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When Only Finitely Many Intermediate Rings Result from Juxtaposing Two Minimal Ring Extensions

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Abstract. Let $R \subset S$ and $S \subset T$ be minimal ring extensions of (commutative) rings. Necessary and sufficient conditions are given for $R \subset T$ to satisfy FIP, that is, to have only finitely many intermediate rings. These criteria are given in terms of the kind of minimal ring extensions that $R \subset S$ and $S \subset T$ are (that is, flat epimorphism, inert, ramified or decomposed). Examples are given to illustrate all the FIP possibilities and all the non-FIP possibilities. An application pays special attention to ring extensions that have exactly one properly included intermediate ring.

1 Introduction

This paper is a sequel to [12]. All rings considered below are commutative with identity; all subrings, inclusions of rings, and ring/algebra homomorphisms are unital. Recall that if $A \subseteq B$ is a ring extension, then $A \subseteq B$ is said to satisfy FCP if each chain of rings contained between A and B is finite; and that $A \subseteq B$ is said to satisfy FIP if there are only finitely many rings contained between A and B. It is clear that FIP $\Rightarrow$ FCP, but the converse is false. Whenever $A \subset B$ satisfies FCP, one has a finite (maximal) chain of rings $A = A_0 \subset \dots \subset A_i \subset A_{i+1} \subset \dots \subset A_n = B$ for some positive integer n, such that $A_i \subset A_{i+1}$ is a minimal ring extension for all $i = 0, \dots, n-1$. (As usual, $\subset$ denotes proper inclusion. Some useful background on minimal ring extensions is given in the next paragraph.) Not all such "compositions" of minimal ring extensions produce a ring extension $A \subset B$ that satisfies FCP. We focus on the case $n = 2$. Suppose that $R \subset S$ and $S \subset T$ are each minimal ring extensions. While [12] studied when the resulting "composition" $R \subset T$ satisfies FCP, this paper is devoted to a comprehensive study of when that composition $R \subset T$ satisfies FIP.

Recall (cf. [13]) that a ring extension $A \subset B$ is a *minimal ring extension* if there does not exist a ring properly contained between A and B. A minimal ring extension $A \subset B$ is either integrally closed (in the sense that A is integrally closed in B) or integral. If $A \subset B$ is a minimal ring extension, it follows from [13, Théorème 2.2 (i) and Lemme 1.3] that there exists a unique maximal ideal M of A (called the *crucial maximal ideal* of $A \subset B$) such that the canonical injective ring homomorphism $A_M \to B_M$ $(:= B_{A \setminus M})$ can be viewed as a minimal ring extension while the canonical ring homomorphism $A_P \to B_P$ is an isomorphism for all prime ideals P of A except M. A minimal ring extension $A \subset B$ is integrally closed if and only if $A \hookrightarrow B$ is a flat epimorphism (in the category of commutative rings). If $A \subset B$ is an integral minimal ring extension with crucial maximal ideal M, there are three possibilities: $A \subset B$ is said to be respectively *inert*, *ramified*, or *decomposed* if B/MB $(= B/M)$ is isomorphic, as an algebra over the field $K := A/M$, to a minimal field extension of K, $K[X]/(X^2)$, or $K \times K$. (As usual, X will denote an indeterminate over the ambient base ring.)

Apart from illustrative examples and remarks, we **fix notation** so that $R \subset S$

and $S \subset T$ are minimal ring extensions, with crucial maximal ideals M and N, respectively. Each of these could be of four types, namely, integrally closed, inert, ramified, or decomposed. Our work will show that there are examples where $R \subset T$ satisfies FIP for each of the resulting 16 cases, but not all of these cases allow the possibility that $R \subset T$ fails to satisfy FIP. Each time we obtain an answer for one of the 16 cases, we follow that theoretical result with at least one remark giving relevant examples where $R \subset T$ satisfies FIP and, if theoretically possible, where $R \subset T$ fails to satisfy FIP. This work is carried out in Sections 2 and 3. A summary of that work is given in Theorem 4.1. The rest of Section 4 pays special attention to ring extensions $R \subset T$ that have exactly one ring properly contained between R and T. (Apart from certain extensions of finite fields, only a couple of sporadic examples of such $R \subset T$ seem to have occurred on the literature. For instance, in the integrally closed case, one could take R to be a two-dimensional valuation domain with quotient field T; and one has [15, Remarks 4.15 (a)] as an example in the integral case.) Section 2 is short, as some of the work in [12] allows us to settle matters when at least one of $R \subset S$ and $S \subset T$ is integrally closed. Consequently, the bulk of this paper is carried out in Section 3, which concerns the 9 cases where both $R \subset S$ and $S \subset T$ are integral (that is, either inert, ramified, or decomposed).

If D is a (commutative integral) domain, it will be convenient to let D' denote the integral closure of D (in its quotient field). If A is a ring, then Spec(A) (resp., Max(A)) denotes the set of prime (resp., maximal) ideals of A. If E is a module over a ring A, then the *support* of E is $\text{Supp}(E) := \text{Supp}_A(E) := \{P \in \text{Spec}(A) \mid E_P \ (:= E_{A\setminus P}) \neq 0\}$; $\text{MSupp}_A(E) := \text{Max}(A) \cap \text{Supp}(E)$; and $\text{L}_A(E)$ denotes the length of the A-module E. If $A \subseteq B$ are rings, then ${}^{+}_{B}A$ denotes the seminormalization of A in B; and $[A, B]$ denotes the set of intermediate rings (that is, the set of rings C such that $A \subseteq C \subseteq B$). Also, as usual, $\mathbb{F}_q$ denotes the finite field of cardinality q. Any unexplained material is standard, as in [16], [18].

2 The essentially known cases

Theorem 2.1 summarizes from [12] that we know under which conditions $R \subset T$ satisfies FIP, provided that at least one of the given minimal ring extensions $R \subset S$ and $S \subset T$ is integrally closed. Then Remark 2.2 develops or collects relevant examples exhibiting all the associated FIP or non-FIP possibilities.

Theorem 2.1. (a) Let $R \subset S$ and $S \subset T$ be minimal ring extensions such that $S \subset T$ is integrally closed. Then $R \subset T$ satisfies FIP.

(b) Let $R \subset S$ and $S \subset T$ be minimal ring extensions such that $R \subset S$ is integrally closed and $S \subset T$ is integral. Let M (resp., N) be the crucial maximal ideal of $R \subset S$ (resp., $S \subset T$). Then $R \subset T$ satisfies FIP if and only if $N \cap R \not\subseteq M$.

Proof. (a) Let the data be as in (a). If $R \subset S$ is integrally closed (resp., integral), then the assertion is contained in part (b) (resp., part (c)) of [12, Proposition 2.1].

(b) The assertion is contained in [12, Theorem 2.3]. □

Remark 2.2. (a) One case of Theorem 2.1 (a) states that if both $R \subset S$ and $S \subset T$ are integrally closed minimal ring extensions, then $R \subset T$ satisfies FIP. One way to illustrate these hypotheses is to take R to be a two-dimensional valuation domain with (unique) height 1 prime ideal P, set $S := R_P$ and take T to be the quotient field of R (cf. [18, Theorem 65; Exercise 29, page 43]). In fact, a characterization of the general situation where both $R \subset S$ and $S \subset T$ are integrally closed minimal ring extensions is, locally, just a pullback of the example that we just gave. We next give those details.

An integrally closed minimal ring extension $A \subset B$ is an example of a "normal pair" (A, B), in the sense of [4]; that is, a ring extension $A \subseteq B$ such that each $C \in [A, B]$ is integrally closed in B. It happens that if both $R \subset S$ and $S \subset T$ are integrally closed minimal ring extensions, then (R, T) is a normal pair, since a "composition" of normal pairs is a normal pair [19, Theorem 5.6, Chapter I] (see also [8, Lemma 6.1]). Now, suppose that both $R \subset S$ and $S \subset T$ are integrally closed minimal ring extensions, with (R, M) quasilocal. Then by the pullback characterization of normal pairs with a quasi-local base ring [8, Theorem 6.8], R has a divided prime ideal Q (that is, $Q \in \text{Spec}(R)$ such that $QR_Q = Q$) such that $T = R/Q$ and $D := R/Q$ is a valuation domain. Observe that T/Q is the quotient field of D. Let $\varphi : T \to T/Q$ be the canonical surjection. As $R \subset S$ and $S \subset T$ are both minimal ring extensions, it follows from a standard homomorphism theorem (cf. [7, Lemma II.3]) that $D \subset S/Q$ and $S/Q \subset T/Q$ are also both minimal ring extensions (and conversely). Then, since D is a valuation domain with overring S/Q and quotient field T/Q, it follows from the facts recalled above from [18] that D must be two-dimensional, with S/Q the localization of D at its height 1 prime ideal P (as in the above example). Since $R = \varphi^{-1}(D)$ and $S = \varphi^{-1}(S/Q) = \varphi^{-1}(D_P)$, we have the pullbacks $R = S \times_{D_P} D$ and $S = T \times_{T/Q} D_P$. By applying a fundamental gluing result [14, Theorem 1.4] to the latter pullback and considering the order-theoretic upshot, we see that S is quasi-local (as were R and T).

Conversely, suppose that (R, T) is a given normal pair with R quasi-local and suppose one has a divided prime ideal Q of R such that $T = R/Q$ and $D := R/Q$ is a valuation domain. Once again, let $\varphi : T \to T/Q$ be the canonical surjection. Suppose that D is two-dimensional with height 1 prime ideal P. Put $S = \varphi^{-1}(D_P)$. Then by the above reasoning, both $R \subset S$ and $S \subset T$ are minimal ring extensions. Moreover, using the well known behavior of integral closure in pullbacks (cf. [14]), both of these extensions are integrally closed.

(b) Next, we give a simple example illustrating the case of Theorem 2.1 (a) where $R \subset S$ is integral (and $S \subset T$ is integrally closed). Let X be an analytic indeterminate over $\mathbb{Q}(\sqrt{2})$, and set $R := \mathbb{Q} + X\mathbb{Q}(\sqrt{2})[[X]]$, $S := \mathbb{Q}(\sqrt{2})[[X]]$ and $T := \mathbb{Q}(\sqrt{2})((X))$. By the well-known description of the overrings and the integral closure for the classical $D + M$ construction (as in [3, Theorem 2.1]), we see that both $R \subset S$ and $S \subset T$ are minimal ring extensions, with $R \subset S$ being integral and $S \subset T$ being integrally closed. (Alternatively, one could use [7, Lemma II.3] and [14] to conclude that $R \subset S$ inherits the property of being an integral minimal ring extension from $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2})$, while $S \subset T$ has the asserted properties because S is a one-dimensional valuation domain with quotient field T.)

(c) To close the section, we give references that show that all the possibilities suggested by the statement of Theorem 2.1 (b) can actually occur. For an example where $R \subset S$ is an integrally closed minimal ring extension and the integral minimal ring extension $S \subset T$ is decomposed (resp., inert; resp., ramified) with $R \subset T$ not satisfying FIP, see Example 3.1 (resp., Example 3.2; resp., Example 3.3) of [12]. On the other hand, see [12, Corollary 4.2] for the corresponding examples where $R \subset T$ satisfies FIP (with $R \subset S$ an integrally closed minimal ring extension and the integral minimal ring extension $S \subset T$ being, as one wishes, decomposed or inert or ramified).

3 The integral cases

We continue to assume that $R \subset S$ and $S \subset T$ are minimal ring extensions and to determine conditions under which $R \subset T$ satisfies FIP. Because of the material in Section 2, we may assume henceforth that both $R \subset S$ and $S \subset T$ are integral. Proposition 3.1 collects a number of useful facts, and Lemma 3.2 applies [8, Theorem 5.18] to the present setting.

Proposition 3.1. Let $R \subset S$ and $S \subset T$ be integral minimal ring extensions, with crucial maximal ideals M and N, respectively. Then:

(a) $R \subset T$ satisfies FIP if and only if $R_M \subset T_M$ satisfies FIP.

(b) Suppose that $N \cap R \neq M$. Then $R \subset T$ satisfies FIP.

(c) Suppose that $N \cap R = M$. Then $R_M \subsetneq S_M$ is a minimal ring extension of the same type (that is, inert, ramified, or decomposed) as $R \subset S$ and has crucial maximal ideal MR_M. Additionally, $S_M \subseteq T_M$ is a minimal ring extension of the same type as $S \subset T$ and has crucial maximal ideal N_M $(= NS_M = NT_M)$.

(d) Put $P := N \cap R$. If $P = M$, then $|[R,T]| = |[R_M, T_M]|$. If $P \neq M$, then $|[R,T]| = |[R_M, T_M]| \cdot |[R_P, T_P]| \geq 4$.

Proof. (a), (b): By considering the chain $R \subset S \subset T$, we see via [8, Corollary 3.2] that $\mathrm{Supp}(S/R) = \{M, N \cap R\}$. (By integrality, this set is also $\mathrm{MSupp}(S/R)$.) In particular, $\mathrm{Supp}(S/R)$ is finite. Therefore, by [8, Proposition 3.7 (a)], $R \subset T$ satisfies FIP if and only if both $R_M \subset T_M$ and $R_{N\cap R} \subset T_{N\cap R}$ satisfy FIP. Hence, to prove (a), it is enough to show that $R_{N\cap R} \subset T_{N\cap R}$ satisfies FIP if $N \cap R \neq M$. However, by a property of the crucial maximal ideal M that was recalled in the Introduction, $N \cap R \neq M$ implies that $R_{N\cap R} = S_{N\cap R}$ canonically, and so $R_{N\cap R} \subset T_{N\cap R}$ is simply $S_{N\cap R} \subset T_{N\cap R}$, which being a minimal ring extension (cf. [13, Lemme 1.3]), certainly satisfies FIP. Finally, to prove (b), it remains only to prove that if $N \cap R \neq M$, then $R_M \subset T_M$ satisfies FIP. However, $N \cap R \neq M$ implies $S_M = T_M$ [8, Lemma 2.5], and so, as above, $R_M \subset T_M$ is simply the minimal ring extension $R_M \subset S_M$ (and hence satisfies FIP).

(c) The assertions concerning $R_M \subset S_M$ were proved in [10, Proposition 4.6]. We next prove the assertions for $S_M \subset T_M$. First, $S_M \subset T_M$ *is* a minimal ring extension, thanks to [8, Lemma 2.5]. Of course, there are now three cases. Suppose first that $S \subset T$ is inert; that is, $S/N \subset T/N$ are fields. Localizing at $R \setminus M$ leads (up to isomorphism) to the integral extension $S_M/N_M \subset T_M/N_M$ of domains. Moreover, these are fields, as the going-up property of integral extensions (cf. [18, Theorem 44]) ensures that N is maximal with respect to being a prime ideal of S (resp., T) that is disjoint from $R \setminus M$. Hence, $S_M \subset T_M$ is inert. In addition, its crucial maximal ideal is N_M, since T being a finitely generated S-module implies that $(S_M : T_M) = (S : T)_{R\setminus M} = N_M$.

The proof in case $S \subset T$ is decomposed (resp., ramified) follows from the generator-and-relations characterization of decomposed (resp., ramified) minimal ring extensions [11, Proposition 2.12]. Indeed, suppose that there exists $q \in T \setminus S$ such that $T = S[q]$ and $Nq \subseteq S$, as well as $q^2 - q \in N$ (resp., as well as $q^2 \in S, q^3 \in S$). Then $q/1 \in T_M \setminus S_M$ is such that $T_M = S_M[q/1]$ and $N_M(q/1) \subseteq S_M$, as well as $(q/1)^2 - q/1 \in N_M$ (resp., as well as $(q/1)^2 \in S_M, (q/1)^3 \in S_M$).

(d) The extension $R \subset T$ is clearly integral. Moreover, it satisfies FCP by [8, Corollary 4.3] (which applies since $R \subset S$ and $S \subset T$ are each integral ring extensions that satisfy FCP). Also, recall from the proof of (a) that $\mathrm{MSupp}(S/R) = \{M, N \cap R\}$. Therefore, an application of [8, Theorem 3.6 (b), (c)] gives the asserted expressions for $|[R,T]|$. It remains only to prove that $|[R_M, T_M]| \cdot |[R_P, T_P]| \geq 4$ if $(N \cap R =)\ P \neq M$. Note that $|[R_M, T_M]| \geq 2$ since $R_M \subset S_M$ is a minimal ring extension. (We need make no comment about $S_M \subseteq T_M$ here.) Thus, it suffices to prove that $|[R_P, T_P]| \geq 2$. Recall from the proof of (a) that $R_P = S_P$ canonically. Thus, we need only prove that $S_P \subset T_P$. In fact, since $R_P = S_P$ canonically and $R \subset S$ is integral, it follows easily that N is the only prime ideal of S that lies over P, and so an application of [8, Lemma 2.4 (b)] yields that $S_P = S_N$ canonically and $T_P = T_N$ canonically. In other words, the canonical ring homomorphisms $S_P \to S_N$ and $T_P \to T_N$ are each isomorphisms. Thus, the ring extension $S_P \subseteq T_P$ can be identified with $S_N \subseteq T_N$, which is a minimal ring extension (since N is the crucial maximal ideal of $S \subset T$), whence $S_P \subset T_P$. ◻

Lemma 3.2. Let (R, M) be a quasi-local ring, and let $R \subseteq S$ and $S \subset T$ be integral minimal ring extensions. Define $J := \mathrm{Rad}(T)$, $A := R + J$, $C := (R : A)$, and for each $i > 0$, $M_i := M + AM^i$, $R_i := R + AM^i$, and, if $R \neq A$, $M_i' := M_i/C$ and $R_i' := R_i/C$. Then $A = {}^{+}_{T}R$. Moreover, either $R = A$, or $M = (R : A)$, or there exists an integer $n > 1$ such that $M^n \subseteq (R : A)$ with $M^{n-1} \not\subseteq (R : A)$. Furthermore, $R \subseteq T$ has FIP if and only if (either R/M is finite or) when R/M is infinite, the following three properties hold:

(i) There exists $\gamma \in T$ such that $T = A[\gamma]$ and γ is algebraic over A;

(ii) Either $R = A$, or $M = (R : A)$, or $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $1 \leq i \leq$

$n-1$;

(iii) If $R \neq A$, then there exists $\alpha \in A$ such that $A = R_1[\alpha]$ and $\alpha^3 \in AM$, and, with $A' := R_1[\alpha^2]$ and $A'' := R + A'M$, there exists $\beta \in A$ such that $A' = A''[\beta]$ and $\beta^3 \in A'M$.

Proof. To indicate how the assertion is just a restatement of [8, Theorem 5.18] adapted to the current hypotheses, one need only check that $R \subset T$ is an integral extension that satisfies FCP. This checking has already been done: see the proof of Proposition 3.1 (d). □

Among other things, Proposition 3.3 resolves 7 of the 9 cases that are before us in this section.

Proposition 3.3. Let $R \subset S$ and $S \subset T$ be integral minimal ring extensions, with crucial maximal ideals M and N, respectively. Then:

(a) Suppose that $^+_{T_M}(R_M) = R_M$ and that $N \cap R = M$. Then $R \subset T$ satisfies FIP if and only if either R/M is finite or there exists $\gamma \in T_M$ such that $T_M = R_M[\gamma]$.

(b) Suppose that both $R \subset S$ and $S \subset T$ are inert and that $N \cap R = M$. Then $^+_{T_M}(R_M) = R_M$, and so $R \subset T$ satisfies FIP if and only if either R/M is finite or there exists $\gamma \in T_M$ such that $T_M = R_M[\gamma]$.

(c) Suppose that $R \subset S$ is decomposed, $S \subset T$ is inert, and $N \cap R = M$. Then $^+_{T_M}(R_M) = R_M$ and $R \subset T$ satisfies FIP.

(d) Suppose that both $R \subset S$ and $S \subset T$ are decomposed and that $N \cap R = M$. Then $^+_{T_M}(R_M) = R_M$ and $R \subset T$ satisfies FIP.

(e) Suppose that $R \subset S$ is inert, $S \subset T$ is decomposed and $N \cap R = M$. Then $^+_{T_M}(R_M) = R_M$ and $R \subset T$ satisfies FIP.

(f) If $R \subset S$ is ramified, then $S \subseteq {}^+_T R$.

(g) If $R \subset S$ is ramified and $S \subset T$ is decomposed, then $R \subset T$ satisfies FIP.

(h) If $R \subset S$ is decomposed and $S \subset T$ is ramified, then $R \subset T$ satisfies FIP.

(i) If $R \subset S$ is ramified and $S \subset T$ is inert, then $R \subset T$ satisfies FIP.

Proof. (a) By Proposition 3.1 (a), $R \subset T$ satisfies FIP if and only if $R_M \subset T_M$ satisfies FIP. As $N \cap R = M$, Proposition 3.1 (c) allows us to replace the tower $R \subset S \subset T$ of integral minimal ring extensions with the tower $R_M \subset S_M \subset T_M$ of integral minimal ring extensions. Thus, without loss of generality, (R, M) is quasi-local with $^+_T R = R$, and our task is to prove that $R \subset T$ satisfies FIP if and only if either R/M is finite or there exists $\gamma \in T$ such that $T = R[\gamma]$. In the notation of Lemma 3.2, $A = {}^+_T R = R$, and so the assertion follows directly from Lemma 3.2.

(b) By Proposition 3.1 (c), the hypotheses imply that $R_M \subset S_M$ and $S_M \subset T_M$ are inert extensions, with crucial maximal ideals MR_M and NS_M, respectively. However, by the INC property of integral ring extensions (cf. [18, Theorem 44]), the facts that $M \in \text{Spec}(S)$ (owing to $R \subset S$ being inert) and $N \cap R = M$ force $N = M$, whence $NS_M = MS_M = MR_M$ and so the Jacobson radical of T_M is MR_M. Therefore, by the first assertion in Lemma 3.2, $^+_{T_M}(R_M) = R_M + MR_M = R_M$. The final assertion follows at once from (a).

(c) Arguing as in the above proof of (a) and taking Proposition 3.1 (c) into account, we may assume that (R, M) is quasi-local. We claim that $^+_T R = R$. Since $R \subset S$ is decomposed, S has exactly two distinct maximal ideals, namely, $N_1 := N$ and (say) N_2. Furthermore, the R-algebra homomorphisms $R/M \to S/N_i$ are isomorphisms (for $i = 1, 2$). As $S \subset T$ is inert, there is a unique maximal ideal (say Q) of T that lies over N_2 and N is the only prime (maximal) ideal of T that meets S in N. If J denotes the Jacobson radical of T, then it follows from the first assertion in Lemma 3.2 that $^+_T R = R + J = R + (Q \cap N) \subseteq R + N \subseteq S$. However, $^+_T R \neq S$ since $R \subset S$ is not subintegral (the point being that the canonical function $\text{Spec}(S) \to \text{Spec}(R)$ is not an injection). Therefore, the minimality of $R \subset S$ implies that $^+_T R = R$, as claimed. Therefore, by (a), we may assume that R/M is infinite.

As the maximal ideals of T are necessarily comaximal, the Chinese Remainder Theorem yields that $T/M \cong T/N \times T/Q$. Since $R \subset S$ is decomposed and has crucial maximal ideal M, we have that $R/M = S/N$ canonically. Hence, $R/M \subset T/N$ is identified with $S/N \subset T/N$, which is a minimal field extension and hence satisfies FIP. On the other hand, since N_2 is not the crucial maximal ideal of $S \subset T$, we have the canonical identifications $S_{N_2} = T_{N_2}$ and, by [8, Lemma 2.4 (b)], $T_{N_2} = T_Q$. Thus, $S_{N_2} = T_Q$. Equating residue fields, we get that $S/N_2 = T/Q$. Recall that $R/M = S/N_2$ canonically since $R \subset S$ is decomposed. The upshot is the identification $T/M = T/N \times T/Q = T/N \times R/M$. We can now use condition (4) in [8, Theorem III.5] to conclude that $R/M \subset T/M$ satisfies FIP (since $R/M \subset T/N$ identifies with the FIP extension $S/N \subset T/N$ and $R/M \subseteq R/M = (R/M)[0]$ trivially satisfies FIP). Therefore, by [7, Proposition II.4], $R \subset T$ also satisfies FIP.

(d) As in the proof of (c), we may assume that (R, M) is quasi-local, and we will first prove that ${}_T^+R = R$. Also as in the proof of (c), the "decomposed" hypothesis for $R \subset S$ (in conjunction with $N \cap R = M$) gives that $\text{Max}(S) = \{N_1, N_2\}$ where $N_1 = N \neq N_2$. Since $S \subset T$ is decomposed and has crucial maximal ideal N, there are exactly three distinct maximal ideals of T. Denote these maximal ideals by P_1 and P_2 which each lie over N and Q which lies over N_2. Also, note that $P_1 \cap P_2 = N$ since $S \subset T$ is decomposed. Then, with J once again denoting the Jacobson radical of T, we see from the first assertion in Lemma 3.2 that ${}_T^+R = R+J = R+(Q\cap P_1\cap P_2) = R+(Q\cap S\cap N) = R+(N_2\cap N) = R+M = R$, as desired. Therefore, by (a), we may assume that R/M is infinite.

As the maximal ideals of T are pairwise comaximal, the Chinese Remainder Theorem yields that $T/M \cong T/P_1 \times T/P_2 \times T/Q$. Since the given minimal ring extensions are decomposed, we have canonical identifications $R/M = S/N_i = T/P_i$ (for $i = 1, 2$) as (R/M)-algebras. Moreover, as in the proof of (c), we see that the identifications $S_{N_2} = T_{N_2} = T_Q$ lead to $S/N_2 = T/Q$. Moreover, since $R \subset S$ is decomposed, we have that $R/M = S/N_2$ canonically, whence $R/M = T/Q$. The upshot is that as (R/M)-algebras, $T/M \cong R/M \times R/M \times R/M$. It now follows easily via condition (4) in [8, Theorem III.5] that $R/M \subset T/M$ satisfies FIP. Hence, by [7, Proposition II.4], $R \subset T$ satisfies FIP.

(e) As in the proof of (c), we may assume that (R, M) is quasi-local, and we will first prove that ${}_T^+R = R$. As in the proof of (b), the "inert" hypothesis implies that $N = M$, and the INC property of $R \subset S$ implies that N is the unique maximal ideal of S. Next, since $S \subset T$ is decomposed and has crucial maximal ideal N, there are exactly two distinct maximal ideals of T, say Q_1 and Q_2; and $Q_1 \cap Q_2 = N = M$. Then, with J denoting the Jacobson radical of T, the first assertion of Lemma 3.2 yields that ${}_T^+R = R + J = R + (Q_1 \cap Q_2) = R + M = R$, as desired. Therefore, by (a), we may assume that R/M is infinite.

Since $\text{Max}(T) = \{Q_1, Q_2\}$, $T/M \cong T/Q_1 \times T/Q_2$. As $S \subset T$ is decomposed with crucial maximal ideal $N = M$, we have canonical identifications $S/M = T/Q_i$ for $i = 1, 2$. Hence, $T/M \cong S/M \times S/M$ as (R/M)-algebras. By [7, Proposition II.4], it suffices to prove that $R/M \subset S/M \times S/M$ satisfies FIP. To do so, we will use the following fact, which is of independent interest but seems to have gone unnoticed in the literature. If K is an infinite field and $K \subseteq L$ is a field extension that satisfies FIP, then $K \hookrightarrow L \times L$ (given by $a \mapsto (a, a)$) also satisfies FIP. (Here is a quick proof of this fact. Since the field K is infinite, it follows from condition (4) in [7, Theorem III.5] that $K \hookrightarrow B := K \times L \times L$ satisfies FIP. Then, applying [1, Proposition 3.3 (a)] to the ideal $I := K \times \{0\} \times \{0\}$, we have that $K/(I \cap K) \subseteq B/I$ satisfies FIP; that is, $K \subseteq L \times L$ satisfies FIP, as asserted.) Applying this fact, with $K := R/M$ and $L := S/M$ completes the proof.

(f) The assertion follows because $R \subset S$ is subintegral (as a consequence of its being ramified) and ${}_T^+R$ is the union of all the subintegral extensions of R that are contained in T.

(g) By Proposition 3.1 (b), we may assume, without loss of generality, that $N \cap R = M$. Note that since $S \subset T$ is decomposed, two distinct prime ideals of

T lie over N. Hence, these prime ideals of T each meet R in M. In particular, $R \subset T$ is not a subintegral extension, and so ${}_T^+R \neq T$. As (f) now ensures that $S \subseteq {}_T^+R \subset T$, the minimality of $S \subset T$ yields that ${}_T^+R = S$. According to [8, Theorem 5.9], it suffices to show that each of the extensions $R \subseteq {}_T^+R$, ${}_T^+R \subseteq {}_T^tR$, and ${}_T^tR \subseteq T$ satisfies FIP. (As usual, ${}_T^tR$ denotes the t-closure of R in T.) The first of these conditions holds because $R \subseteq {}_T^+R = S$ is a minimal ring extension. The final two conditions also hold because of the minimality of $S \subset T$, as ${}_T^tR$ is constrained to be either S or T.

(h) By [8, Lemma 2.8], the hypotheses yield the existence of some (uniquely determined) $S^* \in [R,T]$ such that $R \subset S^*$ is ramified and $S^* \subset T$ is decomposed. Hence an application of (g) completes the proof of (h). (For the sake of a later application, we note also that [8, Lemma 2.8] gives that $|[R,T]| = 4$.)

(i) Once again, Proposition 3.1 (b) allows us to assume that $N \cap R = M$. Also, by (f), $S \subseteq {}_T^+R$. It suffices to prove that ${}_T^+R \neq T$, for one can then repeat the second half of the proof of (g). Hence, it is enough to show that $R \subset T$ is not subintegral. To that end, note that the “inert" hypothesis ensures that $N \in \mathrm{Max}(T)$. As N lies over M, it will be enough to prove that the canonical injective ring homomorphism $f : R/M \to T/N$ is not surjective. In fact, using the canonical isomorphism $R/M \to (R+N)/N$, we see that the image of f is $(R+N)/N \subseteq S/N \subset T/N$, to complete the proof. □

Among the various parts of the next remark, the reader will find illustrations of the “FIP" assertions in Proposition 3.3.

Remark 3.4. (a) We begin by illustrating Proposition 3.1 (b) by giving an example of inert (integral minimal ring) extensions, $R \subset S$ and $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R \neq M$. Take $K \subset L$ to be any minimal field extension. Put $R := K \times K$, $S := K \times L$, and $T := L \times L$. Note that $M := (R : S) = K \times \{0\}$ and $N := (S : T) = \{0\} \times L$. Then $R/M \subseteq S/M$ can be identified with $\{0\} \times K \subset \{0\} \times L$ and, hence, with $K \subset L$. Therefore, $R \subset S$ is an inert extension having crucial maximal ideal M. Similarly, $S \subset T$ is an inert extension having crucial maximal ideal N, since $S/N \subseteq T/N$ can be identified with $K \times \{0\} \subset L \times \{0\}$ and, hence, with $K \subset L$. Also, $N \cap R = \{0\} \times K \not\subseteq K \times \{0\} = M$. Hence, by parts (b) and (d) of Proposition 3.1, $R \subset T$ satisfies FIP and $|[R,T]| \geq 4$. In fact, this second assertion is sharp since the present data satisfy $|[R,T]| = 4$, with $[R,T] = \{R, S, T, L \times K\}$.

(b) Suppose that we are in the situation of Proposition 3.3 (b), namely, where both $R \subset S$ and $S \subset T$ are inert extensions and their crucial maximal ideals satisfy $N \cap R = M$. Then the assertion in Proposition 3.3 (b) (likewise, the assertion in Proposition 3.3 (a)) is best possible. Indeed, examples exist showing that such $R \subset T$ can satisfy (resp., need not satisfy) FIP. The most accessible such examples come from field theory, thanks to the following version of the Primitive Element Theorem: if $K \subseteq L$ is a finite-dimensional field extension (in the sense that $[L : K] < \infty$), then there exists $\gamma \in L$ such that $L = K(\gamma)$ ($= K[\gamma]$) if and only if $K \subseteq L$ satisfies FIP. Take $R \subset S \subset T$ to be a chain of fields $K \subset F \subset L$ such that both $K \subset F$ and $F \subset L$ are minimal (hence, finite-dimensional) field extensions. In other words, both $K \subset F$ and $F \subset L$ are field extensions that are inert (integral minimal ring) extensions. These extensions each have the crucial maximal ideal $\{0\}$, and so the “$N \cap R = M$" condition is also satisfied. Now, if we specialize to the situation where K has characteristic 0 (more generally, where K is a perfect field), classical field theory provides an element $\gamma \in L$ such that $L = K(\gamma)$, and so $K \subset L$ (that is, $R \subset T$) satisfies FIP. Perhaps, the simplest example of this is provided by taking $K \subset F \subset L$ to be $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\sqrt{2}, \sqrt{3})$. However, one can take the chain of fields $K \subset F \subset L$ to be such that $K \subset L$ (that is, $R \subset T$) does not satisfy FIP. Perhaps the best known classical example of this occurs by letting X and Y be (commuting algebraically independent) indeterminates over the finite field $\mathbb{F}_p$ and then taking the chain $K \subset F \subset L$ to be $\mathbb{F}_p(X^p, Y^p) \subset \mathbb{F}_p(X^p, Y) \subset \mathbb{F}_p(X, Y)$.

Each explicit example in the preceding paragraph was such that the (residue field of the) field R was infinite. However, the statement of Proposition 3.1 (b) allows R/M to be finite, in which case (given the other ambient assumptions), $R \subset T$ satisfies FIP. To illustrate this possibility, it suffices to take the above chain $K \subset F \subset L$ of minimal field extensions to consist of finite fields (that is, take K to be a finite field). According to the classical Galois Theory of finite fields, this occurs precisely when there exist prime numbers p, p_1, p_2 and a positive integer n such that $R = K = \mathbb{F}_{p^n}$, $S = F = \mathbb{F}_{p^{np_1}}$, and $T = L = \mathbb{F}_{p^{np_1p_2}}$. The classical theory also gives the following facts (which suggest a theme that will be pursued in Section 4). If $p_1 = p_2$, then $|[R, T]| = 3$, but if $p_1 \neq p_2$, then $|[R, T]| = 4$. In particular, taking the chain $K \subset F \subset L$ to be $\mathbb{F}_2 \subset \mathbb{F}_4 \subset \mathbb{F}_{16}$ leads to $|[R, T]| = 3$.

Ring-theorists should not dismiss the preceding paragraph (or the one before it) as field-theoretic ephemera. To see this, let k be a field that is the "top part" of a chain of minimal field extensions $F_1 \subset F_2 \subset k$. Then let $1 \leq d \leq \infty$ and take (V, M) to be a d-dimensional valuation domain of the form $V = k + M$. Then $A := F_1 + M \subset B := F_2 + M \subset C := k + M$ is a chain of inert (integral minimal ring) extensions, each of which has crucial maximal ideal M, with A, B, C each being d-dimensional domains. Moreover, $|[A, C]| = |[F_1, k]|$ since the assignment $E \mapsto E + M$ gives a bijection $[A, C] \to [F_1, k]$. (These assertions all follow easily from the well known description of the overrings of a classical $D+M$ construction [3, Theorem 3.1].) Thus, by taking $F_1 \subset F_2 \subset k$ to be a suitable chain of finite fields as in the preceding paragraph, we get d-dimensional domains forming a chain $A \subset B \subset C$ of inert extensions where $|[A, C]|$ is either 3 or 4 (and each of these values can be arranged).

(c) Next, we will illustrate Proposition 3.3 (c) by giving an example of a decomposed extension $R \subset S$ and an inert extension $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ and $R \subset T$ satisfies FIP. To that end, take $R := K$ to be a field that has a minimal field extension L. Put $S := K \times K$ and $T := K \times L$. Of course, $R \subset S$ is a decomposed extension with crucial maximal ideal $M := \{0\}$. To see that $S \subset T$ is an inert extension with crucial maximal ideal $N := (S : T) = K \times \{0\}$, observe that $N \in \text{Max}(T)$ and the extension $S/N \subset T/N$ can be identified with $\{0\} \times K \subset \{0\} \times L$, that is, with $K \subset L$. Of course, $N \cap R = M$. It follows from Proposition 3.3 (c) that $R \subset T$ satisfies FIP. One can also show this directly, by applying [7, Theorem III.5], as follows: if K is finite (resp., infinite), use its condition (1), bearing in mind that $[L : K] < \infty$ (resp., condition (4), bearing in mind that $K = K[0]$).

(d) We will illustrate Proposition 3.3 (d) by giving an example of decomposed extensions $R \subset S$ and $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ and $R \subset T$ satisfies FIP. Let K be a field, and put $R := K$, $S := K \times K$, and $T := K \times K \times K$, with $S \hookrightarrow T$ via $(a, b) \mapsto (a, a, b)$. By arguing as in the proof of (c), we are left only with showing that $S \subset T$ is a decomposed extension having crucial maximal ideal $N := (S : T) = \{0\} \times K \in \text{Spec}(S)$ (which is identified with $\{0\} \times \{0\} \times K \subset T$). In fact, $S/N \subseteq T/N$ can be identified with $K \times \{0\} \subset K \times K \times \{0\}$, that is with $K \subset K \times K$, which, being a minimal ring extension, certainly satisfies FIP.

(e) We will illustrate Proposition 3.3 (e) by giving an example of an inert extension $R \subset S$ and a decomposed extension $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ and $R \subset T$ satisfies FIP. First, recall the following fact which was proved in Proposition 3.3 (e). If K is an infinite field and $K \subseteq L$ is a field extension that satisfies FIP, then $K \hookrightarrow L \times L$ (given by $a \mapsto (a, a)$) also satisfies FIP. Now, take $R := K$ to be an infinite field that has a minimal field extension $S := L$. Put $T := L \times L$. Then $R \subset S$ is inert, $S \subset T$ is decomposed, the crucial maximal ideals trivially satisfy $N \cap R = M$, and $R \subset T$ satisfies FIP by virtue of the above-noted fact. (As an interesting sideline, we also note via Proposition 3.3 (e) that there exists $\gamma \in L \times L$ such that $K[\gamma] = L \times L$.)

(f) Next, we will illustrate Proposition 3.3 (g) by giving an example of a ramified extension $R \subset S$ and a decomposed extension $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ (and, necessarily, $R \subset T$ satisfies FIP). Let K be a finite field, and put $R := K$, $S := K[X]/(X^2)$, and $T := K[X]/(X^2) \times K$, with $S \hookrightarrow T$ via $(a + bx) \mapsto (a + bx, a)$, where $a, b \in K$ and $x := X + (X^2)$ satisfies $x \neq 0 = x^2$. Of course, $R \subset S$ is ramified, with crucial maximal ideal $M := \{0\}$. As $N := (S : T) = Sx = Kx$ satisfies $N \cap R = M$, it remains only to observe that $S/N \subset T/N$ can be identified with $K \subset K \times K$. While Proposition 3.3 (g) ensures that $R \subset T$ satisfies FIP, this conclusion also follows directly via condition (1) in [7, Theorem III.5].

(g) We will illustrate Proposition 3.3 (h) by giving an example of a decomposed extension $R \subset S$ and a ramified extension $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ (and, necessarily, $R \subset T$ satisfies FIP). Let K be a finite field, and put $R := K$, $S := K \times K$, and $T := K[X]/(X^2) \times K$. Of course, $R \subset S$ is decomposed, with crucial maximal ideal $M := \{0\}$. As $N := (S : T) = \{0\} \times K$, it remains only to observe that $S/N \subset T/N$ can be identified with $K \subset K[X]/(X^2)$. While Proposition 3.3 (g) ensures that $R \subset T$ satisfies FIP, this conclusion also follows directly via condition (1) in [7, Theorem III.5].

(h) Finally, we will illustrate Proposition 3.3 (i) by giving an example of a ramified extension $R \subset S$ and an inert extension $S \subset T$, with crucial maximal ideals M and N respectively, such that $N \cap R = M$ (and, necessarily, $R \subset T$ satisfies FIP). Let K be a finite field. Then there exists a field L such that $K \subset L$ and $[L : K] = 2$. Put $T := L[X]/(X^2) = L \oplus Lx$, where $x := X + (X^2) \in T$ with $x \neq 0 = x^2$. Then define the rings $R := K \oplus Kx$ and $S := K \oplus Lx$. Observe that $R \subset S \subset T$, with $M := (R : S) = Kx$ and $N := (S : T) = Lx$. In particular, $N \cap R = M$. It is easy to see that $S \subset T$ is inert having crucial maximal ideal N, since $N \in \text{Max}(S) \cap \text{Max}(T)$ and $S/N \subset T/N$ can be identified with the (minimal) field extension $K \subset L$. It remains only to prove that $R \subset S$ is ramified. We will do this somewhat indirectly, by proving that $R \subset S$ is an integral minimal ring extension which is neither decomposed nor inert (and, hence, by the process of elimination, must be ramified).

To prove that $R \subset S$ is a minimal ring extension, we will show that if A is any ring such that $R \subseteq A \subset S$, then $A = R$. Consider the set $V := \{u \in L \mid ux \in A\}$. It is easy to see that $K \subseteq V \subseteq L$, that V is a K-vector subspace of L, and that $A = K + Vx$. Note that $V \subset L$ (for, otherwise, $V = L$, whence $A = K + Lx = S$, a contradiction). As $\dim_K(L) = [L : K] = 2$ and $V \neq L$, it must be that $V = K$. Then $A = K + Kx = R$, as desired. This proves that $R \subset S$ is a minimal ring extension. Moreover, this extension is integral, since S is generated as an R-algebra by the (nilpotent, hence integral) elements in Lx.

By the lying-over property of integral extensions (cf. [18, Theorem 44]), S has only only one maximal ideal, since the quasi-local ring T is integral over S. Therefore, $R \subset S$ cannot be a decomposed extension. It is also not an inert extension, since $M \notin \text{Max}(S)$, the point being that $M = Kx$ is properly contained in the proper ideal Lx of S. This completes the proof that $R \subset T$ is ramified and that the data have all the asserted properties. Incidentally, we observe that, while Proposition 3.3 (g) ensures that $R \subset T$ satisfies FIP, this conclusion also follows directly via condition (1) in [7, Theorem III.5], as $T = L + Lx$ is finite(-dimensional as a vector space over K).

The two remaining cases for (type of $R \subset S$, type of $S \subset T$) are (inert, ramified) and (ramified, ramified), each under the assumption that $N \cap R = M$. Remarkably, both of these cases can be illustrated by examples where $R \subset T$ satisfies FIP and by other examples where $R \subset T$ fails to satisfy FIP. Those examples satisfying FIP (resp., non-FIP) are collected below in Remark 3.6 (resp., Example 3.7). First, in keeping with a promise made in the Introduction, Proposition 3.5 gives necessary and sufficient conditions for each of the contexts (that is, (inert, ramified) and (ramified, ramified)) to allow $R \subset T$ to satisfy FIP. Given the nature of the examples in Remark 3.6 and Example 3.7 (especially the complicated construction of the first example in Example 3.7 (b)), it may not be surprising that the formulations in Proposition 3.5 are somewhat cumbersome, lacking the significantly more succinct nature of the statements for the other 7 cases in Proposition 3.3.

Proposition 3.5. (a) Let $R \subset S \subset T$ be rings such that $R \subset S$ is an inert (integral minimal ring) extension with crucial maximal ideal M, $S \subset T$ is a ramified (integral minimal ring) extension with crucial maximal ideal N, and $N \cap R = M$. Define $J := \text{Rad}(T_M)$, $A := R_M + J$, and for each $i > 0$, $M_i := M_M + AM^i$, $R_i := R_M + AM^i$, $M'_i := M_i/M_M$ and $R'_i := R_i/M_M$. Then $A = {}^+_{T_M}(R_M)$. Furthermore, $R \subseteq T$ has FIP if and only if (either R/M is finite or) when R/M is infinite, the following two properties hold: there exists $\gamma \in T_M$ such that $T_M = A_M[\gamma]$; and there exists $\alpha \in A$ such that $A = R_1[\alpha]$ and $\alpha^3 \in AM$, and, with $A' := R_1[\alpha^2]$ and $A'' := R_M + A'M$, there exists $\beta \in A$ such that $A' = A''[\beta]$ and $\beta^3 \in A'M$.

(b) Let $R \subset S \subset T$ be rings such that $R \subset S$ and $S \subset T$ are each ramified (integral minimal ring) extensions, whose respective crucial maximal ideals M and N satisfy $N \cap R = M$. Define $J := \text{Rad}(T_M)$, $A := R_M + J$, $C := (R_M : A)$, and for each $i > 0$, $M_i := M_M + AM^i$, $R_i := R_M + AM^i$, $M'_i := M_i/C$ and $R'_i := R_i/C$. Then $A = {}^+_{T_M}(R_M)$. Moreover, either $M_M = (R_M : A)$ or there exists an integer $n > 1$ such that $M_M^n \subseteq (R_M : A)$ with $M_M^{n-1} \not\subseteq (R_M : A)$. Furthermore, $R \subseteq T$ has FIP if and only if (either R/M is finite or) when R/M is infinite, the following three properties hold: there exists $\gamma \in T_M$ such that $T_M = A_M[\gamma]$; either $M_M = (R_M : A)$ or $\mathrm{L}_{R_M}(M_i/M_{i+1}) = 1$ for all $1 \le i \le n-1$; and there exists $\alpha \in A$ such that $A = R_1[\alpha]$ and $\alpha^3 \in AM$, and, with $A' := R_1[\alpha^2]$ and $A'' := R_M + A'M$, there exists $\beta \in T$ such that $A' = A''[\beta]$ and $\beta^3 \in A'M$.

Proof. By parts (a) and (b) of Proposition 3.1, we may assume that (R, M) is quasi-local. The assertions are direct applications of Lemma 3.2, whose notation we use freely here, once we prove in part (a) (resp., in part (b)) that $(R : A) = M$ (resp., that $A = T$).

(a) Since (R, M) is quasi-local and $R \subset S$ is inert, the usual argument involving INC shows that (S, M) is quasi-local. In particular, $N = M$. As $S \subset T$ is ramified, T is quasi-local and its unique maximal ideal M' satisfies $(M')^2 \subseteq M \subset M'$. The Jacobson radical of T is then $J = M'$, and so $A = R + J = R + M'$. It follows that $A \neq R$, since $M \subset M'$ (in conjunction with $M' \cap R = M$) ensures that $M' \not\subseteq R$. Also, since $A \neq R$, we see that $(R : A) = M$ (since $MM' \subseteq (M')^2 \subseteq M \subset R$). Then the assertion follows from Lemma 3.2, noticing that condition (ii) has been verified.

(b) Since both $R \subset S$ and $S \subset T$ are subintegral, so is $R \subset T$. It follows that $A = {}^+_T R = T$. Then the assertion follows from Lemma 3.2. □

We next collect the two remaining examples that satisfy FIP.

Remark 3.6. (a) It is easy to construct an example of rings $R \subset S \subset T$ such that $R \subset S$ is inert, $S \subset T$ is ramified (resp., decomposed; resp., inert), the crucial maximal ideals satisfy $N \cap M = R$, and $R \subset T$ satisfies FIP. Indeed, let $K \subset L$ be a minimal (field) extension of finite fields, and take $R := K$, $S := L$, and $T := L[X]/(X^2)$ (resp., $L \times L$; resp., a minimal field extension of the finite field L). The first several assertions are clear, and the "FIP" assertion follows directly from condition (1) in [7, Theorem III.5] since $\dim_K(T) < \infty$.

(b) It is harder to construct a chain of ramified (integral minimal ring) extensions $R \subset S \subset T$ such that the crucial maximal ideals satisfy $N \cap M = R$ and $R \subset T$ satisfies FIP, but we do so next. Let K be a field. Put $R := K$ and $S := K[X]/(X^2) = K[x] = K + Kx$, where $x := X + (X^2)$ satisfies $x \neq 0 = x^2$. Of course, $R \subset S$ is ramified, with crucial maximal ideal $M := \{0\}$. Let Y be an indeterminate over S, and put $T := S[Y]/(Y^3, Y^2 - x)$. Observe that $T = K[y]$, where $y := Y + (Y^3, Y^2 - x)$ satisfies $y^3 = 0$ and $y^2 = x$. It is important to verify that $S \hookrightarrow T$; that is, the canonical R-algebra homomorphism $S \to T$ is an injection. This comes down to showing that if $a, b \in K$ with $a+bx \in (Y^3, Y^2-x)S[Y]$, then $a = 0 = b$. Applying the S-algebra homomorphism that sends Y to 0, we get that $a + bx \in xS = Kx$, whence $a = 0$. Then $bx \in (Y^3, Y^2 - x)S[Y]$ leads easily to $b = 0$, as desired, and so we can view $S \subseteq T$. Similar reasoning allows us to conclude that $Y \notin K + (Y^3, Y^2 - x)S[Y]$ and $Y^2 \notin K + KY + (Y^3, Y^2 - x)S[Y]$. Thus, as a K-vector space, $T = K \oplus Ky \oplus Ky^2 = K \oplus Kx \oplus Ky = S \oplus Ky$; and the multiplication among generators satisfies $x^2 = 0$, $y^2 = x$ and $y^3 = yx = 0$. It is now easy to check that $N := (S : T) = y^2S = y^2K = xK$. Clearly, $N \cap R = M$. To show that $S \subset T$ is ramified with crucial maximal ideal N, it is enough to prove that $(K \cong S/N \subset)\ T/N \cong K[W]/(W^2)$ for some indeterminate W. In fact, this holds since $T/N = K[y]/y^2K = K \oplus Kz$, where $z := y + y^2K$ satisfies $0 \neq z = z^2$. It remains only to prove that $R \subset T$ satisfies FIP. If K is finite (resp., infinite) this follows from condition (1) (resp., condition (3)) in [7, Theorem III.5].

We come now to somewhat more intricately constructed examples. They show that FIP can fail to be satisfied in each of the two contexts that (for the moment) remain open before us.

Example 3.7. (a) There exists a chain of rings $R \subset S \subset T$ such that $R \subset S$ is an inert (integral minimal ring) extension and $S \subset T$ is a ramified (integral minimal ring) extension, whose respective crucial maximal ideals M and N satisfy $N \cap R = M$, and $R \subset T$ does not satisfy FIP.

(b) There exists a chain of rings $R \subset S \subset T$ such that $R \subset S$ and $S \subset T$ are each ramified (integral minimal ring) extensions, whose respective crucial maximal ideals M and N satisfy $N \cap R = M$, and $R \subset T$ does not satisfy FIP.

Proof. (a) Let K be an infinite field and $K \subset L$ a minimal field extension. Take $R := K$, $S := L$, and $T := L[X]/(X^2)$. Of course, $R \subset S$ is inert, with crucial maximal ideal $M := \{0\}$; and $S \subset T$ is ramified, whose crucial maximal ideal $N := \{0\}$ trivially satisfies $N \cap R = M$. By inspecting the conditions (especially condition (3)) in [7, Theorem III.5] (and bearing in mind that T is quasi-local), we see that the conclusion that $R \subset T$ fails to satisfy FIP is equivalent to the assertion that there does not exist an element $\alpha \in T$ such that $T = K[\alpha]$ and $\alpha^3 = 0$. Suppose, on the contrary, that such α exists. We can write $\alpha = a + bx$ for some $a, b \in K$. As $\alpha^3 = a^3 + 3a^2bx$, the condition that $\alpha^3 = 0$ is equivalent to $a = 0$. Then $T = K[bx] \supset K$ implies that $b \neq 0$, and so $S = K[x] = K[bx] = T$, the desired contradiction.

(b) We offer two different constructions that each prove (b). The first of these slightly modifies the construction from Remark 3.6 (b). Let K be an infinite field. Put $R := K$ and $S := K[X]/(X^2) = K[x] = K + Kx$, where $x := X + (X^2)$ satisfies $x \neq 0 = x^2$. Of course, $R \subset S$ is ramified, with crucial maximal ideal $M := \{0\}$. Let Y be an indeterminate over S, and (this is where the construction deviates from the earlier one) put $T := S[Y]/(Y^2, xY)$. It is important to notice that $S \hookrightarrow T$ (that is, the canonical R-algebra homomorphism $S \to T$ is an injection): this holds since $a, b \in K$ with $a + bx \in (Y^2, xY)$ easily implies that $0 = a = b$. Observe that $y := Y + (Y^2, xY)$ satisfies $0 = y^2 = xy$. Additively, we have that $T = S \oplus Sy = K + Kx + Ky$. We claim that this last sum is a direct sum: as a K-vector space, $T = K \oplus Kx \oplus Ky$. To prove this claim, it suffices to verify that $Y \notin K + Kx + (Y^2, xY)$, and this can be done as in the proof of Example 3.6 (b) by using suitable substitutions (in this case, $Y \mapsto 0$) and analyzing appropriate coefficients (in this case, the coefficient of Y, noticing also that $1 \notin xS$). It follows that $\dim_K(T) = 3$. Notice that the multiplication among generators satisfies $0 = x^2 = y^2 = xy$. Next, consider $N := (S : T) = Kx$. Then $S/N \subset T/N$ can be identified with $K \subset (K + Kx + Ky)/(Kx) \cong K + Kz$, where $z := y + Kx$ satisfies $z \neq 0 = z^2$. Consequently, $S \subset T$ is a ramified extension with crucial maximal ideal N; and of course, $N \cap R = M$. It remains to prove that $R \subset T$ does not satisfy FIP.

Although T need not be quasi-local, we claim that T cannot take the form $K[\alpha] \times K_2 \times \cdots \times K_n$ for some element α such that $\alpha^3 = 0$, some integer $n \geq 2$, and some fields K_j such that $K \subseteq K_j$ satisfies FIP for each j. Indeed, if one had such a description, then T would be semi-quasi-local, whereas the infinitely many maximal ideals of $S[Y]$ that contain Y lead to infinitely many maximal ideals of $S[Y]/(Y^2, xY) = T$ (cf. [18, page 25]), the desired contradiction, thus proving the claim. Therefore, if $R \subset T$ satisfies FIP, it follows from [7, Theorem III.5] that $T = K[\gamma]$ for some element $\gamma \in T$ such that $\gamma^3 = 0$. To complete the proof, we need only show that the existence of such γ leads to a contradiction.

Assume that $T = K[\gamma]$. Since $T = K + Kx + Ky$, $\gamma = c + ax + by$ for some $a, b, c \in K$. Then $\delta := \gamma - c = ax + by \in Kx + Ky$ satisfies $K[\delta] = T$. We have that $\delta^2 = a^2x^2 + 2abxy + b^2y^2 = 0$ since $0 = x^2 = xy = y^2$. It follows that $T = K[\delta] = K + K\delta$, whence $\dim_K(T) \leq 2$, the desired contradiction (as we showed above that $\dim_K(T) = 3$).

We next indicate a second construction to prove (b). It depends on some recent work of G. Picavet and M. Picavet-L'Hermitte involving idealizations. Once again, we take $R := K$ to be an infinite field. Let V be a two-dimensional K-vector space and fix a one-dimensional K-subspace W of V. Consider the idealizations $S := R(+)W$ and $T := R(+)V$. Note that [6, Remark 2.9] ensures that $R \subset S$ is a minimal ring extension since W is a simple K-module (that is, a one-dimensional vector space over K); and that $R \subset T$ does not satisfy FIP because W has infinitely many R-submodules (that is, K-subspaces). However, by [20, Lemma 2.1], both $R \subset S$ and $R \subset T$ are subintegral extensions. In particular, $R \subset S$ is ramified. To complete the verification of this example, it remains only to show that $S \subset T$ is ramified. To that end, first note that $S \subset T$ is a minimal ring extension, by [20, Proposition 2.8 (3)], which applies because V/W has only finitely many (in fact, two) K-submodules. Next, note that $S \subset T$ is integral, since T is generated as an S-algebra by the (nilpotent, hence integral) elements in $\{0\}(+)V$. Hence, we need only prove that $S \subset T$ is neither inert nor decomposed. This, in turn, follows from the fact that T is a quasi-local ring whose maximal ideal $\{0\}(+)V$ properly contains $(S : T) = \{0\}(+)W$, which is the unique maximal ideal of S. □

Next, we assess the current state of our program and offer an opinion, buttressed by an example, predicting what may be found by anyone who takes this program "one step further."

Remark 3.8. (a) We have now effectively determined which maximal chains of rings of the form $R \subset S \subset T$ lead to $R \subset T$ satisfying FIP. Because of the following example, anyone who would take this program "one step further" may expect to find relatively more non-FIP behavior. Let X be an analytic indeterminate over a field k. For each positive integer i, the extension $A := k + X^{i+1}k[[X]] \subset B := k + X^i k[[X]]$ is a ramified (integral minimal ring) extension. (Indeed, $I := X^{i+1}k[[X]] = (A : B)$ and the extension $A/I \subset B/I$ can be identified with $k \subset k[Y]/(Y^2)$, where $Y := X^i + I$ is an indeterminate over k.) Consider the tower

$$k + X^4k[[X]] \subset k + X^3k[[X]] \subset k + X^2k[[X]] \subset k + X^1k[[X]] = k[[X]]$$

of ramified extensions. Now, suppose further that the field k is infinite. Then $C := k + X^4k[[X]] \subset D := k[[X]]$ does not satisfy FIP. To see this, consider $J := (C : D) = X^4k[[X]]$. It is enough to show that $\overline{C} := C/J \subset \overline{D} := D/J$ does not satisfy FIP (cf. [7, Proposition II.4]). Note that $\overline{D} = \overline{C}[x]$, where $x := X + J \in D$ is a nilpotent element of nilpotency index 4. As $\overline{C} \cong k$, an application of [1, Lemma 3.6 (a)] therefore shows that $\overline{C} \subset \overline{D}$ does not satisfy FIP, as desired. We believe that the relative ease with which this example has been built stands in contrast to the comparative difficulty of building the first (and possibly even the second) example in Example 3.7 (b).

(b) The example in (a) should also be contrasted with the following example. Assume now that the (not necessarily infinite) field k has characteristic 2; and X

still denotes an analytic indeterminate over k. Consider the tower

$$R := k + X^2k + X^4k[[X]] \subset S := k + X^2k[[X]] \subset T := k[[X]].$$

Reasoning as in (a), one checks easily that $R \subset S$ (resp., $S \subset T$) is a ramified integral minimal ring extension with crucial maximal ideal $M := X^2k + X^4k[[X]]$ (resp., $N := X^2k[[X]]$). We claim that $R \subset T$ satisfies FIP. One way to show this is to apply Lemma 3.2. Note that M is the unique maximal ideal of R and that $(R : T) = X^4k[[X]] = M^2$, while $M \not\subseteq (R : T)$. Also, ${}^+_TR = A = R + N = T$ (as we continue to use the notation from Lemma 3.2 here). Hence, $n = 1$. If k is finite, the claim follows from Lemma 3.2. If k is infinite, the claim follows from Lemma 3.2, in conjunction with the following observations: (i) holds with $\gamma := 0$; (ii) holds vacuously since $n = 1$; since $A_1 = S$ (and $A_2 = R$), we can take $\alpha := X$; and then since $A' = S[X^2] = S$ and $A'' = R + SM = R$, we can satisfy (iii) by taking $\beta := X^3$.

To close this section, we give a result that serves to somewhat balance the thrust of Remark 3.8 (a). Recall that if B is a (commutative) algebra over a ring A, then B is said to be a *separable* A-algebra if B is projective over $B \otimes_A B$ (via $(\sum_{i=1}^n x_i \otimes y_i) \cdot z = \sum_{i=1}^n x_i y_i z$ for all $x_i, y_i, z \in B$). It is well known that if A is a field, then an A-algebra B is a separable A-algebra if and only if B is isomorphic (as an A-algebra) to a direct product of finitely many finite-dimensional separable field extensions of A. Note that Proposition 3.9 (b) generalizes Proposition 3.3 (d) (while giving a new proof for it).

Proposition 3.9. (a) If $A \subset B$ is a decomposed (integral minimal ring) extension, then B is a separable A-algebra.
(b) Let $A = R_0 \subset \ldots \subset R_n = B$ be a finite-length tower of rings such that $R_{i-1} \subset R_i$ is a decomposed extension for all $i = 1, \ldots, n$. Then B is a separable algebra over A. If, in addition, the crucial maximal ideal of $R_{i-1} \subset R_i$ lies over the crucial maximal ideal M of $A \subset R_1$ for all $i = 2, \ldots, n$, then $A \subset B$ satisfies FIP.

Proof. (a) By [10, Proposition 4.6] (or Proposition 3.1 (c)) and the implication (b) $\Rightarrow$ (a) in [5, Theorem 7.1, page 72], we may assume, without loss of generality, that (A, M) is quasi-local. Then B has exactly two maximal ideals, say N_1 and N_2; $N_1 \cap N_2 = M$; and the canonical (A/M)-algebra homomorphisms $A/M \to B/N_i$ are isomorphisms (for $i = 1, 2$). As N_1 and N_2 are comaximal in B, the Chinese Remainder Theorem yields that $B/M \cong B/N_1 \times B/N_2$ ($\cong A/M \times A/M$) as (A/M)-algebras. Thus, by the above remarks, B/M is a separable algebra over A/M. Therefore, by the implication (c) $\Rightarrow$ (a) in [5, Theorem 7.1, page 72], B is a separable algebra over A.

(b) It is known that if $\Lambda \subset \Gamma \subset \Omega$ are rings such that Γ is a separable Λ-algebra and Ω is a separable Γ-algebra, then Ω is a separable Λ-algebra [5, Proposition 1.12, page 46]. In view of (a), it now follows by an easy induction that B is a separable algebra over A. Moreover, B is a finite-type algebra and integral (hence module-finite) over A (since each R_i is of finite-type and integral over R_{i-1}). Henceforth, we assume the additional condition that all the crucial maximal ideals lie over M. Then, reasoning as in the proof of Proposition 3.3 (d) (and using an easy induction), we see that the Jacobson radical of B is M. In particular, M is a common ideal of A and B. If A/M is finite, it follows, via condition (1) in [7, Theorem III.5], that $A/M \subset B/M$ satisfies FIP (since B/M is a finite-dimensional vector space over A/M), and so $A \subset B$ satisfies FIP by [7, Proposition II.4]. Thus, without loss of generality, A/M is infinite.

Note that B_M is a finite-type separable algebra over A_M [5, Corollary 1.7, page 44]. Hence, since A/M is infinite, it follows from [17, Lemma 3.1] there exists $\xi \in B_M$ such that $B_M = A_M[\xi]$. As we have seen that M is the Jacobson radical of B, the fact that B has only finitely many maximal ideals leads to the Jacobson radical of B_M being MA_M (since the formation of rings of fractions commutes with finite intersections). Therefore, by the first assertion in [8, Theorem 5.18], ${}^{+}_{B_M}(A_M) = A_M$. Hence, in view of the existence of ξ, it now follows from [8, Theorem 5.18] that $A_M \subset B_M$ satisfies FIP. (To apply the cited result, one also needs to note that $A \subset B$ satisfies FCP; that, in turn, is an easy consequence of [8, Corollary 4.3].) Next, note via [8, Corollary 3.2] and the assumption concerning crucial maximal ideals, that $\mathrm{MSupp}(B/A) = \{M\}$. Consequently, by [8, Proposition 3.7 (a)], $A \subset B$ satisfies FIP ($\Leftrightarrow$ $A_M \subset B_M$ satisfies FIP). □

4 Summary and an application

Theorem 4.1 summarizes the earlier material in this paper by essentially listing all the possible kinds of pairs of minimal ring extensions $R \subset S$ and $S \subset T$ that can be "composed" so that $R \subset T$ has FIP (that is, so that $|[R,T]| < \infty$).

Theorem 4.1. Let $R \subset S$ and $S \subset T$ be minimal ring extensions, with crucial maximal ideals M and N, respectively. Then $R \subset T$ satisfies FIP if and only if (exactly) one of the following conditions holds:

(i) Both $R \subset S$ and $S \subset T$ are integrally closed.

(ii) $R \subset S$ is integral and $S \subset T$ is integrally closed.

(iii) $R \subset S$ is integrally closed, $S \subset T$ is integral, and $N \cap R \not\subseteq M$.

(iv) Both $R \subset S$ and $S \subset T$ are integral and $N \cap R \neq M$.

(v) Both $R \subset S$ and $S \subset T$ are inert, $N \cap R = M$, and either R/M is finite or there exists $\gamma \in T_M$ such that $T_M = R_M[\gamma]$.

(vi) $R \subset S$ is decomposed, $S \subset T$ is inert and $N \cap R = M$.

(vii) Both $R \subset S$ and $S \subset T$ are decomposed and $N \cap R = M$.

(viii) $R \subset S$ is inert, $S \subset T$ is decomposed, and $N \cap R = M$.

(ix) $R \subset S$ is ramified, $S \subset T$ is decomposed, and $N \cap R = M$.

(x) $R \subset S$ is decomposed, $S \subset T$ is ramified, and $N \cap R = M$.

(xi) $R \subset S$ is ramified, $S \subset T$ is inert, and $N \cap R = M$.

(xii) $R \subset S$ is inert, $S \subset T$ is ramified, $N \cap R = M$, and the two conditions stated in Proposition 3.5 (a) hold.

(xiii) Both $R \subset S$ and $S \subset T$ are ramified, $N \cap R = M$, and the two conditions stated in Proposition 3.5 (b) hold.

Proof. Combine the appropriate parts of Theorem 2.1, Proposition 3.1, Proposition 3.3 and Proposition 3.5. □

While Theorem 4.1 has given a complete answer to the question at hand, we wish to stress that various FIP- (and FCP-) theoretic studies remain to be pursued in other contexts. Among these, we mention here only the context of ϕ-rings, for which [2] initiated such studies in some important special cases.

The rest of this section refers to the "application" mentioned in its title. For motivation, note that a minimal ring extension $A \subset B$ can be characterized as a ring extension $A \subseteq B$ such that $|[A,B]| = 2$. Thus, one way to contemplate ring extensions that are, in some sense, only "one step more complex than" a minimal ring extension would be to study (necessarily proper) ring extensions $R \subset T$ such that $|[R,T]| = 3$. Any such $R \subset T$ must accommodate some (uniquely determined) $S \in [R,T]$ such that both $R \subset S$ and $S \subset T$ are minimal ring extensions. Given the above focus on such "composable" pairs of minimal ring extensions, one could combine that earlier work (as summarized in Theorem 4.1) to give a weak, formal characterization of "$|[R,T]| = 3$" by appending some common verbiage, such as "each ring in $[R,T]$ is comparable with S", to each of the conditions listed in Theorem 4.1. Such a compilation would frankly be of little use. We prefer, instead, to collect a few comments about how each of the conditions listed in Theorem 4.1 relates to the "$|[R,T]| = 3$" question. Remark 4.2 emphasizes the contexts studied in Section 2, while Remark 4.3 focuses on certain chains of the (inert, inert) kind.

Remark 4.2. (a) The first paragraph of Remark 2.2 (a) showed one way to find data $R \subset S \subset T$, with R quasi-local, as in Theorem 4.1 (i) such that $|[R,T]| = 3$. The second and third paragraphs of Remark 2.2 (a) developed enough material concerning normal pairs to show that all examples of integrally closed extensions $R \subset S \subset T$ that illustrate Theorem 4.1 (i) such that $R \subset T$ satisfies FIP, with R quasi-local, can be obtained via pullback from the construction given in the first paragraph of Remark 2.2 (a). To make an example that is constructed in this way satisfy $|[R,T]| = 3$, it is necessary and sufficient for the pullback to feature a divided prime ideal Q of R such that $D := R/Q$ is a two-dimensional valuation domain with (quotient field) $T = R/Q$.

However, in general, when both $R \subset S$ and $S \subset T$ are integrally closed minimal ring extensions such that $|[R,T]| = 3$, we do not know (even for domains) whether R must be quasi-local. A partial result in this vein is given next.

Let R be a finite-dimensional semi-quasi-local Prüfer domain such that there exist integrally closed minimal ring extensions $R \subset S$ and $S \subset T$ with $|[R,T]| = 3$. Then we are in the situation described in the first paragraph of Remark 2.2 (a). In other words, R is a (quasi-local) two-dimensional valuation domain and (up to R-algebra isomorphism) S is the localization of R at its height 1 prime ideal and T is the quotient field of R. For a proof, note that the classification of the minimal ring extensions of a domain (cf. [11, Corollary 2.5]) ensures that (up to R-algebra isomorphism) S is an overring of R and, hence, also a finite-dimensional semi-quasi-local Prüfer domain [16, Theorem 26.1 (1), (2)]. Similarly, T is a finite-dimensional semi-quasi-local Prüfer domain. Note that if P and Q are distinct maximal ideals of R, then R_P and R_Q are incomparable. Hence, we see, from the condition that $|[R,T]| = 3$, that R has a unique maximal ideal (that is, R is a valuation domain). Then, since $|[R,T]| = 3$ and each (valuation) overring of R is a localization at some prime ideal of R (cf. [18, Theorem 65]), it follows that R has a unique nonzero non-maximal prime ideal, and the assertion becomes clear.

(b) We next comment on the conditions in Theorem 4.1 (ii). Suppose that $R \subset S$ is an integral minimal ring extension and $S \subset T$ is an integrally closed minimal ring extension. Then $|[R,T]| = 3 \Leftrightarrow$ whenever $u \in T$ is not integral over R, then $R[u] = T$. For a proof, note first that S is the integral closure of R in T. Thus, the only way for $|[R,T]| \neq 3$ would be to have some ring $A \in [R,T]$ fail to be comparable with S because A contains an element u that is not integral over R while S is not contained in A. This is equivalent to requiring the existence of $u \in T$ such that u is not integral over R and $S \not\subseteq R[u]$. Ruling *this* out means that whenever $u \in T$ is not integral over R, one has $S \subset R[u] \subseteq T$ (whence $R[u] = T$ by the minimality of $S \subset T$).

(c) We next show that the data in Remark 2.2 (b) satisfy the criterion established in (b) for the context of Theorem 4.1 (ii). As before, X denotes an analytic indeterminate over $\mathbb{Q}(\sqrt{2})$, and we take $R := \mathbb{Q} + X\mathbb{Q}(\sqrt{2})[[X]]$, $S := \mathbb{Q}(\sqrt{2})[[X]]$ and $T := \mathbb{Q}(\sqrt{2})((X))$. Our task is to show that if $u \in T$ is not integral over R, then $R[u] = T$. As S is the integral closure of R in T, we can write $u = f/X^n$, for some $f \in S$ and minimal positive integer n. As $u \notin S$ and X does not divide f in S, we see that f is a unit of S. Accordingly, by the theory of G-domains [18, Theorem 19], $S[u] = T$. Let c denote the (nonzero) constant term of f. If $c \in \mathbb{Q}$, then f is a unit of R and a direct application of [18, Theorem 19] gives $R[u] = T$. Thus, we may suppose that $c \in \mathbb{Q}(\sqrt{2}) \setminus \mathbb{Q}$ and need only prove that $S \subseteq R[u]$. In fact, $R[u]$ contains $\mathbb{Q}(c) + X\mathbb{Q}(\sqrt{2})[[X]] = \mathbb{Q}(\sqrt{2}) + X\mathbb{Q}(\sqrt{2})[[X]] = S$, as desired. Hence, by (a), $|[R,T]| = 3$.

(d) Next, we show that if $R \subset S$ is an integral minimal ring extension and $S \subset T$ is an integrally closed minimal ring extension, then it need not be the case that $|[R,T]| = 3$. For example, consider $R := \mathbb{Z}[2i]$; $S := R' = \mathbb{Z}[i]$, the ring of Gaussian integers; and $T := \cap_{P \neq Q} S_P$, where the index set for this intersection consists of all the prime ideals P of S other than $Q := 3S$. In fact, by classical algebraic number theory [21, Theorems 6-2-1 and 6-1-1], 3 is inert in the Gaussian numbers, and so (since S is a Dedekind domain) $Q \in \mathrm{Spec}(S)$

and Q is the only prime ideal of S that lies over $3\mathbb{Z}$. Consequently, since every overring of S is an intersection of localizations (at prime ideals) of S (by [16, Theorem 26.1 (2)]), it follows that there is no ring properly contained between S and T. On the other hand, $1/3 \in T \setminus S$, and so $S \subset T$ is a minimal ring extension; moreover, this extension is integrally closed (since S is integrally closed and T is an overring of S). Also, it is easy to check that $R \subset S$ is an integral minimal ring extension. It remains only to produce a ring in $[R,T] \setminus \{R,S,T\}$. Consider $A := R[1/3] \in [R,T]$. Observe that $1/3 \in A \setminus (R \cup S)$; and $i \in T \setminus A$. This completes the verification of the example.

(e) The conditions in Theorem 4.1 (iii) do not permit $|[R,T]|$ to be 3. To see why this is so, suppose that the minimal ring extension $R \subset S$ is integrally closed, the minimal ring extension $S \subset T$ is integral, and that their crucial maximal ideals satisfy $N \cap R \not\subseteq M$. Then the proof of the Crosswise exchange Lemma [8, Lemma 2.7] produces a ring in $[R,T] \setminus \{R,S,T\}$, and so $|[R,T]| \neq 3$.

(f) By Proposition 3.1 (d) (or arguing as in (e) via the Crosswise exchange Lemma), we see that the conditions in Theorem 4.1 (iv) lead to $|[R,T]| \geq 4$.

(g) As classical field extensions suffice to show the diversity of FIP-related behavior in the (inert, inert) context, we pass now to the (decomposed, inert) conditions in Theorem 4.1 (vi). We will show that those conditions admit an example where $|[R,T]| = 3$. To do so, we use the data in Remark 3.4 (c), specializing to the fields $K := \mathbb{R}$ and $L := \mathbb{C}$, so that $R = \mathbb{R}$, $S = \mathbb{R} \times \mathbb{R}$ and $T = \mathbb{R} \times \mathbb{C}$. To show that $|[R,T]| = 3$ (that is, $|[\mathbb{R}, \mathbb{R} \times \mathbb{C}]| = 3$), it suffices to prove that if B is any $\mathbb{R}$-subalgebra of $\mathbb{R} \times \mathbb{C}$ such that $\dim_{\mathbb{R}}(B) = 2$, then $B = \mathbb{R} \times \mathbb{R}$.

Suppose not. Then there are two cases. In the first case, there exists $\xi = (r,s) \in B$ such that $r, s \in \mathbb{R}$ and $r \neq s$. Then $\alpha := (r-s, 0) = \xi - (s,s) \in B \setminus R$. Put $A := R[\alpha]$. By considering vector space dimensions over $\mathbb{R}$, we have $A = B$. Next, we will show that $S \subseteq A$, that is, that $\mathbb{R} \times \mathbb{R} \subseteq A$. To do so, we will show that if $a, b \in \mathbb{R}$, then $(a,b) \in A$, or equivalently, that $(a-b, 0) \in A$. This, in turn, is clear since $\rho := (a-b)(r-s)^{-1} \in \mathbb{R}$ satisfies $(a-b,0) = \rho\alpha \in \mathbb{R}[\alpha] = A$, as required. Thus, by considering vector space dimensions, $A = S$. It follows that $B = S$, the desired contradiction.

In the remaining case, there exists $\eta = (u,v) \in B$ such that ($u \in \mathbb{R}$ and) $v \in \mathbb{C} \setminus \mathbb{R}$. Write $v = x + yi$ with $x, y \in \mathbb{R}$ (and $y \neq 0$). We have $\eta - (u,u) = (0, (x-u) + yi) \in B$. Thus, without loss of generality, $u = 0$ and (by *abus de langage*) $\eta = (0, x + yi) \in B$ (still with $x, y \in \mathbb{R}$ and $y \neq 0$). There are now two subcases. In the first of these, $x \neq 0$. Then $\delta := (-x, yi) \in B$, and so since T has no nonzero nilpotents, $\delta^2 = (x^2, -y^2)$ is a nonzero element of B. Equivalently, $0 \neq (x^2 + y^2, 0)$ is a nonzero element of B. It follows that $(1,0) \in B$, whence $B = \mathbb{R}[(1,0)] = \mathbb{R} \oplus \mathbb{R}(1,0)$. Therefore, $B = \{(\lambda + \mu, \lambda) \in \mathbb{R} \times \mathbb{R} \mid \lambda, \mu \in \mathbb{R}\}$; that is, $B = \mathbb{R} \times \mathbb{R}$, the desired contradiction.

In the remaining subcase, $x = 0$, so that $\eta = (0, yi) \in B$ with $0 \neq y \in \mathbb{R}$. Then $(1,0) = (1,1) - (y^{-1}\eta)^2 \in B$, so that by considering vector space dimensions, we have $B = \mathbb{R}[(1,0)]$. As in the previous subcase, this leads to the desired contradiction.

(h) The conditions in Theorem 4.1 (vii) lead to $|[R,T]| \geq 5$. Indeed, by Proposition 3.1 (c) and the first assertion in Proposition 3.1 (d), we may assume, without loss of generality, that (R,M) is quasi-local. Then recall from the proof of Proposition 3.3 (d) that T/M can be identified with $R/M \times R/M \times R/M$. Since $|[R,T]| = |[R/M, T/M]|$ by a standard homomorphism theorem, it is enough to show that $|[R/M, T/M]| \geq 5$. To that end,
let the elements a, b and c run independently through R/M, and consider the five sets $R/M = \{(a,a,a)\}, \{(b,a,a)\}, \{(a,b,a)\}, \{(a,a,b)\}$ and $\{(a,b,c)\} = T/M$.

(i) The conditions in Theorem 4.1 (viii) do not permit $|[R,T]|$ to be 3 (or 4). Indeed, as in the proof of (h), we may assume, without loss of generality, that (R,M) is quasi-local. Then recall from the proof of Proposition 3.3 (e) that

$T/M \cong S/M \times S/M$. Viewing this isomorphism as an identification, we have

$$[R/M, T/M] \supseteq \{R/M, S/M, T/M, R/M \times S/M, S/M \times R/M\}$$

and so $|[R,T]| = |[R/M, T/M]| \geq 5$.

(j) The conditions in Theorem 4.1 (ix) admit an example where $|[R,T]| > 3$. To show this, we use the data in Remark 3.4 (f), with the minimal field extension $K \subset L$ taken to be $\{0,1\} = \mathbb{F}_2 \subset \mathbb{F}_4$. Then $R = \mathbb{F}_2$, $S = \mathbb{F}_2[X]/(X^2)$ and $T = \mathbb{F}_2[X]/(X^2) \times \mathbb{F}_2$ (with $S \hookrightarrow T$ via $a + bx \mapsto (a + bx, a)$ as before). It is straightforward to verify that $B := \{(0,0), (1,1), (1,0), (0,1)\}$ is an R-subalgebra of T which is not in $\{R, S, T\}$.

(k) As we saw in the proof of Proposition 3.1 (h), it follows from [8, Lemma 2.8] that the hypotheses of Theorem 4.1 (x) ensure that $|[R,T]| \geq 4$. For instance, by comparing parts (f) and (g) of Remark 3.4, we see that the data in Remark 3.4 (g) satisfy $K[X]/(X^2) \in [R,T] \setminus \{R, S, T\}$.

(l) The conditions in Theorem 4.1 (xi) do permit $|[R,T]|$ to be 3. In fact, we prove that, with $K := \mathbb{F}_2$, the data in Remark 3.4 (h) satisfy $|[R,T]| = 3$. As the cardinalities of R, S and T are $4, 8$ and 16, respectively, it follows from elementary group theory that we need only show that S is the only member S^* of $[R,T]$ that has cardinality 8. Without loss of generality, $S^* \not\subseteq S$. Then, with $x := X + (X^2)$ as usual, pick $\xi := \lambda + \mu x \in S^*$ for some $\lambda \in L \setminus K$, $\mu \in L$. Then by adding ξ to the elements of R, we get

$$S^* = \{0, 1, x, 1 + x, \lambda + \mu x, 1 + \lambda + \mu x, \lambda + (1 + \mu)x, 1 + \lambda + (1 + \mu)x\}.$$

As $\lambda^2 = \xi^2 \in S^*$ (and $\lambda^2 \neq 0, 1, \lambda$), we see by the process of elimination that $\lambda^2 = \lambda + 1$, with $\mu \in K$. Replacing ξ with $\xi - \mu x$, we may suppose that $\mu = 0$. It is evident that the displayed set of 8 elements is not a ring since it is not closed under multiplication (for instance, because it does not include the product of x and λ), the desired contradiction, which completes the proof.

(m) If the conditions in Theorem 4.1 (xiii) hold (so that, in particular, $R \subset T$ satisfies FIP) and the field R/M is infinite, then $|[R,T]| = 1 + \ell[R,T]$, where $\ell[R,T]$ denotes the maximal length of a chain composed of members of $[R,T]$. In other words, if $R \subset S$ and $S \subset T$ are ramified extensions whose crucial maximal ideals satisfy $N \cap R = M$ and $R \subset T$ satisfies FIP, then $|[R,T]|$ is the cardinality of a maximal chain composed of members of $[R,T]$. (For an example illustrating this assertion, one can use the data in Remark 3.6 (b), taking the field K to be infinite.) For a proof, use parts (a), (c) and (d) of Proposition 3.1, in conjunction with [9, Lemma 4.5], to reduce to the case where (R, M) is quasi-local; then, since $R \subset T$ is a (composition of) subintegral extension(s), an application of [9, Proposition 4.13] completes the proof. It follows that under the stated conditions, if $R \subset S \subset T$ is one of the chains in $[R,T]$ having the greatest possible length, then it is the only such chain and $|[R,T]| = 3$.

On the other hand, it is easy to give an example where the conditions in Theorem 4.1 (xiii) hold, the field R/M is finite and $|[R,T]| = 3$. To do so, one can once again use the data in Remark 3.6 (b), but this time taking the field $K := \mathbb{F}_2$. Indeed, by the proof of Remark 3.6 (b), we have that $R = K$, $S = K[x] = K[y^2]$ and $T = K \oplus Ky \oplus Ky^2$, with $y^3 = 0$. Thus, it suffices to show that $A := K[y + y^2] \in \{R, S, T\}$. A moment's thought shows that $y^2 = (y + y^2)^2 \in A$, whence $A = T$, to complete the proof.

Recall that the Introduction mentioned an example of a ring extension $R \subset T$ from [15] such that $|[R,T]| = 3$. The focus of [15] was on λ-*extensions*, that is, ring extensions $A \subseteq B$ such that $[A, B]$ is linearly ordered by inclusion. The final remark mentions three other results from [15] that are related to the "$|[R,T]| = 3$" question.

Remark 4.3. (a) If $K \subseteq L$ is a field extension, we say that K is *purely inseparably closed in* L if no element of $L \setminus K$ is purely inseparable over K. According to [15, Proposition 3.17 (2)], if $K \subseteq L$ is a λ-extension of fields, then L is algebraic over K and either K is purely inseparably closed in L or L is purely inseparable over K.

One family of field extensions $K \subseteq L$ such that K is purely inseparably closed in L is provided by the (algebraic) Galois field extensions. One consequence of [15, Theorem 3.36] is that if $K \subseteq L$ is a finite-dimensional Galois field extension, then $|[K, L]| = 3 \Rightarrow$ the Galois group of L/K is cyclic of order p^n for some prime number p and positive integer n. An easy consequence is that a finite-dimensional Galois field extension $K \subseteq L$ satisfies $|[K, L]| = 3 \Leftrightarrow$ the Galois group of L/K is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$ for some prime number p. By the Fundamental Theorem of Galois Theory, it follows that if $K \subseteq L$ is a finite-dimensional Galois field extension such that $|[K, L]| = 3$, then $[L : K] = p^2$ for some prime number p. This generalizes the result in Remark 3.4 (b) that if q is any prime-power and p is any prime number, then $|[\mathbb{F}_q, \mathbb{F}_{qp^2}]| = 3$ (and that this is the only kind of extension of finite fields that has exactly one properly intermediate field).

(b) The "p^2-dimensional" theme continues to the "purely inseparable" context, according to the following consequence of [15, Proposition 3.24]. If $K \subseteq L$ is a purely inseparable field extension of characteristic $p > 0$ such that $|[K, L]| = 3$, then $L = K^{p^{-2}} := \{u \in L \mid u^{p^2} \in K\}$, the element of $[K, L] \setminus \{K, L\}$ is $K^{p^{-1}} := \{v \in L \mid v^p \in K\}$, and $[L : K] = p^2$.

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