

Proper classes related with complements and supplements

RAFAIL ALÍZADE and ENGİN MERMUT

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Abstract. In this survey article, we shall give some properties of classes of short exact sequences of modules related with complement and supplement submodules; some of them form proper classes in the sense of Buchsbaum and some do not. Neat subgroups in abelian groups can be generalized to modules in several different ways; these give us the proper classes of short exact sequences of modules projectively generated or injectively generated or flatly generated by simple modules which approximate the proper classes defined using supplement submodules or complement submodules. The classes of short exact sequences of modules defined using weak supplements or submodules that have a supplement or small submodules do not always form a proper class, but for hereditary rings we can determine the smallest proper class containing them. For the proper classes we are interested in, we describe better their interrelations and their homological objects, like projectives, injectives, coprojectives, coinjectives with respect to the proper class, when the ring is a Dedekind domain. There are many natural questions for proper classes that needs to be further investigated.

1 Introduction

In this survey article, we shall describe the relative homological algebra approach to complements and supplements in terms of classes of short exact sequences of modules which may or may not form a proper class in the sense of Buchsbaum; for the concept of proper classes of short exact sequences of objects in an abelian category, see [39, Ch. XII, §4]. When a class of short exact sequences of modules do not form a proper class, a natural question is to determine the smallest proper class containing it. For completeness, the terminology and notation for proper classes of short exact sequences of modules have been gathered at the end in the Appendix Section A to this article; our notation for proper classes is as in [56] and [52]. The reason for using proper classes is to formulate easily and explicitly some problems of interest (for relative injectivity, projectivity, flatness), and to use the present technique for them for further investigations of the relations between them along these lines. After explaining our motivation from abelian groups, we shall list the definition of all the classes of short exact sequences of modules defined in this article by some kinds of submodules (whose definitions are given below) related with complements and supplements that we are interested in.

Throughout the article, R denotes an arbitrary ring with unity and an R -module or module means a unital left R -module; $R\text{-Mod}$ denotes the category of all left R -modules. $\mathcal{A}b$ denotes the category of abelian groups. \mathbb{Z} denotes the ring of integers and \mathbb{Q} denotes the field of rational numbers.

1.1 Complements, supplements, coclosed submodules, weak supplements, Rad-supplements, coatomic supplements

Let A be a submodule of a module B . It would be best if A is a *direct summand* of B , that is, if there exists another submodule K of B such that $B = A \oplus K$ which means $B = A + K$ and $A \cap K = 0$. When A is not a direct summand, retaining at least one of these conditions with a maximality or minimality condition give rise to the concepts *complement* and *supplement*; others are obtained by some similar conditions.

- (i) A is said to be a *complement in B* or is said to be a *complement submodule of B* if A is a

complement of some submodule K of B , that is, $K \cap A = 0$ and A is *maximal* with respect to this property. A submodule A of a module B is said to be *closed in B* if A has no proper essential extension in B , that is, there exists no submodule \tilde{A} of B such that $A \subsetneq \tilde{A}$ and $A \trianglelefteq \tilde{A}$ ($A \trianglelefteq \tilde{A}$ means that A is *essential* in \tilde{A} , that is, for every non-zero submodule X of \tilde{A} , we have $A \cap X \neq 0$). We also say in this case that A is a *closed submodule* of B and it is known that closed submodules and complement submodules in a module coincide. See the monograph [23] for a survey of results in the related concepts.

- (ii) Dually, A is said to be a *supplement in B* or A is said to be a *supplement submodule of B* if A is a supplement of some submodule K of B , that is, $B = K + A$ and A is *minimal* with respect to this property; equivalently, $K + A = B$ and $K \cap A \ll A$ ($K \cap A \ll A$ means that $K \cap A$ is *small* (=superfluous) in A , that is, for no proper submodule X of A , $K \cap A + X = A$). For the definitions and related properties, see [60, §41]; the monograph [17] focuses on the concepts related with supplements.
- (iii) A is said to be a *weak supplement in B* or A is said to be a *weak supplement submodule of B* if A is a weak supplement of some submodule K of B , that is, $B = K + A$ and $K \cap A \ll B$.
- (iv) A is said to be a *Rad-supplement in B* or A is said to be a *Rad-supplement submodule of B* if A is a Rad-supplement of some submodule K of B , that is, $B = K + A$ and $K \cap A \subseteq \text{Rad}(A)$.
- (v) A is said to be a *coatomic supplement in B* if A is a *coatomic supplement* of a submodule K of B , that is, $B = K + A$ and $K \cap A$ is a coatomic module (A module M is said to be *coatomic* if $\text{Rad}(M/U) \neq M/U$ for every proper submodule U of M , or equivalently, every proper submodule of M is contained in a maximal submodule of M ; coatomic modules appear in the theory of supplemented, semiperfect, and perfect modules, see [62] and [64]).
- (vi) Given submodules $K \subseteq A \subseteq B$, the inclusion $K \subseteq A$ is called *cosmall in B* if $A/K \ll B/K$; A is called *coclosed in B* if A has no proper submodule K for which the inclusion $K \subseteq A$ is cosmall in B (see [17, 3.1 and 3.6]).

1.2 Motivating ideas in abelian groups: Neat subgroups

A subgroup A of an abelian group B is said to be a *neat subgroup* if $A \cap pB = pA$ for all prime numbers p ([35], [30, p. 131]). This is a weakening of the condition for being a pure subgroup: A subgroup A of an abelian group B is said to be a *pure subgroup* if $A \cap nB = nA$ for all integers n . See [30, Ch. 5 and §53] for the very important notion of purity in abelian groups, pure-exact sequences of abelian groups, pure-projectivity and pure-injectivity and the functor Pext ; this is one of the main motivations for the relative homological algebra approach.

For a subgroup A of an abelian group B , the following are equivalent (see [41, Theorem 4.1.1]):

- (i) A is neat subgroup of B , that is, $A \cap pB = pA$ for all prime numbers p .
- (ii) The sequence

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z}) \otimes A \xrightarrow{1_{\mathbb{Z}/p\mathbb{Z}} \otimes i_A} (\mathbb{Z}/p\mathbb{Z}) \otimes B$$

obtained by applying the functor $(\mathbb{Z}/p\mathbb{Z}) \otimes -$ to the inclusion monomorphism $i_A : A \rightarrow B$ is exact for all prime numbers p .

- (iii) The sequence

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, B/A) \longrightarrow 0$$

obtained by applying the functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$ to the canonical epimorphism $B \rightarrow B/A$ is exact for all prime numbers p .

- (iv) The sequence

$$\text{Hom}_{\mathbb{Z}}(B, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0$$

obtained by applying the functor $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/p\mathbb{Z})$ to the inclusion monomorphism $i_A : A \rightarrow B$ is exact for all prime numbers p .

- (v) A is a complement of a subgroup K of B , that is, $A \cap K = 0$ and A is maximal with respect to this property (equivalently, A is a closed subgroup of B , that is, A has no proper essential extension in B).

1.3 The classes of short exact sequences related with complements and supplements

For ease of reference, we give below the list of the classes of short exact sequences of modules that we shall deal in this article in the next sections; the main motivation is the characterization of neat subgroups given in the previous Subsection 1.2. Some of them are proper classes and some are not as explained in the later sections.

Each of the following are classes of short exact sequences of (left) R -modules; to emphasize the ring R , we shall sometimes write the index R on the left, like in ${}_R\text{Compl}$, but *throughout the article for the arbitrary ring R , we shall usually not write it*. When we use some specific ring like the ring \mathbb{Z} of integers and consider abelian groups, we shall of course write this explicitly, like in ${}_{\mathbb{Z}}\text{Compl}$.

For a given class \mathcal{M} of modules, $\pi^{-1}(\mathcal{M})$ [$\iota^{-1}(\mathcal{M})$] denotes the proper class *projectively generated* [resp. *injectively generated*] by \mathcal{M} . For a given class \mathcal{M} of *right* R -modules, $\tau^{-1}(\mathcal{M})$ denotes the proper class *flatly generated* by the class \mathcal{M} of *right* R -modules. See the Appendix Section A. We are using the letters π , ι and τ for describing the concepts related with the relative versions of ‘projective’, ‘injective’ and ‘flat’ as in [56] and [52]. A class of short exact sequences of modules is usually given by describing it as the class of all short exact sequences

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{1.1}$$

of R -modules such that $\text{Im}(f)$ is a submodule of the stated kind; some of the classes below are given by stating this condition on $\text{Im}(f)$.

- (i) *Compl*: $\text{Im}(f)$ is a complement submodule of B .
- (ii) *Suppl*: $\text{Im}(f)$ is a supplement submodule of B .
- (iii) $\tau\text{-Compl} = \pi^{-1}(\{N \in R\text{-Mod} \mid \tau(N) = N\})$ for an idempotent preradical τ on $R\text{-Mod}$.
- (iv) $\tau\text{-Suppl} = \iota^{-1}(\{N \in R\text{-Mod} \mid \tau(N) = 0\})$ for a radical τ on $R\text{-Mod}$.
- (v) $\text{Neat} = \text{Neat}^\pi = \pi^{-1}(\{\text{all simple } R\text{-modules}\}) = \text{Soc-Compl}$.
- (vi) $\text{Neat}^\tau = \tau^{-1}(\{\text{all simple right } R\text{-modules}\})$.
- (vii) $\text{Neat}^\iota = \iota^{-1}(\{\text{all simple } R\text{-modules}\})$.
- (viii) $\text{Co-Neat} = \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) = \text{Rad-Suppl}$.
- (ix) $\mathcal{P}\text{-Pure} = \tau^{-1}(\{R/P \mid P \in \mathcal{P}\})$, where \mathcal{P} is the collection of all left primitive ideals of R .
- (x) *Co-Closed*: $\text{Im}(f)$ is a coclosed submodule of B .
- (xi) *S*: $\text{Im } f$ has a supplement in B (such short exact sequences are called κ -elements by Zöschinger in [63]);
- (xii) *WS*: $\text{Im } f$ has (is) a weak supplement in B ;
- (xiii) *Small*: $\text{Im } f$ is a small submodule of B ;
- (xiv) *CAS*: $\text{Im } f$ has a coatomic supplement in B .
- (xv) $\overline{\text{WS}}$ consists of the so called *extended weak supplement* short exact sequences of modules, where a short exact sequence E as in (1.1) is said to be *extended weak supplement* if there is a short exact sequence

$$E' : \quad 0 \longrightarrow A \xrightarrow{u} B' \longrightarrow C' \longrightarrow 0$$

of modules such that $\text{Im } u$ has (is) a weak supplement in B' (that is, E' is in the class *WS*) and there is a homomorphism $h : C \rightarrow C'$ such that $E = h^*(E')$, where $h^* = \text{Ext}_R^1(h, 1_A)$, that is, there is a commutative diagram as follows:

$$\begin{array}{ccccccccc} E : & 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ & & & 1_A & & & & h & & \\ E' : & 0 & \longrightarrow & A & \xrightarrow{u} & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

2 The proper classes $Compl$, $Suppl$, $Neat$, $CoNeat$, $\mathcal{P}\text{-Pure}$ and $Co\text{-Closed}$

The classes ${}_RCompl$ and ${}_RSuppl$ form proper classes as has been shown more generally by [32, Theorem 1], [33, Theorem 1], [55, Proposition 4 and Remark after Proposition 6]. In [55], following the terminology in abelian groups, the term ‘high’ is used instead of complements and ‘low’ for supplements. [32, 33] use the terminology ‘high’ and ‘cohigh’ for complements and supplements, and give more general definitions for proper classes of complements and supplements related to another given proper class (motivated by the considerations as pure-high extensions and neat-high extensions in [34]); ‘weak purity’ in [32] is what we denote by ${}_RCompl$. See also [27, Theorem 2.7.15 and Theorem 3.1.2] or [60, 10.5 and 20.7] for the proofs of ${}_RCompl$ and ${}_RSuppl$ being proper classes.

Denote by ${}_ZNeat$, the proper class of all short exact sequences (1.1) of abelian groups and abelian group homomorphisms where $\text{Im}(f)$ is a neat subgroup of B ; call such short exact sequences *neat-exact sequences* of abelian groups (like the terminology for pure-exact sequences of abelian groups). The characterization of neat subgroups of abelian groups given in Subsection 1.2 can be described in terms of proper classes as follows. The proper class ${}_ZCompl = {}_ZNeat$ is projectively generated, flatly generated and injectively generated by the simple abelian groups $\mathbb{Z}/p\mathbb{Z}$, p prime number:

$$\begin{aligned} {}_ZCompl &= {}_ZNeat = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) \\ &= \tau^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}) = \iota^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\}). \end{aligned}$$

The second equality ${}_ZNeat = \pi^{-1}(\{\mathbb{Z}/p\mathbb{Z} \mid p \text{ prime}\})$ is the motivation to define for any ring R

$$\begin{aligned} {}_RNeat &= \pi^{-1}(\{\text{all simple } R\text{-modules}\}) \\ &= \pi^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\}), \end{aligned}$$

following [56, 9.6] (and [55, §3]). For a submodule A of an R -module B , say that A is a *neat submodule* of B if A is a ${}_RNeat$ -submodule, that is, if for every simple R -module S , the sequence $\text{Hom}_R(S, B) \rightarrow \text{Hom}_R(S, B/A) \rightarrow 0$ obtained by applying the functor $\text{Hom}_R(S, -)$ to the canonical epimorphism $B \rightarrow B/A$ is exact.

We always have ${}_RCompl \subseteq {}_RNeat$ for any ring R (by [55, Proposition 5]). [32, Theorem 5] gives a characterization of this equality in terms of the ring R :

Theorem 2.1. (see [41, Theorem 3.3.2]): ${}_RCompl = {}_RNeat$ if and only if R is a left C -ring.

The notion of C -ring has been introduced in [50]: A ring R is said to be a *left C -ring* if for every (left) R -module B and for every essential proper submodule A of B , $\text{Soc}(B/A) \neq 0$, that is B/A has a simple submodule. Clearly, if R is a left semiartinian ring (R that is, if $\text{Soc}(R/I) \neq 0$ for every proper left ideal I of R), then R is a left C -ring. A commutative Noetherian ring in which every *nonzero* prime ideal is maximal is also a C -ring. So, of course, in particular a Dedekind domain and therefore a principal ideal domain is also a C -ring. A commutative domain R is a C -ring if and only if every nonzero torsion module has a simple submodule and such rings have been considered in [26, Theorem 4.4.1] when dealing with torsion-free covering modules over a commutative domain.

Because of the characterization of neat subgroups of abelian groups in Subsection 1.2, there are several reasonable ways to generalize the concept of neat subgroups to modules and a natural question is when these are equivalent (see Section 3). Another natural generalization of neat subgroups to modules is what is called \mathcal{P} -purity. Denote by \mathcal{P} the collection of all *left primitive ideals* of the ring R ; recall that a (two-sided) ideal P of R is said to be a *left primitive ideal* if it is the annihilator of a simple (left) R -module. We say that a submodule A of an R -module B is \mathcal{P} -pure in B if $A \cap PB = PA$ for all $P \in \mathcal{P}$. Denote by $\mathcal{P}\text{-Pure}$ the proper class of all short exact sequences (1.1) of R -modules such that $\text{Im}(f)$ is \mathcal{P} -pure in B . By Example A.1-(4), the proper class $\mathcal{P}\text{-Pure}$ is a flatly generated proper class:

$$\mathcal{P}\text{-Pure} = \tau^{-1}(\{R/P \mid P \in \mathcal{P}\}).$$

In [42], the relation of \mathcal{P} -purity with complements and supplements has been investigated and the structure of c -injective modules (= $Compl$ -injective modules) over Dedekind domains has been given. If the ring R is commutative, then \mathcal{P} is just the set of all maximal ideals of R .

We clearly have,

$$\begin{aligned} {}_R\mathcal{N}eat &= \pi^{-1}(\{\text{all semisimple } R\text{-modules}\}) \\ &= \pi^{-1}(\{M \mid \text{Soc } M = M, M \text{ an } R\text{-module}\}), \end{aligned}$$

where $\text{Soc } M$ is the socle of M , that is the sum of all simple submodules of M . Dualizing this, we define the proper class ${}_{\mathcal{C}o}\mathcal{N}eat_{R\text{-}Mod}$ by

$$\begin{aligned} {}_R\mathcal{C}o\mathcal{N}eat &= \iota^{-1}(\{\text{all } R\text{-modules with zero radical}\}) \\ &= \iota^{-1}(\{M \mid \text{Rad } M = 0, M \text{ an } R\text{-module}\}). \end{aligned}$$

If A is a ${}_{\mathcal{C}o}\mathcal{N}eat$ -submodule of an R -module B , we say that A is a *co-neat submodule* of B , or that the submodule A of the module B is *co-neat in* B .

For the motivation to consider these proper classes $\mathcal{C}ompl$, $\mathcal{S}uppl$, $\mathcal{N}eat$, $\mathcal{C}o\mathcal{N}eat$ and for further related results, see [41], [5], [1], [17, §10 and 20.7-8], [16] and [44].

Every module M with $\text{Rad } M = 0$ is $\mathcal{S}uppl$ -injective that is M is injective with respect to every short exact sequence in $\mathcal{S}uppl$. Thus supplement submodules are co-neat submodules by the definition of co-neat submodules (see [41, Proposition 3.4.1]), and we have for any ring R

$$\mathcal{S}uppl \subseteq \mathcal{C}o\mathcal{N}eat \subseteq \iota^{-1}(\{\text{all (semi-)simple } R\text{-modules}\}).$$

The class $\mathcal{C}o\mathcal{N}eat$ has an interpretation in terms of supplements. Being a co-neat submodule is like being a supplement: For a submodule A of a module B , A is co-neat in B if and only if A is a Rad-supplement in B , that is, there exists a submodule $K \subseteq B$ such that

$$A + K = B \quad \text{and} \quad A \cap K \subseteq \text{Rad } A$$

(see [41, Proposition 3.4.2] or [17, 10.14] or [1, 1.14]). This characterization will be the particular case $\tau = \text{Rad}$ in Proposition 4.1 and this is the reason for considering Rad-supplements and in general τ -supplements; see Section 4 for the proper class $\tau\text{-}\mathcal{S}uppl$ defined using τ -supplements for a radical τ on $R\text{-}Mod$ and the proper class $\tau\text{-}\mathcal{C}ompl$ for an idempotent preradical τ on $R\text{-}Mod$. For more results on co-neat submodules see [41], [5], [17, §10 and 20.7–8] and [1].

For a semilocal ring R ,

$$\mathcal{C}o\mathcal{N}eat = \iota^{-1}(\{\text{all (semi-)simple } R\text{-modules}\}),$$

and for a left perfect ring R ,

$$\mathcal{S}uppl = \mathcal{C}o\mathcal{N}eat = \iota^{-1}(\{\text{all (semi-)simple } R\text{-modules}\})$$

(see [41, Theorem 3.8.7 and Corollary 3.8.8]).

A natural question for a proper class is when it equals $\mathcal{S}plit$ (=the class of all splitting short exact sequences of modules) or when it equals $\mathcal{A}bs$ (=the class of all short exact sequences of modules). For the proper classes $\mathcal{C}ompl$ and $\mathcal{S}uppl$, we know that:

Theorem 2.2. (by [23, 13.5] and [47, Corollary 2.5]) *For a ring R , the following are equivalent:*

- (i) $\mathcal{C}ompl = \mathcal{S}plit$ (equivalently, all left R -modules are extending (CS)),
- (ii) R is left perfect and $\mathcal{S}uppl = \mathcal{S}plit$ (equivalently, all left R -modules are lifting),
- (iii) R is (left and right) artinian serial and $J^2 = 0$ for the Jacobson radical J of R .

2.1 The relations between these proper classes over Dedekind domains

In abelian groups, we have:

Theorem 2.3. ([41, Theorems 4.6.5 and 4.4.4] and [5, Theorem 5.3])

- (i) ${}_{\mathbb{Z}}\mathcal{S}uppl \subsetneq {}_{\mathbb{Z}}\mathcal{C}o\mathcal{N}eat \subsetneq {}_{\mathbb{Z}}\mathcal{N}eat = {}_{\mathbb{Z}}\mathcal{C}ompl$.
- (ii) *The inductive closure of the proper class ${}_{\mathbb{Z}}\mathcal{S}uppl$, that is, the smallest inductively closed proper class containing ${}_{\mathbb{Z}}\mathcal{S}uppl$, is flatly generated by all simple abelian groups and so it is ${}_{\mathbb{Z}}\mathcal{C}ompl = {}_{\mathbb{Z}}\mathcal{N}eat$.*

In abelian groups, the functor $\text{Ext}_{{}_{\mathbb{Z}}\mathcal{C}ompl}$ is factorizable, in the sense defined in Subsection A.1, like $\text{Ext}_{{}_{\mathbb{Z}}\mathcal{P}ure}$ in Example A.2; but the proper class ${}_{\mathbb{Z}}\mathcal{S}uppl$ behaves badly in this sense:

Theorem 2.4. [41, Theorems 4.1.5, 4.5.3]

(i) ([30, Exercise 53.4] and [45, Theorem 5.1]) For all abelian groups A and C ,

$$\text{Ext}_{\mathbb{Z}\text{Compl}}(C, A) = \bigcap_{p \text{ prime}} p \text{Ext}_{\mathbb{Z}}(C, A) = \text{Rad}(\text{Ext}_{\mathbb{Z}}(C; A)),$$

where the functor $\text{Rad} : \mathcal{A}b \rightarrow \mathcal{A}b$ which associates with each abelian group M , its Frattini subgroup (which is its radical as a \mathbb{Z} -module and which equals) $\text{Rad}(M) = \bigcap_{p \text{ prime}} pM$.

So the functor $\text{Ext}_{\mathbb{Z}\text{Compl}}$ is factorizable.

(ii) The functor $\text{Ext}_{\mathbb{Z}\text{Suppl}}$ is not factorizable as

$$\mathbb{Z}\text{-Mod} \times \mathbb{Z}\text{-Mod} \xrightarrow{\text{Ext}_{\mathbb{Z}}} \mathcal{A}b \xrightarrow{H} \mathcal{A}b$$

for any functor $H : \mathcal{A}b \rightarrow \mathcal{A}b$ on the category of abelian groups.

For Dedekind domains we have the following inclusion relations for the proper classes Compl , Suppl , Neat and Co-Neat , and like in abelian groups $\text{Ext}_{\text{Compl}} = \text{Ext}_{\text{Neat}}$ is factorizable and when also $\text{Rad } R = 0$, $\text{Ext}_{\text{Suppl}}$ and $\text{Ext}_{\text{Co-Neat}}$ are not factorizable:

Theorem 2.5. [41, Theorems 5.2.2, 5.2.3, 5.4.6 and 5.4.8] If R is a Dedekind domain which is not a field, we have:

(i) $\text{Compl} = \text{Neat}$ is also flatly generated by all simple R -modules and injectively generated by all simple R -modules.

(ii) (By [45, Theorem 5.1]) $\text{Ext}_{\text{Compl}} = \text{Ext}_{\text{Neat}}$ is factorizable as

$$R\text{-Mod} \times R\text{-Mod} \xrightarrow{\text{Ext}_R} R\text{-Mod} \xrightarrow{\text{Rad}} R\text{-Mod}$$

using the functor $\text{Rad} : R\text{-Mod} \rightarrow R\text{-Mod}$, that is, for all R -modules A and C ,

$$\text{Ext}_{\text{Compl}}(C, A) = \text{Ext}_{\text{Neat}}(C, A) = \text{Rad}(\text{Ext}_R(C, A)).$$

(iii) If $\text{Rad } R = 0$, then

$$\text{Suppl} \subsetneq \text{Co-Neat} \subsetneq \text{Neat} = \text{Compl},$$

and the functors $\text{Ext}_{\text{Suppl}}$ and $\text{Ext}_{\text{Co-Neat}}$ are not factorizable as

$$R\text{-Mod} \times R\text{-Mod} \xrightarrow{\text{Ext}_R} R\text{-Mod} \xrightarrow{H} R\text{-Mod}$$

for any functor $H : R\text{-Mod} \rightarrow R\text{-Mod}$.

(iv) If $\text{Rad } R \neq 0$, then

$$\text{Suppl} \subsetneq \text{Co-Neat} = \text{Neat} = \text{Compl},$$

2.2 The proper class Co-Closed

For the proof that the class Co-Closed defined using coclosed submodules forms a proper class, see [44, 5.9] or [48, Theorem 4.6.4] or [36, Theorem 3.11]; what is being used for that is essentially [65, Lemma A.4] and [17, 3.7]. Zöschinger calls Co-Closed -coinjective modules *weakly injective* (=modules that are coclosed in every module containing them) in [65]. In his recent article [66], he deals with Compl -coprojectives which he calls *weakly flat*; over a commutative Noetherian ring R , he also shows that closed submodules are coclosed if and only if coclosed submodules are closed if and only if R is a distributive ring (that is, $(I + J) \cap K = (I \cap K) + (J \cap K)$ for all ideals I, J, K of R). In [36], Suppl -coprojectives and Co-Closed -coprojectives are considered, they call them absolute co-supplement and absolute co-coclosed modules (this absolute terminology come from the notion of absolutely pure modules which means Pure -coinjectives; so for example by absolutely supplement modules, they mean Suppl -coinjectives).

3 Proper classes generated by simple modules

The following three proper classes defined below that are projectively, flatly or injectively generated by simple (left or right) modules are natural ways to extend the concept of neat subgroups to modules; so we name all of them using ‘neat’:

- (i) $\mathcal{N}eat = \mathcal{N}eat^\pi = \pi^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\})$ is the proper class projectively generated by all simple R -modules.
- (ii) $\mathcal{N}eat^\tau = \tau^{-1}(\{R/P \mid P \text{ is a maximal right ideal of } R\})$ is the proper class flatly generated by all simple *right* R -modules.
- (iii) $\mathcal{N}eat^\iota = \iota^{-1}(\{R/P \mid P \text{ is a maximal left ideal of } R\})$ is the proper class injectively generated by all simple R -modules.

Note that when R is a commutative ring, the last two proper classes above coincide with $\mathcal{P}\mathcal{P}ure$:

$$\mathcal{N}eat^\iota = \mathcal{N}eat^\tau = \mathcal{P}\mathcal{P}ure \quad \text{if } R \text{ is a commutative ring.}$$

The last equality is obvious (see Example A.1-(4)); for the first one see for example [18, Proposition 3.1] or [31]. The equality $\mathcal{N}eat^\iota = \mathcal{P}\mathcal{P}ure$ holds more generally for rings R such that R/P is an Artinian ring for every left primitive ideal P of R ; see [42, Corollary 2.6].

By the relations given in Section 2, we have:

- (i) $\mathcal{C}ompl \subseteq \mathcal{N}eat$ and $\mathcal{S}uppl \subseteq \mathcal{C}o\mathcal{N}eat \subseteq \mathcal{N}eat^\iota$ for every ring R .
- (ii) If the ring R is commutative, then $\mathcal{C}o\mathcal{N}eat \subseteq \mathcal{N}eat^\iota = \mathcal{N}eat^\tau = \mathcal{P}\mathcal{P}ure$.
- (iii) If R is a semilocal ring, then $\mathcal{C}o\mathcal{N}eat = \mathcal{N}eat^\iota$.
- (iv) If R is a left perfect ring, then $\mathcal{S}uppl = \mathcal{C}o\mathcal{N}eat = \mathcal{N}eat^\iota$.

Note that for a commutative ring R , [31] calls the short exact sequences in $\mathcal{N}eat^\tau$ co-neat (and since the ring is commutative $\mathcal{N}eat^\tau = \mathcal{P}\mathcal{P}ure = \mathcal{N}eat^\iota$). In [15], the short exact sequences in $\mathcal{N}eat^\tau$ are called s -pure, following the terminology in [18]. In [14], the short exact sequences in $\mathcal{N}eat^\iota$ are called co-neat. But we reserve the word co-neat in the way we have defined; being a co-neat submodule is equivalent to being a Rad-supplement.

3.1 Commutative domains where $\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$: N -domains

A natural question to ask is when neatness and \mathcal{P} -purity coincide. Suppose that the ring R is commutative. Then \mathcal{P} is the collection of all *maximal ideals* of R . Recently László Fuchs has characterized the commutative domains for which these two notions coincide; see [31]. Fuchs calls a ring R to be an N -domain if R is a commutative domain such that neatness and \mathcal{P} -purity coincide, that is, $\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$. Unlike expected, Fuchs shows that N -domains are not just Dedekind domains; they are exactly the commutative domains whose all *maximal ideals* are projective (and so all maximal ideals are invertible ideals and finitely generated). For a commutative domain R , Fuchs has proved that $\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$ if and only if the projective dimension of every simple module is ≤ 1 . Note that the projective dimension of the simple R -modules is important when the ring R is commutative Noetherian because then the global dimension of the ring is the supremum of the projective dimensions of all simple R -modules. So among commutative Noetherian domains, the N -domains are just Dedekind domains.

Motivated by Fuchs’ result for commutative domains, one wonders whether for some class of commutative rings larger than commutative domains, neatness and \mathcal{P} -purity coincide if and only if all the maximal ideals of the ring are projective and finitely generated. In [58], the answer is shown to be yes if every maximal ideal of the commutative ring contains a regular element (that is an element that is not a zero-divisor) so that the maximal ideals of R that are invertible in the total quotient ring of R will be just projective ones as in the case of commutative domains (see for example [37, §2C]). Indeed, this condition is also weakened and it suffices to require that the socle of the commutative ring R is zero, that is, R contains no simple submodules. A bit less to assume is that the commutative ring R contains no simple submodules that are not direct summands of R . See [58, Ch. 4].

It is known that a proper class of short exact sequences of modules that is projectively generated by a set of finitely presented modules is flatly generated by ‘the’ Auslander-Bridger transpose of these finitely presented modules. So to generalize the sufficiency of the Fuchs’ characterization of N -domains to all commutative rings, it is shown in [58, §4.4] that for a commutative ring R , an Auslander-Bridger transpose of a finitely presented simple R -module S of projective

dimension 1 is isomorphic to S . This enables us to prove that if R is a commutative ring such that every maximal ideal of R is finitely generated and projective, then neatness and \mathcal{P} -purity coincide.

3.2 The Auslander-Bridger Transpose of Finitely Presented Simple Modules

See Subsection A.2 for the Auslander-Bridger transpose of finitely presented modules. We can extend the results in [31] for the Auslander-Bridger transpose of finitely presented simple modules of projective dimension ≤ 1 over a commutative domain to commutative rings:

Theorem 3.1. [58, Theorem 4.4.4] *Let R be a commutative ring and P be a finitely generated maximal ideal of R that is projective. Take the following presentation of the simple R -module $S = R/P$ (where f is the inclusion monomorphism and g is the natural epimorphism):*

$$\gamma: \quad P \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

- (i) *If S is projective, then $S^* \neq 0$ and $\text{Tr}_\gamma(S) = 0$.*
- (ii) *If S is not projective, then $S^* = 0$ and $\text{Tr}_\gamma(S) \cong \text{Ext}_R^1(S, R) \cong S$.*
- (iii) *S is projective if and only if $S^* \neq 0$.*

Nunke shows that $I^{-1}/R \cong R/I$ if I is a non-zero ideal of a Dedekind domain R (see [45, Lemma 4.4]). For an invertible maximal ideal P of a commutative ring R , we show next that $P^{-1}/R \cong R/P$ where the invertibility is in the total quotient ring of R , that is, the localization of R with respect to the set of all regular elements of R , and P^{-1} consists of all q in the total quotient ring of R such that $qP \subseteq R$.

Proposition 3.2. [58, Proposition 4.4.5] *If R is a commutative ring and P is a maximal ideal of R that is invertible in the total ring of quotients of R , then for the simple R -module $S = R/P$ and for the presentation*

$$\gamma: \quad P \xrightarrow{f} R \xrightarrow{g} S \longrightarrow 0$$

of S (where f is the inclusion monomorphism and g is the natural epimorphism), we have

$$\text{Tr}_\gamma(S) \cong P^{-1}/R \cong S = R/P.$$

Using Theorem 3.1, we obtain the following sufficient condition for $\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$ over commutative rings as in Fuchs' characterization of N -domains:

Theorem 3.3. [58, Theorem 4.5.1] *If R is a commutative ring such that every maximal ideal of R is finitely generated and projective, then $\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$.*

For the converse, we have the following:

Theorem 3.4. [58, Corollary 4.5.4] *The following are equivalent for a commutative ring R such that for each simple R -module S , $S^* = 0$ or S is projective:*

- (i) *$\mathcal{N}eat = \mathcal{P}\mathcal{P}ure$.*
- (ii) *Every maximal ideal P of R is projective and finitely generated.*

3.3 Noetherian distributive rings

Turning back to our motivation in abelian groups for the characterizations of neat subgroups given in Subsection 1.2, we ask for which commutative rings R it is true that

$$\text{Compl} = \mathcal{N}eat = \mathcal{P}\mathcal{P}ure,$$

that is, the proper class Compl is projectively generated and flatly generated and injectively generated by all simple modules (since $\mathcal{P}\mathcal{P}ure = \mathcal{N}eat^\tau = \mathcal{N}eat^t$ for a commutative ring). The answer is given in [25] by using the characterization for commutative Noetherian distributive rings in [66] which we mentioned in Subsection 2.2.

Theorem 3.5. [25] *For a commutative ring R , $\text{Compl} = \mathcal{N}eat = \mathcal{P}\mathcal{P}ure$ if and only if R is a Noetherian distributive ring.*

3.4 Coprojectives and coinjectives with respect to the proper classes generated by simple modules

The notion of max-injectivity (a weakened injectivity in view of Baer's criterion) has been studied by several authors; see, for example, [19], [20] and [59]: A module M is said to be *maximally injective* or *max-injective* if for every maximal left ideal P of R , every homomorphism $f : P \rightarrow M$ can be extended to a homomorphism $g : R \rightarrow M$ (max-injective modules are called *m-injective modules* in [19]). A module M is max-injective if and only if $\text{Ext}_R^1(S, M) = 0$ for every simple module S (see [19, Theorem 2]). Since $\mathcal{N}eat$ is the proper class projectively generated by all simple modules, we have then that max-injective modules are just $\mathcal{N}eat$ -coinjective modules. It has been proved by Patrick F. Smith that for a ring R , $\text{Soc}(R/I) \neq 0$ for every essential proper left ideal I of R (that is, R is a left C -ring) if and only if every max-injective module is injective [53, Lemma 4]. This result is also stated in [22] and for its proof the reference to [53] has been given. A proof of this result with our interest in the proper classes $\mathcal{N}eat$ and $\mathcal{C}ompl$ and with further observations is given in [48, §4.2]. A module M is $\mathcal{C}ompl$ -coinjective if and only if M is a complement submodule (=closed submodule) of its injective envelope $E(M)$. Since M is essential in $E(M)$, we obtain that $\mathcal{C}ompl$ -coinjective modules are just injective modules. If R is a left C -ring, then $\mathcal{C}ompl = \mathcal{N}eat$ by Theorem 2.1 and so all max-injective modules(= $\mathcal{N}eat$ -coinjective modules) are just injective modules. Conversely, if all $\mathcal{N}eat$ -coinjective modules are injective, then R is a left C -ring. We have the following characterizations of left C -rings:

Theorem 3.6. [48, Theorem 4.2.18] *For a ring R , the following are equivalent:*

- (i) R is a left C -ring;
- (ii) $\mathcal{C}ompl = \mathcal{N}eat$;
- (iii) All max-injective (= $\mathcal{N}eat$ -coinjective) modules are injective;
- (iv) $\{\text{all simple modules}\}^\perp = \{\text{all injective modules}\}$, where for a class \mathcal{M} of modules, \mathcal{M}^\perp consists of all modules L such that $\text{Ext}_R^1(M, L) = 0$ for every $M \in \mathcal{M}$.
- (v) $N = \bigoplus_{\substack{P \subseteq_e R \\ \text{max.}}} R/P$, where the direct sum is over all maximal left ideals of R , is a Whitehead test module for injectivity, that is, for every module M , $\text{Ext}_R^1(N, M) = 0$ implies M is injective (see [57]).

Coinjective modules and coprojective modules with respect to a proper class appear naturally in many contexts, like the above max-injective modules. In cotorsion theory, this relation is given in [44, Ch. 6]. For some kinds of relative injectivity problems, these ideas also help to see the way. Some notions can be formulated with this language. In [9], the notion of subinjectivity and indigence has been introduced; it turns out that for modules M and N , M is N -subinjective if and only if N is $\iota^{-1}(\{M\})$ -coinjective, where $\iota^{-1}(\{M\})$ is the proper class injectively generated by the module M . For a proper class \mathcal{A} , a module N is \mathcal{A} -coinjective if and only if the short exact sequence starting with the inclusion monomorphism $N \hookrightarrow E(N)$, where $E(N)$ is the injective envelope of N , is in the proper class \mathcal{A} . Of course, every injective module is \mathcal{A} -coinjective. One obvious extreme case to consider is when all \mathcal{A} -coinjectives consist of just all injective modules. For example, just by definition, a module M is an indigent module if and only if $\iota^{-1}(\{M\})$ -coinjective modules are just injective modules because the subinjectivity domain $\underline{\mathcal{I}n}^{-1}(M)$ equals all $\iota^{-1}(\{M\})$ -coinjective modules. In this coinjective terminology, for a module M and a class \mathcal{C} of R -modules, every module $C \in \mathcal{C}$ is M -subinjective means M is $\iota^{-1}(\mathcal{C})$ -coinjective, where $\iota^{-1}(\mathcal{C})$ is the proper class injectively generated by the class \mathcal{C} of modules. One can consider for example what happens when \mathcal{C} is the class of all simple modules. Another related notion introduced in [3] is test modules for injectivity by subinjectivity (shortly t.i.b.s.): a module N is said to be a t.i.b.s. if the only N -subinjective modules are injective modules.

For further investigation on the coinjectives and coprojectives of the proper classes generated by simple modules, see [15], [14] and [13]. In our terminology:

- (i) Absolutely s -pure modules in [15] are $\mathcal{N}eat^r$ -coinjective modules.
- (ii) Neat-flat modules in [15] and [13] are $\mathcal{N}eat$ -coprojective modules.
- (iii) Co-neat-flat modules in [14] are $\mathcal{N}eat^t$ -coprojective modules; by co-neat submodules, they mean $\mathcal{N}eat^t$ -submodules.

4 The proper classes τ -*Compl* and τ -*Suppl*

For definitions and elementary properties of preradicals, see [54, Ch. VI], [11] or [17, §6]. A preradical τ for $R\text{-Mod}$ is defined to be a subfunctor of the identity functor on $R\text{-Mod}$. Let τ be a preradical for $R\text{-Mod}$. The following module classes are defined: the preradical or (pre)torsion class of τ is

$$\mathbb{T}_\tau = \{N \in R\text{-Mod} \mid \tau(N) = N\}$$

and the preradical free or (pre)torsion free class of τ is

$$\mathbb{F}_\tau = \{N \in R\text{-Mod} \mid \tau(N) = 0\}.$$

τ is said to be idempotent if $\tau(\tau(N)) = \tau(N)$ for every R -module N . τ is said to be a radical if $\tau(N/\tau(N)) = 0$ for every R -module N . For example, Soc is an idempotent preradical and Rad is a radical on $R\text{-Mod}$.

In the definition of co-neat submodules, using any radical τ on $R\text{-Mod}$ instead of Rad, the following result is obtained. It gives us the definition of a τ -supplement in a module because the last condition is like the usual supplement condition except that, instead of $U \cap V \ll V$, the condition $U \cap V \subseteq \tau(V)$ is required.

Proposition 4.1. (see [17, 10.11] or [1, 1.11]) *Let τ be a radical for $R\text{-Mod}$. For a submodule $V \subseteq M$, the following statements are equivalent.*

- (i) *Every module N with $\tau(N) = 0$ is injective with respect to the inclusion $V \hookrightarrow M$.*
- (ii) *There exists a submodule $U \subseteq M$ such that $U + V = M$ and $U \cap V = \tau(V)$.*
- (iii) *There exists a submodule $U \subseteq M$ such that $U + V = M$ and $U \cap V \subseteq \tau(V)$.*

If these conditions are satisfied, then V is called a τ -supplement in M .

The usual definitions are then given for a radical τ on $R\text{-Mod}$ as follows. For submodules U and V of a module M , the submodule V is said to be a τ -supplement of U in M or U is said to have a τ -supplement V in M if $U + V = M$ and $U \cap V \subseteq \tau(V)$. A module M is called a τ -supplemented module if every submodule of M has a τ -supplement in M . For $\tau = \text{Rad}$, the above definitions give Rad-supplement submodules of a module, Rad-supplemented modules, etc. By these definitions, a submodule V of a module M is a co-neat submodule of M if and only if V is a Rad-supplement of a submodule U of M in M . In [16], some properties of Rad-supplemented modules and in general τ -supplemented modules have been investigated; it was shown that every left R -module is Rad-supplemented if and only if R/W is left perfect where W is the sum of all left ideals I of R such that $\text{Rad } I = I$.

For a radical τ on $R\text{-Mod}$, we have by Proposition 4.1 that the proper class

$$\tau\text{-Suppl} = \iota^{-1}(\mathbb{F}_\tau) = \iota^{-1}(\{N \in R\text{-Mod} \mid \tau(N) = 0\})$$

consists of all short exact sequences (1.1) of modules such that $\text{Im}(f)$ is a τ -supplement in B .

Similarly we have:

Proposition 4.2. (see [17, 10.6] or [1, 1.6]) *Let τ be an idempotent preradical for $R\text{-Mod}$. For a submodule $V \subseteq M$, the following statements are equivalent.*

- (i) *Every module N with $\tau(N) = N$ is projective with respect to the natural epimorphism $M \rightarrow M/V$.*
- (ii) *There exists a submodule $U \subseteq M$ such that $V \cap U = 0$ and $\tau(M/V) = (U + V)/V \cong U$.*
- (iii) *There exists a submodule $U \subseteq M$ such that $V \cap U = 0$ and $\tau(M/V) \subseteq (U + V)/V \cong U$.*

If these conditions are satisfied, then V is called a τ -complement in M .

For an idempotent radical τ on $R\text{-Mod}$, we have by Proposition 4.2 that the proper class

$$\tau\text{-Compl} = \pi^{-1}(\mathbb{T}_\tau) = \pi^{-1}(\{N \in R\text{-Mod} \mid \tau(N) = N\})$$

consists of all short exact sequences (1.1) of modules such that $\text{Im}(f)$ is a τ -complement in B .

Carlos Federico Preisser Montañón in his PhD Thesis [44] studied homological objects of τ -Suppl and τ -Compl and related other proper classes. He proved that a module I is τ -Suppl-injective if and only if I is a direct summand of a module of the form $E \oplus F$ with E injective and $\tau(F) = 0$. Similarly τ -Compl-projective modules are exactly direct summands of the modules of the form $P \oplus Q$ with P projective and $\tau(Q) = 0$. Every τ -torsion module N (that is,

$\tau(N) = N$ is τ -*Compl*-coinjective. If $\tau(R) = 0$, then τ -*Compl*-coprojective modules are projective. If every injective module is τ -torsion, then every τ -*Compl*-coinjective module is injective. Every τ -torsionfree module N (that is, $\tau(N) = 0$) is τ -*Compl*-coprojective. He also studied the proper class *Co-Closed* induced by coclosed submodules and proved that every *Co-Closed*-coprojective module that has a projective cover is projective. If the module N is fully non-small (see [17, 8.11]), then N is *Co-Closed*-coinjective. Generalizing the notion of δ -small submodule introduced by Zhou in [61], he defines \mathcal{C} -small submodules and \mathcal{C} -supplement submodules and the corresponding proper class, where \mathcal{C} is a class of submodules closed under submodules and factor modules. Dually, he defines \mathcal{C} -essential submodules, \mathcal{C} -complements and the corresponding proper class. His another important contribution is to relate the language of proper classes of short exact sequences of modules with cotorsion pairs, covers and envelopes; see the monograph [26] for the latter concepts.

5 The smallest proper class containing \mathcal{WS} over a hereditary ring

Unfortunately not all classes of short exact sequences of modules generated by various types of supplements are proper. In these cases, the smallest proper classes containing these classes are studied. The definition of the classes *Small*, \mathcal{S} , \mathcal{WS} , \mathcal{CAS} and $\overline{\mathcal{WS}}$ have been given in Subsection 1.3.

If X is a *Small*-submodule of an R -module Y , then Y is a supplement of X in Y , so X is an \mathcal{S} -submodule of Y . If U is an \mathcal{S} -submodule of an R -module Z , then a supplement V of U in Z is also a weak supplement, therefore U is a \mathcal{WS} -submodule of Z . These arguments give the relation $\text{Small} \subseteq \mathcal{S} \subseteq \mathcal{WS}$ for any ring R . Our next example shows that neither of the classes *Small*, \mathcal{S} or \mathcal{WS} need to be a proper class in general. Therefore, it makes sense to study the smallest proper classes generated by them. It turns out that for left hereditary rings, these three classes generate the same proper class denoted by $\overline{\mathcal{WS}}$. The class $\overline{\mathcal{WS}}$ consisting of the so called *extended weak supplement* short exact sequences of modules is described below; when R is a left hereditary ring, $\overline{\mathcal{WS}}$ is a proper class and we shall also see the homological objects of it. The class \mathcal{CAS} is shown to be a proper class in [24].

Example 5.1. Let $R = \mathbb{Z}$ and consider the composition $\beta \circ \alpha$ of the monomorphisms $\alpha : 2\mathbb{Z} \rightarrow \mathbb{Z}$ and $\beta : \mathbb{Z} \rightarrow \mathbb{Q}$ where α and β are the corresponding inclusion maps. Then the short exact

sequence $0 \rightarrow 2\mathbb{Z} \xrightarrow{\beta \circ \alpha} \mathbb{Q} \rightarrow \mathbb{Q}/2\mathbb{Z} \rightarrow 0$ is in *Small* and therefore in \mathcal{WS} , but the short exact sequence $0 \rightarrow 2\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0$ is not in \mathcal{WS} as $\text{Rad } \mathbb{Z} = 0$.

For a homomorphism $f : A \rightarrow B$ and a module C , we will denote the induced homomorphisms $\text{Ext}_R^1(1_C, f) : \text{Ext}_R^1(C, A) \rightarrow \text{Ext}_R^1(C, B)$ and $\text{Ext}_R^1(f, 1_C) : \text{Ext}_R^1(B, C) \rightarrow \text{Ext}_R^1(A, C)$ by f_* and f^* as in [39, Ch. 3, §2]. Example 5.1 shows that $\text{Ext}_{\mathcal{WS}}(\cdot, \cdot)$ need not be a subfunctor of $\text{Ext}_R^1(\cdot, \cdot)$ since the elements from \mathcal{WS} are not preserved with respect to the first variable. We begin by extending the class \mathcal{WS} to the class $\overline{\mathcal{WS}}$, which consists of all images of \mathcal{WS} -elements of $\text{Ext}(C', A)$ under $h^* : \text{Ext}(C', A) \rightarrow \text{Ext}(C, A)$ for all homomorphisms $h : C \rightarrow C'$.

A short exact sequence $E : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is said to be *extended weak supplement* if there is a short exact sequence

$$E' : 0 \rightarrow A \xrightarrow{u} B' \rightarrow C' \rightarrow 0$$

of modules such that $\text{Im } u$ has (is) a weak supplement in B (that is, E' is in \mathcal{WS}) and there is a homomorphism $h : C \rightarrow C'$ such that $E = h^*(E')$, that is, there is a commutative diagram as follows:

$$\begin{array}{ccccccccc} E : & 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & & \parallel & & \downarrow & & \downarrow & & \\ & & & 1_A & & & & h & & \\ E' : & 0 & \longrightarrow & A & \xrightarrow{u} & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

We denote by $\overline{\mathcal{WS}}$ the class of all extended weak supplement short exact sequences.

Theorem 5.2. [4, Theorem 3.12 and Corollary 3.13] *If R is a left hereditary ring, then $\overline{\mathcal{WS}}$ is a proper class and*

$$\langle \text{Small} \rangle = \langle \mathcal{S} \rangle = \langle \mathcal{WS} \rangle = \overline{\mathcal{WS}}.$$

It turns out that when the ring is a Dedekind domain, extended weak supplement has a description in terms of some type of supplement:

Theorem 5.3. [4, Theorem 5.3] *Over a Dedekind domain R , the class \mathcal{CAS} coincides with the class $\overline{\mathcal{WS}}$.*

Now we focus on the study of the main homological objects of the proper class $\overline{\mathcal{WS}}$ when R is a left hereditary ring, that is, $\overline{\mathcal{WS}}$ -injective, $\overline{\mathcal{WS}}$ -projective, $\overline{\mathcal{WS}}$ -coinjective and $\overline{\mathcal{WS}}$ -coprojective modules. We begin by studying $\overline{\mathcal{WS}}$ -coinjective modules; this will allow us to prove that the proper class $\overline{\mathcal{WS}}$ has global dimension bounded by 1.

Theorem 5.4. [4] *Let R be a left hereditary ring. For an R -module A , the following are equivalent:*

- (i) A is $\overline{\mathcal{WS}}$ -coinjective.
- (ii) There is a submodule N of A such that N is small in the injective envelope $E(A)$ of A and A/N is injective.
- (iii) A has a weak supplement in its injective envelope $E(A)$.
- (iv) A has a weak supplement in some injective module I .

If R is a left hereditary ring, the class of $\overline{\mathcal{WS}}$ -coinjective R -modules is closed under extensions and factor modules. This class contains torsion-free modules with finite rank. Over a Dedekind domain, coatomic modules and bounded modules are $\overline{\mathcal{WS}}$ -coinjective ([4]). Recall that a module M over a commutative ring R is said to be *bounded* if $rM = 0$ for some $0 \neq r \in R$.

$\overline{\mathcal{WS}}$ -coinjective modules need not be injective or coatomic. For example, in abelian groups, the group J_p of p -adic numbers is $\overline{\mathcal{WS}}$ -coinjective but not coatomic.

Over a discrete valuation ring R , if A is a reduced torsion module, B is a bounded submodule of A and A/B is divisible, then A is also bounded ([21, Lemma 4.5]). If M is a torsion and reduced module over a discrete valuation ring, then M is $\overline{\mathcal{WS}}$ -coinjective if and only if M is coatomic ([4]).

Theorem 5.5. [6, Theorem 2] *If \mathcal{J} is a class of modules closed under extensions, then $\underline{k}(\mathcal{J})$ consists of all short exact sequences $f_*(E)$ where E is a short exact sequence of the form*

$$E : \quad 0 \longrightarrow J \longrightarrow B \longrightarrow C \longrightarrow 0$$

with $J \in \mathcal{J}$ and $f : J \rightarrow A$ is a homomorphism.

A module M is called *small* if M can be embedded as a small submodule into some submodule N , or equivalently if M is small in its injective envelope $E(M)$ (see [38]).

The following proposition shows that $\overline{\mathcal{WS}}$ is a coinjectively generated proper class when R is a left hereditary ring:

Proposition 5.6. [4]

- (i) For a left hereditary ring R , $\overline{\mathcal{WS}} = \underline{k}(Sm)$, where Sm is the class of all small modules.
- (ii) Over a Dedekind domain R , $\overline{\mathcal{WS}} = \underline{k}(coA)$ where coA is the class of all coatomic modules.

If R is a left hereditary ring, then $\text{gl. dim } \underline{k}(\mathcal{J}) \leq 1$ for every coinjectively generated class $\underline{k}(\mathcal{J})$ (see [2]). So using Proposition 5.6, we obtain the following evaluation for the global dimension of the class $\overline{\mathcal{WS}}$.

Theorem 5.7. [4] *For a left hereditary ring R , $\text{gl. dim } \overline{\mathcal{WS}} \leq 1$.*

We do not know whether there are $\overline{\mathcal{WS}}$ -injective modules which are not injective in general, but for Dedekind domains we have the following proposition:

Proposition 5.8. [4] *Over a Dedekind domain R , the only $\overline{\mathcal{WS}}$ -injective (= \mathcal{WS} -injective) modules are the injective R -modules.*

We finish this section with some results about $\overline{\mathcal{WS}}$ -projective and $\overline{\mathcal{WS}}$ -coprojective modules. By Proposition 5.6 and by the dual statement of [49, Proposition 1.2], $\overline{\mathcal{WS}}$ is generated by the short exact sequences of the form

$$0 \longrightarrow M \longrightarrow I \longrightarrow B \longrightarrow 0$$

where M is a small module and I is injective. By [49, Proposition 2.4], we have the following criteria for the $\overline{\mathcal{WS}}$ -projective modules:

Proposition 5.9. [4] *For a left hereditary ring R , a module C is \overline{WS} -projective if and only if $\text{Ext}_R(C, M) = 0$ for every small module M . Over a Dedekind domain R , a module C is \overline{WS} -projective if and only if $\text{Ext}_R(C, M) = 0$ for every coatomic module M .*

Corollary 5.10. [4] *For a left hereditary ring R , every finitely presented module is \overline{WS} -coprojective.*

Corollary 5.11. [4] *If R is a discrete valuation ring, $\mathcal{CAS} = \overline{WS} = \underline{k}(\mathcal{CA}) = \mathcal{WS}$ and so \mathcal{WS} is also a proper class.*

A Terminology and notation for proper classes

For the definition of the $\text{Ext}_R^1(C, A)$ directly as the equivalence classes of extensions of an R -module A by an R -module C , and for the related notation, and for the abelian groups $\text{Ext}_R^n(C, A)$ for all $n \in \mathbb{Z}^+$, see [39, Ch. III]. $\text{Ext}_R^1(C, A)$ consists of the equivalence classes of short exact sequences of R -modules starting with A and ending with C , and we shall identify the equivalence class $[E]$ of a short exact sequence E of R -modules starting with A and ending with C with any short exact sequence in the equivalence class $[E]$. We shall usually write $\text{Ext}_R(C, A)$ for $\text{Ext}_R^1(C, A)$. For rings R, S, T and bimodules ${}_T C_R$ and ${}_S A_R$, the abelian group $\text{Ext}_R^n(C; A)$ can be made an S - T -bimodule ${}_S [\text{Ext}_R^n(C, A)]_T$; see [39, §V.3]. Similarly for bimodules ${}_R C_T$ and ${}_R A_S$, the abelian group $\text{Ext}_R^n(C; A)$ can be made a T - S -bimodule ${}_T [\text{Ext}_R^n(C, A)]_S$.

For *proper classes* of short exact sequences modules, see [39, Ch. 12, §4], [56], [52], [43], [17, §10] and [1].

Let \mathcal{A} be a class of short exact sequences in $R\text{-Mod}$. If a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{A.1}$$

belongs to \mathcal{A} , then f is said to be an \mathcal{A} -monomorphism and g is said to be an \mathcal{A} -epimorphism; both maps are said to be \mathcal{A} -proper and the short exact sequence is said to be an \mathcal{A} -proper short exact sequence. The class \mathcal{A} is said to be *proper* in the sense of Buchsbaum ([12]) if it satisfies the following conditions (see [39, Ch. 12, §4] or [56] or [52]):

- (i) If a short exact sequence E is in \mathcal{A} , then \mathcal{A} contains every short exact sequence isomorphic to E .
- (ii) \mathcal{A} contains all splitting short exact sequences.
- (iii) The composite of two \mathcal{A} -monomorphisms is an \mathcal{A} -monomorphism if this composite is defined. The composite of two \mathcal{A} -epimorphisms is an \mathcal{A} -epimorphism if this composite is defined.
- (iv) If $g \circ f$ is an \mathcal{A} -monomorphism, then f is an \mathcal{A} -monomorphism. If $g \circ f$ is an \mathcal{A} -epimorphism, then g is an \mathcal{A} -epimorphism.

For a proper class \mathcal{A} , call a submodule A of a module B an \mathcal{A} -submodule of B , if the inclusion monomorphism $i_A : A \rightarrow B, i_A(a) = a, a \in A$, is an \mathcal{A} -monomorphism.

A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ of modules is determined by the monomorphism f or the epimorphism g uniquely up to isomorphism since it is isomorphic to the short exact sequence

$$0 \longrightarrow \text{Im } f \xrightarrow{i} B \xrightarrow{p} B/\text{Im } f \longrightarrow 0,$$

where i is the inclusion map and p is the natural epimorphism. So giving a proper class \mathcal{A} of short exact sequences of modules means choosing for every R -module B , some kind of submodules, which we call \mathcal{A} -submodules.

We shall use the letters π, ι and τ for describing the concepts related with the relative versions of ‘projective’, ‘injective’ and ‘flat’ as in [56] and [52].

Denote by \mathcal{A} a proper class of short exact sequences in $R\text{-Mod}$. An R -module M is said to be \mathcal{A} -projective [\mathcal{A} -injective] if it is projective [resp. injective] with respect to all short exact sequences in \mathcal{A} , that is, $\text{Hom}(M, E)$ [resp. $\text{Hom}(E, M)$] is exact for every E in \mathcal{A} . Denote all \mathcal{A} -projective [\mathcal{A} -injective] modules by $\pi(\mathcal{A})$ [resp. $\iota(\mathcal{A})$]. For a given class \mathcal{M} of modules, denote by $\pi^{-1}(\mathcal{M})$ [$\iota^{-1}(\mathcal{M})$], the largest proper class \mathcal{A} for which each $M \in \mathcal{M}$ is \mathcal{A} -projective [resp. \mathcal{A} -injective]; it is called the proper class *projectively generated* [resp. *injectively generated*]

by \mathcal{M} . A *right* R -module M is said to be \mathcal{A} -flat if M is flat with respect to every short exact sequence $E \in \mathcal{A}$, that is, $M \otimes E$ is exact for every E in \mathcal{A} . Denote all \mathcal{A} -flat *right* R -modules by $\tau(\mathcal{A})$. For a given class \mathcal{M} of *right* R -modules, denote by $\tau^{-1}(\mathcal{M})$ the class of all short exact sequences E of R -modules and R -module homomorphisms such that $M \otimes E$ is exact for all $M \in \mathcal{M}$. $\tau^{-1}(\mathcal{M})$ is the largest proper class \mathcal{A} of (left) R -modules for which each $M \in \mathcal{M}$ is \mathcal{A} -flat. It is called the proper class *flatly generated* by the class \mathcal{M} of *right* R -modules. When the ring R is commutative, there is no need to mention the sides of the modules. See [56, §2-3] and [52, §1-3,8] for these concepts in relative homological algebra in categories of modules. We shall use the same notation for proper classes of *right* R -modules. For example if \mathcal{M} is a class of *right* R -modules, then $\pi^{-1}(\mathcal{M})$ denotes the proper class of *right* R -modules projectively generated by \mathcal{M} . A module M is called \mathcal{A} -coprojective if every short exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

is in \mathcal{A} . A module M is called \mathcal{A} -coinjective if every short exact sequence of the form

$$0 \longrightarrow M \longrightarrow B \longrightarrow C \longrightarrow 0$$

is in \mathcal{A} . It is easily checked that a module M is called \mathcal{A} -coinjective if and only if M is an \mathcal{A} -submodule of its injective envelope $E(M)$. If $\mathcal{A} = \pi^{-1}(\mathcal{M})$ for a class \mathcal{M} of modules, then for an R -module A , the condition $\text{Ext}_R^1(M, A) = 0$ for all $M \in \mathcal{M}$ is equivalent to A being \mathcal{A} -coinjective (see [52, Proposition 9.5] and [41, Proposition 2.6.7]). Similarly, if $\mathcal{A} = \iota^{-1}(\mathcal{M})$ for a class \mathcal{M} of modules, then for an R -module C , the condition $\text{Ext}_R^1(C, M) = 0$ for all $M \in \mathcal{M}$ is equivalent to C being \mathcal{A} -coprojective (see [52, Proposition 9.4] and [41, Proposition 2.6.5]).

Let \mathcal{M} and \mathcal{J} be classes of modules. The smallest proper class $\bar{k}(\mathcal{M})$ such that all modules in \mathcal{M} are $\bar{k}(\mathcal{M})$ -coprojective is said to be the proper class *coprojectively generated* by \mathcal{M} . The proper class $\underline{k}(\mathcal{J})$ *coinjectively generated* by \mathcal{J} is defined dually.

For a class \mathcal{A} of short exact sequences of R -modules and for R -modules A, C , denote by $\text{Ext}_{\mathcal{A}}^1(C, A)$ or just by $\text{Ext}_{\mathcal{A}}(C, A)$, the equivalence classes of all short exact sequences in \mathcal{A} which start with A and end with C , i.e. a short exact sequence in \mathcal{A} of the form (A.1). If \mathcal{A} is a proper class, then this turns out to be a subgroup of $\text{Ext}_R^1(C, A)$ and a bifunctor $\text{Ext}_{\mathcal{A}}^1 : R\text{-Mod} \times R\text{-Mod} \rightarrow \mathcal{A}b$ is obtained which is a subfunctor of Ext_R^1 (see [39, Ch. 12, §4-5]). Conversely, given a class \mathcal{A} of short exact sequences, if $\text{Ext}_{\mathcal{A}}^1$ is a subfunctor of Ext_R^1 , $\text{Ext}_{\mathcal{A}}^1(C, A)$ is a subgroup of $\text{Ext}_R^1(C, A)$ for all R -modules A, C and if the composition of two \mathcal{A} -monomorphisms (or \mathcal{A} -epimorphisms) is an \mathcal{A} -monomorphism (an \mathcal{A} -epimorphism respectively), then \mathcal{A} is a proper class (see [46, Theorem 1.1]).

Using the functor $\text{Ext}_{\mathcal{A}}$ for a proper class \mathcal{A} , the \mathcal{A} -projectives, \mathcal{A} -injectives, \mathcal{A} -coprojectives, \mathcal{A} -coinjectives are simply described as extreme ends for the subgroup $\text{Ext}_{\mathcal{A}}(C, A) \subseteq \text{Ext}_R(C, A)$ being 0 or the whole of $\text{Ext}_R(C, A)$:

- (i) An R -module C is \mathcal{A} -projective if and only if $\text{Ext}_{\mathcal{A}}(C, A) = 0$ for all R -modules A .
- (ii) An R -module C is \mathcal{A} -coprojective if and only if $\text{Ext}_{\mathcal{A}}(C, A) = \text{Ext}_R(C, A)$ for all R -modules A .
- (iii) An R -module A is \mathcal{A} -injective if and only if $\text{Ext}_{\mathcal{A}}(C, A) = 0$ for all R -modules C .
- (iv) An R -module A is \mathcal{A} -coinjective if and only if $\text{Ext}_{\mathcal{A}}(C, A) = \text{Ext}_R(C, A)$ for all R -modules C .

For a proper class \mathcal{A} , and for each $n \in \mathbb{Z}^+$, the congruence classes of the \mathcal{A} -proper n -fold exact sequences give the abelian group $\text{Ext}_{\mathcal{A}}^n(C, A)$ (see [39, Ch. 12, §4-5] and [44, §17]).

For a proper class \mathcal{A} of short exact sequences of R -modules, the *global dimension* of \mathcal{A} is defined as follows: If there is no $n \in \mathbb{Z}^+ \cup \{0\}$ such that $\text{Ext}_{\mathcal{A}}^{n+1}(C, A) = 0$ for all R -modules A and C , then $\text{gl. dim } \mathcal{A} = \infty$; otherwise, it is defined to be

$$\text{gl. dim } \mathcal{A} = \min(\{n \in \mathbb{Z}^+ \cup \{0\} \mid \text{Ext}_{\mathcal{A}}^{n+1}(C, A) = 0 \text{ for all } R\text{-modules } A \text{ and } C\}).$$

A proper class \mathcal{A} is said to be *inductively closed* if for every direct system $\{E_i (i \in I); \pi_i^j (i \leq j)\}$ in \mathcal{A} , the direct limit $E = \varinjlim E_i$ is also in \mathcal{A} (see [29] and [52, §8]). For a proper class \mathcal{A} , the smallest inductively closed proper class containing \mathcal{A} is called the *inductive closure* of \mathcal{A} .

Two functors that are frequently used for relations between relative injectivity, projectivity and flatness are the R -dual functor

$$(-)^* = \text{Hom}_R(-, R) : R\text{-Mod} \longrightarrow \text{Mod-}R$$

and the character module functor

$$(-)^b = \text{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z}) : R\text{-Mod} \longrightarrow \text{Mod-}R.$$

For a (left) R -module M , its R -dual $M^* = \text{Hom}_R(M, R)$ is a *right* R -module. The character module functor $(-)^b : R\text{-Mod} \longrightarrow \text{Mod-}R$ uses the injective cogenerator \mathbb{Q}/\mathbb{Z} for $\mathbb{Z}\text{-Mod}$: For a (left) R -module M , $M^b = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a *right* R -module.

For a functor T from a category \mathcal{A} of left or right R -modules to a category \mathcal{B} of left or right S -modules (where R, S are rings), and for a given class \mathcal{F} of short exact sequences in \mathcal{B} , let $T^{-1}(\mathcal{F})$ be the class of those short exact sequences of \mathcal{A} which are carried into \mathcal{F} by the functor T . If the functor T is left or right exact, then $T^{-1}(\mathcal{F})$ is a proper class; see [56, Proposition 2.1].

Example A.1. The third purity example below (generalized from pure subgroups of abelian groups) is the main motivation for relative homological algebra; this is the reason why proper classes are also called purities (see [43] for ω -purities).

- (i) ${}_R\text{Split}$ is the smallest proper class consisting of all splitting short exact sequences of R -modules.
- (ii) ${}_R\text{Abs}$ is the largest proper class consisting of all short exact sequences of R -modules.
- (iii) ${}_R\text{Pure}$ is the classical *Cohn's purity*:

$$\begin{aligned} {}_R\text{Pure} &= \pi^{-1}(\{\text{all finitely presented } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all finitely presented } \textit{right } R\text{-modules}\}) \\ &= \tau^{-1}(\{\text{all } \textit{right } R\text{-modules}\}) \\ &= [(-)^b]^{-1}(\textit{Split}). \\ &= \iota^{-1}(\{M^b \mid M \text{ is a finitely presented } \textit{right } R\text{-module}\}) \end{aligned}$$

See for example [28, §1.4] for the proof of first four of these equalities. See [52, Proposition 6.2] for the last equality. The second equality above that allows us to pass from a proper class projectively generated by a class of finitely presented modules to a flatly generated proper class is a general idea; what is being used in this passage is the Auslander-Bridger transpose of finitely presented modules; see the below Subsection A.2. \textit{Pure} is the smallest inductively closed proper class; see [52, §6]. The above equalities give us some present technique to work on some of the relations between the proper classes related with neat submodules. See [31] for the use of this technique for some relations between neat and \mathcal{P} -pure short exact sequences of modules. It is well known that \textit{Pure} -coprojectives are just all flat modules (this is the reason why \mathcal{A} -coprojectives are also called \mathcal{A} -flat for a proper class \mathcal{A}). The coprojectives of this proper class is the class of all flat modules; the second proof of the Flat Cover Conjecture in [10] proves that for a proper class \mathcal{A} that is flatly generated by some set of finitely presented modules, if \mathcal{M} is the class of all \mathcal{A} -coprojective modules, then every modules has an \mathcal{M} -cover; see also [51] for details of this second proof.

- (iv) Given a collection \mathcal{I} of *right* ideals of R , denote by $\mathcal{I}^\tau = \mathcal{I}\text{-Pure}$ the proper class consisting of all short exact sequences $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ such that for all ideals I in \mathcal{I} , $A' \cap IB = IA'$ holds for $A' = \text{Im}(f)$. This condition is equivalent to $(R/I) \otimes A' \longrightarrow (R/I) \otimes B$ being monic (by for example [52, Lemma 6.1]). So this proper class is

$$\mathcal{I}^\tau = \mathcal{I}\text{-Pure} = \tau^{-1}(\{R/I \mid I \in \mathcal{I}\}).$$

When a class of \mathcal{E} of short exact sequences of modules does not form a proper class, the smallest proper class containing that class \mathcal{E} is studied. The intersection of all proper classes containing \mathcal{E} is clearly a proper class and it is denoted by $\langle \mathcal{E} \rangle$. We say that $\langle \mathcal{E} \rangle$ is the proper class generated by \mathcal{E} (see [49]). Clearly, $\langle \mathcal{E} \rangle$ is the smallest proper class containing \mathcal{E} .

A.1 Factorizable $\text{Ext}_{\mathcal{A}}$ with respect to a proper class \mathcal{A}

For a proper class \mathcal{A} of R -modules, let us say that $\text{Ext}_{\mathcal{A}}$ is *factorizable* as

$$R\text{-Mod} \times R\text{-Mod} \xrightarrow{\text{Ext}_R} \mathcal{A}b \longrightarrow \mathcal{A}b ,$$

if it is a composition $H \circ \text{Ext}_R$ for some functor $H : \mathcal{A}b \longrightarrow \mathcal{A}b$, that is, the diagram

$$\begin{array}{ccc} R\text{-Mod} \times R\text{-Mod} & \xrightarrow{\text{Ext}_{\mathcal{A}}} & \mathcal{A}b \\ & \searrow \text{Ext}_R & \nearrow H \\ & & \mathcal{A}b \end{array}$$

is commutative: for all R -modules A, C ,

$$\text{Ext}_{\mathcal{A}}(C, A) = H(\text{Ext}_R(C, A)).$$

When the ring R is *commutative*, since the functor Ext_R can be considered to have range $R\text{-Mod}$, we say that $\text{Ext}_{\mathcal{A}}$ is *factorizable* as

$$R\text{-Mod} \times R\text{-Mod} \xrightarrow{\text{Ext}_R} R\text{-Mod} \longrightarrow R\text{-Mod} ,$$

if it is a composition $H \circ \text{Ext}_R$ for some functor $H : R\text{-Mod} \longrightarrow R\text{-Mod}$, that is, the diagram

$$\begin{array}{ccc} R\text{-Mod} \times R\text{-Mod} & \xrightarrow{\text{Ext}_{\mathcal{A}}} & R\text{-Mod} \\ & \searrow \text{Ext}_R & \nearrow H \\ & & R\text{-Mod} \end{array}$$

is commutative: for all R -modules A, C ,

$$\text{Ext}_{\mathcal{A}}(C, A) = H(\text{Ext}_R(C, A)).$$

For the ring $R = \mathbb{Z}$, since we identify the categories $\mathcal{A}b$ and $\mathbb{Z}\text{-Mod}$, both the above two definitions coincide so that for a proper class \mathcal{A} of \mathbb{Z} -modules (abelian groups), we just say that $\text{Ext}_{\mathcal{A}}$ is *factorizable* in the above cases.

Example A.2. Let $U : \mathcal{A}b \longrightarrow \mathcal{A}b$ be the Ulm functor which associates to each abelian group A , its Ulm subgroup $U(A) = \bigcap_{n=1}^{\infty} nA$. $\text{Ext}_{\mathbb{Z}\mathcal{P}ure}$ is denoted by Pext in [30, §53], and by [45, Theorem 5.1], it is shown that for abelian groups A, C ,

$$\text{Ext}_{\mathbb{Z}\mathcal{P}ure}(C, A) = \text{Pext}(C, A) = \bigcap_{n=1}^{\infty} n \text{Ext}_{\mathbb{Z}}(C, A) = U(\text{Ext}_{\mathbb{Z}}(C; A)).$$

Thus, $\text{Ext}_{\mathbb{Z}\mathcal{P}ure}$ is factorizable.

A.2 The Auslander-Bridger Transpose of Finitely Presented Modules

Let M be a *finitely presented* R -module. Take a *projective presentation* of it, that is, take an exact sequence

$$\gamma : \quad P_1 \xrightarrow{f} P_0 \xrightarrow{g} M \longrightarrow 0$$

where P_0 and P_1 are *finitely generated projective* R -modules. Apply the functor $(-)^* = \text{Hom}_R(-, R)$ to this projective presentation:

$$0 \longrightarrow \text{Hom}_R(M, R) \xrightarrow{g^*} \text{Hom}_R(P_0, R) \xrightarrow{f^*} \text{Hom}_R(P_1, R)$$

Fill the right side of this sequence of *right* R -modules by the module $\text{Tr}_\gamma(M) = \text{Coker}(f^*) = P_1^*/\text{Im}(f^*)$ to obtain the exact sequence

$$\gamma^* : \quad P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{\sigma} \text{Tr}_\gamma(M) \longrightarrow 0, \quad (\text{A.2})$$

where σ is the canonical epimorphism. Since P_0^* and P_1^* are finitely generated projective *right* R -modules, the exact sequence (A.2) is a projective presentation for the finitely presented *right* R -module $\text{Tr}_\gamma(M)$ which is called the *Auslander-Bridger transpose* of the finitely presented R -module M with respect to the projective presentation γ . See [7] and [8].

Two R -modules A and B are said to be *projectively equivalent* if there exist projective R -modules P and Q such that $A \oplus P \cong B \oplus Q$. This is an equivalence relation on the class of (finitely generated) R -modules. An Auslander-Bridger transpose of a finitely presented R -module M is unique up to projective equivalence, that is, if γ and ρ are two projective presentations of a finitely presented R -module M , then $\text{Tr}_\gamma(M)$ is projectively equivalent to $\text{Tr}_\rho(M)$. For a detailed proof of this, see [40, §1] or [58, Theorem 4.2.4]). We shall just write $\text{Tr}(M)$ for an Auslander-Bridger transpose of the finitely presented R -module M keeping in mind that this is unique up to projective equivalence. Similarly, the Auslander-Bridger transpose of *right* R -modules are defined. Using the Auslander-Bridger transpose, we have a passage between relative flatness and relative projectivity (see [58, §4.3] and [52, §6]):

Theorem A.3. (by for example [52, Corollary 5.1]) *Let M be a finitely presented right R -module and E a short exact sequence of R -modules. Then the sequence $M \otimes_R E$ is exact if and only if $\text{Hom}_R(\text{Tr}(M), E)$ is exact.*

This gives us that a proper class that is projectively generated by a set \mathcal{M} of finitely presented modules is flatly generated by the Auslander-Bridger transposes of the modules in \mathcal{M} :

Theorem A.4. [52, Theorem 8.3] *Let \mathcal{M} be a set of finitely presented right R -modules. Let $\text{Tr}(\mathcal{M}) = \{\text{Tr}(M) \mid M \in \mathcal{M}\}$. We may assume that $\text{Tr}(\text{Tr}(\mathcal{M})) = \mathcal{M}$. Then we have*

$$\tau^{-1}(\mathcal{M}) = \pi^{-1}(\text{Tr}(\mathcal{M})) \quad \text{and} \quad \pi^{-1}(\mathcal{M}) = \tau^{-1}(\text{Tr}(\mathcal{M})).$$

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Author information

RAFAİL ALİZADE, Yaşar University, Department of Mathematics, Selçuk Yaşar Campus, Üniversite Cad.,No. 35-37, 35100, Bornova, İzmir, Turkey.
E-mail: rafail.alizade@yasar.edu.tr

ENGİN MERMUT, Dokuz Eylül Üniversitesi, Tınaztepe Yerleşkesi, Fen Fakültesi, Matematik Bölümü, 35160, Buca/İzmir, TURKEY.
E-mail: engin.mermut@deu.edu.tr

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