{c_{11}} & 0 & pI_{c_{11}} & 0 \\ 0 & \| & J_{c_{11}} & \| & 0 & 0 & 0 & I_{c_{11}} & 0 & pI_{c_{11}} \end{bmatrix}$$

M can be one of three matrices.

$$\begin{bmatrix} 1 & 0 & 0 & \| & 0 & 0 & 0 & \| & 1 & 0 & 0 \\ 0 & 1 & 0 & \| & 0 & 0 & p & \| & 0 & 1 & 0 \\ 0 & 0 & p & \| & 0 & 1 & 0 & \| & 1 & 0 & p \\ 0 & 0 & 0 & \| & 1 & 0 & 0 & \| & 0 & 1 & p \end{bmatrix} = M_{G_{41}}, \quad \begin{bmatrix} 1 & 0 & \| & 0 & \| & 1 \\ 0 & p & \| & 1 & \| & 1 \end{bmatrix} = M_{G_{21}},$$

or

$$\begin{bmatrix} 1 & \| & 0 & p & \| & 1 & 0 \\ 0 & \| & 1 & 0 & \| & 1 & p \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 1 & p & \| & 1 & 0 & \| & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix} = M_{G_{23}}.$$

Case $M = M_{22}$**.** Here pX_{41221} can be annihilated and we get

$$M_2 = \begin{bmatrix} I_{a_{1211}} & \| & 0 & \| & I_{a_{1211}} & 0 & 0 & 0 \\ I_{a_{221}} & \| & pJ_{a_{221}} & \| & \mathbf{0} & I_{a_{221}} & 0 & 0 \\ pI_{b_1} & \| & J_{b_1} & \| & 0 & I_{b_1} & 0 & pX_{41222} \\ pI_{b_{21}} & \| & J_{b_{21}} & \| & I_{b_{21}} & 0 & pI_{b_{21}} & 0 \end{bmatrix}.$$

There are three possibilities:

$$\begin{bmatrix} 1 & 0 & \| & 0 & \| & 1 & 0 \\ 0 & p & \| & 1 & \| & 1 & p \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \| & 1 & 0 & \| & 0 \\ 0 & p & \| & 1 & p & \| & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix} = M_{G_{23}},$$

$$\begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 & 0 \\ 0 & p & \| & 1 & 0 & \| & 1 & p \end{bmatrix} = M_{G_{24}} \text{ and } \begin{bmatrix} 1 & 0 & \| & 0 & p & \| & 1 \\ 0 & p & \| & 1 & 0 & \| & 1 \end{bmatrix} = M_{G_{23}}$$

We have now shown that any indecomposable group $G \in \mathrm{hc}'(3, p^2)$ is nearly isomorphic to one of the groups listed in Theorem 6.3 □

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