Palestine Journal of Mathematics

Vol. 2(2) (2013) , 144–158

On Energy Stability for a Thermal Convection in Viscous Fluids with Memory

Giovambattista Amendola, Mauro Fabrizio and Adele Manes

Communicated by Marcelo Cavalcanti

MSC 2010: 45K05, 35Q79, 80A17

Keywords and phrases: Bénard Problem, fluid with memory, exponential stability

Work performed with support from the Italian I.N.D.A.M. and M.I.U.R.

Abstract. Taking into account a new free energy expressed by the minimal state of a viscous fluid with memory, we prove an existence and uniqueness theorem for the related thermal convection problem, defined on arbitrary bounded domains of the three-dimensional space. Some conditions, related with the Rayleigh number, allow us to prove the exponential decay in the linearized problem.

1 Introduction

In recent years, energy stability results for some special classes of viscoelastic fluids have been studied by many authors. Assuming a linearized expression for a Newtonian viscous term, the energy stability of shear flows was derived by Preziosi and Rionero in [16]. Moreover, Lozinsky and Owens [14] and Doering and others [9] have investigated nonlinear energy stability for a fluid of Oldroyd-B type. In particular, Lozinsky and Owens have derived an a priori estimate for plane channel flow, whereas Doering and others have shown that in some cases a nonlinear energy functional cannot be constructed to yield a decay result. Finally, a comprehensive review of these issues can be found in [18].

In the paper [2] the problem of heating a viscoelastic fluid has been studied, by considering the Bénard problem for a viscous fluid with fading memory and employing the Boussinesq approximation in the temperature effects. In this work, for an incompressible viscoelastic fluid with memory, the existence and uniqueness of the solution for the linearized system of equations has been proved under a suitable restriction for the Rayleigh number.

Specifically, in this paper we consider the similar problem presented in [2], by means of a new free energy recently introduced for viscoelastic solids in [13] and [8] and for viscoelastic fluids in [1]. Such a functional has the property of being defined in a very much large domain of minimal states (see, for example, [3]). So in this framework, we generalize Slemrod's stability results [17], as well as an analogous problem considered in [2]. Hence, by the semigroup theory (see [6], [8] and [11]) the exponential stability is shown in a large domain of initial conditions defined by the notion of minimal state.

The layout of the paper is as follows. In Section 2, after introducing the constitutive equation for viscoelastic fluids, we recall the related thermodynamic restriction. Hence, in Section 3, some free energies and the internal dissipation properties are recalled. In Section 4, the basic equations are given in the linear approximation and some results, derived in the previous work [2], are also recalled. Finally, in Section 5, by using the new free energy, we prove an existence and uniqueness theorem for the linearized thermal problem and the conditions for the exponential stability.

2 Incompressible fluids with fading memory

Consider simple incompressible viscous fluids with memory, characterized by the following equations

$$\mathbf{T}(\mathbf{x},t) = -p(\mathbf{x},t)\mathbf{I} + 2\int_0^\infty \mu(s)\mathbf{D}^t(\mathbf{x},s)ds, \tag{2.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2.2}$$

where $\mathbf{T}$ is the stress tensor, p denotes the pressure, $\mathbf{I}$ is the unit tensor, $\mathbf{D}^t(\mathbf{x},s) = \mathbf{D}(\mathbf{x},t-s)$ $\forall s \in \mathbb{R}^+$ is the history up to time t of the strain rate tensor $\mathbf{D} = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right) := \nabla\tilde{\mathbf{v}}$ and μ is the relaxation function, which describes the fading memory and is such that the Second Law of Thermodynamics holds.

The extra stress $\mathbf{T}_E$, defined by

$$\mathbf{T}_E(\mathbf{x},t) = \mathbf{T}(\mathbf{x},t) + p(\mathbf{x},t)\mathbf{I}, \tag{2.3}$$

taking account of (2.1) and integrating by parts, can be expressed by one of the following two forms

$$\mathbf{T}_E(\mathbf{x},t) = 2\int_0^\infty \mu(s)\mathbf{D}^t(\mathbf{x},s)ds = 2\int_0^\infty \mu'(s)\mathbf{E}_r^t(\mathbf{x},s)ds, \tag{2.4}$$

where $\mathbf{E}_r^t(\mathbf{x},s)$ denotes the *relative history* of the infinitesimal strain tensor $\mathbf{E}$ with respect to the configuration at time t, defined by

$$\mathbf{E}_r^t(\mathbf{x},s) = \mathbf{E}^t(\mathbf{x},s) - \mathbf{E}(t) \qquad \forall s \in \mathbb{R}^+ = (0,\infty), \tag{2.5}$$

$\mathbf{E}^t(\mathbf{x},s) = \mathbf{E}(\mathbf{x},t-s)\,\forall s \in \mathbb{R}^+$ being the history up to time t of $\mathbf{E}$.

The dependence on $\mathbf{x}$ will be often omitted later on.

In [12] the authors have derived the constraint imposed by Thermodynamics on the constitutive equation, by proving the following theorem.

Theorem 2.1. *The constitutive equation (2.1) for linear viscoelastic fluids is compatible with the Second Law of Thermodynamics if and only if for every relaxation function $\mu \in L^1(\mathbb{R}^+)$, so that $\int_0^\infty \mu(s)ds \neq 0$, the following inequality*

$$\hat{\mu}_c(\omega) := \int_0^\infty \mu(s)\cos\omega s\,ds > 0 \qquad \forall\omega \in \mathbb{R} \tag{2.6}$$

holds.

Here $\hat{\mu}_c(\omega)$ denotes the half-range Fourier cosine transforms of μ.

We now consider isotropic, homogeneous and incompressible viscoelastic fluids in the linear, isothermal approximation, characterized by the constitutive equation (2.1), in which $\mu \in L^1(\mathbb{R}^+)$ and $\int_0^\infty \mu(s)ds \neq 0$; thus, Theorem 2.1 holds. Moreover, we suppose that in $(2.4)_2$ the kernel $\mu'(\cdot) \in L^1(\mathbb{R}^+)$ and the shear relaxation function

$$\mu(s) := \int_\infty^s \mu'(\xi)d\xi \qquad \forall s \in \mathbb{R}^+ \tag{2.7}$$

is such that $\mu(\cdot) \in H^2(\mathbb{R}^+)$. Hence, we have

$$\lim_{s\to\infty}\mu(s) = 0. \tag{2.8}$$

In what follows we shall assume the following inequalities

$$\mu(s) > 0, \quad \mu'(s) < 0, \quad \mu''(s) \geq 0 \qquad \forall s \in \mathbb{R}^+ \tag{2.9}$$

and

$$0 \neq |\mu'(0)| < \infty. \tag{2.10}$$

The infinitesimal strain tensor $\mathbf{E} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) := \nabla \tilde{\mathbf{u}}$ and the strain rate tensor $\mathbf{D}$, we have already introduced, yield

$$\mathbf{E}_r^t(s) = \nabla \tilde{\mathbf{u}}_r^t(s), \qquad \mathbf{D}^t(s) = \nabla \tilde{\mathbf{v}}^t(s) \tag{2.11}$$

and allow us to write (2.4) in the form

$$\mathbf{T}_E(t) = 2\int_0^\infty \mu'(s)\nabla \tilde{\mathbf{u}}_r^t(s)ds = 2\int_0^\infty \mu(s)\nabla \tilde{\mathbf{v}}^t(s)ds. \tag{2.12}$$

3 Free energies

In linear viscoelasticity there is a family $\mathcal{F}$ of free energies (see e.g. Coleman and Owen [5]), which is a convex set, with a minimum and a maximum element denoted by ψ_m and ψ_M, respectively. Some expressions for the free energies have been derived for viscoelastic fluids in [1], by adapting their forms introduced for viscoelastic solids; in such a paper, for any of these free energies the related internal dissipation function $D(t)$, which is a non-negative function because of the Second Law of Thermodynamics, has been also evaluated.

In particular, for our fluids we recall the maximum free energy, considered in [10] and expressed by

$$\psi_M\left(\mathbf{E}^t\right) = -\frac{1}{2}\int_0^\infty \int_0^\infty \mu'\left(|s-s'|\right)\nabla \mathbf{u}_r^t(s)\cdot \nabla \mathbf{u}_r^t(s')dsds', \tag{3.1}$$

together with the representation of the minimum free energy proposed by Bruer-Onat in [4] in the form

$$\Psi_m(\dot{\mathbf{E}}_m^t) = \frac{1}{2}\int_0^\infty \int_0^\infty \mu(|s_1-s_2|)\nabla \mathbf{v}_m(s_1)\cdot \nabla \mathbf{v}_m\left(s_2\right)ds_1ds_2, \tag{3.2}$$

where $\nabla \mathbf{v}_m$ is the optimal process which yields the maximum recoverable work. Unfortunately, this functional is not a state function.

In [2], in order to derive a stability theorem for their fluids, the authors have considered the Graffi-Volterra functional, frequently used in applications, and given by

$$\Psi_G(\nabla \mathbf{u}_r^t) = -\frac{1}{2}\int_0^\infty \mu'(s)\nabla \mathbf{u}_r^t(s)\cdot \nabla \mathbf{u}_r^t(s)ds, \tag{3.3}$$

under the hypothesis that the kernel satisfies (2.9).

In this paper we shall be concerned with the new free energy, considered, in particular, in [13] and [8] for viscoelastic solids and denoted by ψ_F. The form assumed by such a functional for our fluids, already derived in [1], is given by

$$\psi_F(t) = -\frac{1}{4}\int_0^\infty \frac{1}{\mu'(\tau)}\breve{\mathbf{I}}_{(1)}^t(\tau)\cdot \breve{\mathbf{I}}_{(1)}^t(\tau)d\tau, \tag{3.4}$$

where, taking into account what we have observed to write (2.12), the function $\breve{\mathbf{I}}^t(\tau)$ is given by

$$\breve{\mathbf{I}}^t(\tau) = -2\int_0^\infty \mu'(\tau+s)\mathbf{E}_r^t(s)ds = -2\int_0^\infty \mu(\tau+s)\nabla \tilde{\mathbf{v}}^t(s)ds \quad \forall \tau \in \mathbb{R}^+ \tag{3.5}$$

and $\breve{\mathbf{I}}_{(1)}^t(\tau)$ denotes its derivative with respect to τ, that is

$$\breve{\mathbf{I}}_{(1)}^t(\tau) := \frac{\partial}{\partial \tau}\breve{\mathbf{I}}^t(\tau) = -2\int_0^\infty \mu'(\tau+s)\nabla \tilde{\mathbf{v}}^t(s)ds. \tag{3.6}$$

Since the functional (3.4) must give a non-negative quantity, still now we must assume that relations (2.9)-(2.10) hold.

We observe that no problem arises in (3.4) from the presence of $1/\mu'(\tau)$, which diverges as τ tends to infinity, because this factor is multiplied by other factors, which assure the existence of the integral under consideration. Moreover, the domain of definition of the functional ψ_F is characterized by the following space of functions

$$H'_F(\mathbb{R}^+) = \left\{ \breve{\mathbf{I}}^t; \left| \int_0^\infty \frac{1}{\mu'(\tau)} \breve{\mathbf{I}}^t_{(1)}(\tau) \cdot \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau \right| < \infty \right\}, \tag{3.7}$$

which yields a very much larger space with respect to that one corresponding to the Graffi-Volterra free energy ψ_G.

From $(3.5)_2$-(3.6), taking into account $(2.12)_2$, it follows that

$$\breve{\mathbf{I}}^t(0) = -2 \int_0^\infty \mu(s) \nabla \tilde{\mathbf{v}}^t(s) ds \equiv -\mathbf{T}_E(t), \tag{3.8}$$

$$\breve{\mathbf{I}}^t_{(1)}(0) = -2 \int_0^\infty \mu'(s) \nabla \tilde{\mathbf{v}}^t(s) ds. \tag{3.9}$$

Finally we recall the internal dissipation related to this last free energy, which has been derived in [1] and has the following expression

$$D_F(t) = \frac{1}{4} \int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2} \breve{\mathbf{I}}^t_{(1)}(\tau) \cdot \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau - \frac{1}{4} \frac{1}{\mu'(0)} \breve{\mathbf{I}}^t_{(1)}(0) \cdot \breve{\mathbf{I}}^t_{(1)}(0) \geq 0, \tag{3.10}$$

which is a non-negative quantity for all histories because of the hypotheses $(2.9)_3$, on the second derivative of $\mu(s)$, and $(2.9)_2$-(2.10), on $\mu'(0)$.

4 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be an arbitrary bounded domain occupied by an incompressible fluid with memory effects. In order to study the thermal convection in such a fluid, we consider the Boussinesq approximation related to the density, that is by assuming the density constant except in the body force due to gravity.

Let $(\mathbf{v}(\mathbf{x}, t), \theta(\mathbf{x}, t), p(\mathbf{x}, t))$ denote a perturbation to a steady solution, where $\mathbf{v}$, θ and p denote velocity, temperature and pressure. This perturbation must satisfy a system of differential equations, which can be linearized and, with a few change with respect to the general case studied in [18], due to the presence of the memory effects, put in a non-dimensional form already derived in [2]. We get

$$\mathbf{v}_t = -\nabla p + \int_0^t \mu(s) \nabla^2 \mathbf{v}(t-s) ds + R\theta \mathbf{k} + \nabla \cdot \mathbf{F}, \tag{4.1}$$

$$\mathrm{Pr}\, \theta_t = \nabla^2 \theta + R \mathbf{v} \cdot \mathbf{k} + r. \tag{4.2}$$

In these equations R is the Rayleigh number, Pr is the Prandtle number and $\mathbf{k} = (0, 0, 1)$; moreover, we have put

$$\mu(0) = 1 \tag{4.3}$$

and introduced the tensor function $\mathbf{F}(\mathbf{x}, t)$ such that

$$\nabla \cdot \mathbf{F}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t) + \int_0^\infty \mu(s+t) \nabla^2 \mathbf{v}_0(-s) ds,$$

where $\mathbf{f}$ and r are the perturbations to the corresponding quantities in the steady state.

To these equations we associate the following boundary conditions

$$\mathbf{v}(\mathbf{x},t)=\mathbf{0}, \quad \theta(\mathbf{x},t)=0 \qquad \forall \mathbf{x}\in\partial\Omega \tag{4.4}$$

and we must add the initial conditions. For this purpose, we recall that it is always possible to change variables from the most general initial conditions, having the form

$$\begin{aligned} \mathbf{v}(\mathbf{x},0) &= \mathbf{v}_0(\mathbf{x}),\ \theta(\mathbf{x},0)=\theta_0(\mathbf{x}),\ \mathbf{v}(\mathbf{x},-s)=\mathbf{v}_0(\mathbf{x},-s)\ \forall s\in\mathbf{R}^+, && (4.5)\\ \mathbf{f}(\mathbf{x},0) &= \mathbf{f}_0(\mathbf{x}),\ r(\mathbf{x},0)=r_0(\mathbf{x}), && (4.6)\end{aligned}$$

with a suitable modification of the sources, in order to reduce them to the following ones

$$\begin{aligned} \mathbf{v}(\mathbf{x},0) &= \mathbf{0},\ \theta(\mathbf{x},0)=0,\ \mathbf{v}^{t=0}(\mathbf{x},s)=\mathbf{v}^0(\mathbf{x},s) \quad \forall s\in\mathbf{R}^+, && (4.7)\\ \mathbf{f}(\mathbf{x},0) &= \mathbf{0},\ r(\mathbf{x},0)=0. && (4.8)\end{aligned}$$

Taking into account (2.2), we consider the space

$$\dot{H}_0^1(\Omega)=\left\{\mathbf{v}\in H_0^1(\Omega);\nabla\cdot\mathbf{v}=\mathbf{0}\right\}; \tag{4.9}$$

moreover, we consider the following function spaces

$$\begin{aligned} \mathcal{K}_{\mathbf{v}}(\mathbb{R}^+,\Omega) &= \Big\{\mathbf{v}\in H^{1/2}(\mathbb{R}^+;L^2(\Omega))\cap L^2(\mathbb{R}^+;\dot{H}_0^1(\Omega));\\ &\qquad \int_{-\infty}^{\infty}\int_\Omega \hat{\mu}_c(\omega)|\nabla\hat{\mathbf{v}}_+(\mathbf{x},\omega)|^2 d\mathbf{x}d\omega<\infty\Big\}, && (4.10)\\ \mathcal{K}_\vartheta(\mathbb{R}^+,\Omega) &= \Big\{\theta\in H^{1/2}(\mathbb{R}^+;L^2(\Omega))\cap L^2(\mathbb{R}^+;H_0^1(\Omega))\Big\}, && (4.11)\end{aligned}$$

and

$$\begin{aligned} \mathcal{M}_{\mathbf{F}}(\mathbb{R}^+,\Omega) &= \big\{\mathbf{F}\in L^2(\mathbb{R}^+;L^2(\Omega));\\ &\qquad \int_{-\infty}^{\infty}\int_\Omega \frac{1}{\hat{\mu}_c(\omega)}|\hat{\mathbf{F}}_+(\mathbf{x},\omega)|^2 d\mathbf{x}d\omega<\infty\Big\}, && (4.12)\\ \tilde{\mathcal{M}}_{\mathbf{F}}(\mathbb{R}^+,\Omega) &= \big\{\mathbf{F}\in L^2(\mathbb{R}^+;L^2(\Omega));\\ &\qquad \int_{-\infty}^{\infty}\int_\Omega \frac{1+|\omega|}{\hat{\mu}_c(\omega)}|\hat{\mathbf{F}}_+(\mathbf{x},\omega)|^2 d\mathbf{x}d\omega<\infty\Big\}, && (4.13)\end{aligned}$$

where $\hat{\mu}_c$, $\hat{\mathbf{v}}_+$ and $\hat{\mathbf{F}}_+$ denote the Fourier transforms.

In [2] the authors have proved the existence and uniqueness theorem for this problem, by means of the following definition.

Definition 4.1. A pair $(\mathbf{v},\theta)\in\mathcal{K}_{\mathbf{v}}(\mathbb{R}^+,\Omega)\times\mathcal{K}_\vartheta(\mathbb{R}^+,\Omega)$ is called a weak solution of the problem (4.1)-(4.2), with the boundary conditions (4.4), the initial data (4.7)-(4.8), and $\mathbf{F}\in\mathcal{M}_{\mathbf{F}}(\mathbb{R}^+,\Omega)$ and $r\in L^2(\mathbb{R}^+;L^2(\Omega))$, if it satisfies the equations

$$\begin{aligned} &\int_0^\infty\int_\Omega\left[\mathbf{v}\cdot\mathbf{w}_t-\int_0^t\mu(s)\nabla\mathbf{v}(t-s)ds\cdot\nabla\mathbf{w}+R\theta\mathbf{k}\cdot\mathbf{w}\right]d\mathbf{x}dt\\ &\qquad =\int_0^\infty\int_\Omega \mathbf{F}\cdot\nabla\mathbf{w}\,d\mathbf{x}dt, && (4.14)\end{aligned}$$

$$\int_0^\infty\int_\Omega(\Pr\theta\varphi_t-\nabla\theta\cdot\nabla\varphi+R\mathbf{v}\cdot\mathbf{k}\varphi)\,d\mathbf{x}dt=\int_0^\infty\int_\Omega r\varphi\,d\mathbf{x}dt \tag{4.15}$$

for all $\mathbf{w}\in\mathcal{K}_{\mathbf{v}}(\mathbb{R}^+,\Omega)$ and $\varphi\in\mathcal{K}_\vartheta(\mathbb{R}^+,\Omega)$, such that $\mathbf{w}(\mathbf{x},0)=\mathbf{0}$ and $\varphi(\mathbf{x},0)=0$.

By using the Fourier transforms and applying the Parseval-Plancherel theorem to (4.14)-(4.15), the following theorem has been proved in [2].

Theorem 4.2. *If the kernel $\mu \in H^1(\mathbb{R}^+; L^\infty(\Omega))$ and satisfies the condition (2.6) and, moreover, $\mathbf{F} \in \tilde{\mathcal{M}}_{\mathbf{F}}(\mathbb{R}^+, \Omega)$ and $r, \dot{r} \in L^2(\mathbb{R}^+; L^2(\Omega))$, then, for sufficiently small values of R, there exists a unique weak solution $(\mathbf{v}, \theta) \in \mathcal{K}_{\mathbf{v}}(\mathbb{R}^+, \Omega) \times \mathcal{K}_{\vartheta}(\mathbb{R}^+, \Omega)$ defined in Definition 4.1.*

5 Existence, uniqueness and exponential stability of the solution

We now consider the linearized system of equations in the non-dimensional form (4.1)-(4.2) in order to derive an existence and uniqueness theorem for its solution and to study the exponential stability of the solution.

For this purpose we assume that the kernel $\mu(s) \in H^2(\mathbb{R}^+)$ satisfies (2.9)-(2.10) and that there exists a positive constant ξ such that the following conditions

$$\mu'(s) + \xi\mu(s) \leq 0, \quad \mu''(s) + \xi\mu'(s) \geq 0 \qquad \xi > 0 \tag{5.1}$$

hold.

The non-dimensional linearized system (4.1)-(4.2) with (2.2), by using (2.12), can be rewritten as follows

$$\frac{\partial}{\partial t}\mathbf{v} = -\nabla p + \nabla \cdot \mathbf{T}_E + R\theta\mathbf{k}, \tag{5.2}$$

$$\frac{\partial}{\partial t}\theta = \frac{1}{\mathrm{Pr}}\left(\nabla^2\theta + R\mathbf{v}\cdot\mathbf{k}\right), \tag{5.3}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{5.4}$$

$$\frac{\partial}{\partial t}\mathbf{T}_E = 2\int_0^\infty \mu'(s)\nabla\tilde{\mathbf{v}}_r^t(s)ds, \tag{5.5}$$

$$\frac{\partial}{\partial t}\breve{\mathbf{I}}^t_{(1)}(\tau) = \frac{\partial \breve{\mathbf{I}}^t_{(1)}(\tau)}{\partial \tau} - 2\mu'(\tau)\nabla\tilde{\mathbf{v}}(t), \tag{5.6}$$

$$\frac{\partial}{\partial t}\mathbf{v}_r^t(s) = -\frac{\partial \mathbf{v}_r^t(s)}{\partial s} - \frac{\partial \mathbf{v}(t)}{\partial t}, \tag{5.7}$$

where the definitions of $\mathbf{u}_r^t(s)$ and $\mathbf{v}_r^t(s)$ are analogous to that one in (2.5) and we have considered the supplies $\mathbf{f} = \mathbf{0}$ and $r = 0$. This system (5.2)-(5.7) is supplemented the boundary conditions (4.4).

We note that (5.5) follows from $(2.12)_1$ because

$$\frac{\partial}{\partial t}\mathbf{T}_E(t) = \frac{\partial}{\partial t}\left[2\int_0^\infty \mu'(s)\nabla\tilde{\mathbf{u}}_r^t(s)ds\right] = 2\int_0^\infty \mu'(s)\nabla\tilde{\mathbf{v}}_r^t(s)ds;$$

(5.6) follows from (3.6) since, by integrating by parts, we have

$$\begin{aligned}\frac{\partial \breve{\mathbf{I}}^t_{(1)}(\tau)}{\partial t} &= 2\int_0^\infty \mu'(\tau+s)\frac{\partial}{\partial s}\nabla\tilde{\mathbf{v}}^t(s)ds = -2\int_0^\infty \mu''(\tau+s)\nabla\tilde{\mathbf{v}}^t(s)ds \\ &\quad -2\mu'(\tau)\nabla\tilde{\mathbf{v}}(t) = \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t_{(1)}(\tau) - 2\mu'(\tau)\nabla\tilde{\mathbf{v}}(t);\end{aligned} \tag{5.8}$$

(5.7) follows from the definition of $\mathbf{v}_r^t(s)$, analogous to that one in (2.5), by differentiating with respect to t.

Now, let us introduce the quantity

$$\chi = (\mathbf{v}, \theta, \mathbf{T}_E, \breve{\mathbf{I}}^t_{(1)}, \mathbf{v}^t_r) \tag{5.9}$$

such that

$$\chi \in \mathcal{H} = \dot{H}^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times \mathcal{B}(\Omega, \mathbb{R}^+) \times \mathcal{D}(\Omega, \mathbb{R}^+), \tag{5.10}$$

where

$$\mathcal{B}(\Omega, \mathbb{R}^+) = \left\{ \breve{\mathbf{I}}^t_{(1)}(\mathbf{x}, \cdot) : \mathbb{R}^+ \to V^3;\ -\int_\Omega \int_0^\infty \frac{1}{\mu'(\tau)} \left[\breve{\mathbf{I}}^t_{(1)}(\mathbf{x}, \tau) \right]^2 d\tau d\mathbf{x} < \infty \right\} \tag{5.11}$$

and

$$\mathcal{D}(\Omega, \mathbb{R}^+) = \left\{ \mathbf{v}^t_r(\mathbf{x}, \cdot) : \mathbb{R}^+ \to V^3;\ -\int_\Omega \int_0^\infty \mu'(s) \left[\mathbf{v}^t_r(\mathbf{x}, s) \right]^2 ds d\mathbf{x} < \infty \right\} \tag{5.12}$$

are Hilbert spaces with inner products defined by

$$\langle \breve{\mathbf{I}}^t_{(1)_1}, \breve{\mathbf{I}}^t_{(1)_2} \rangle_{\mathcal{B}} = -\frac{1}{2} \int_\Omega \int_0^\infty \frac{1}{\mu'(\tau)} \breve{\mathbf{I}}^t_{(1)_1}(\tau) \cdot \breve{\mathbf{I}}^t_{(1)_2}(\tau) d\tau d\mathbf{x}, \tag{5.13}$$

$$\langle \mathbf{v}^t_{1_r}, \mathbf{v}^t_{2_r} \rangle_{\mathcal{D}} = -\int_\Omega \int_0^\infty \mu'(s) \mathbf{v}^t_{1_r}(s) \cdot \mathbf{v}^t_{2_r}(s) ds d\mathbf{x}, \tag{5.14}$$

respectively. We observe that the last of these inner products can be expressed in terms of the quantity $\nabla \mathbf{v}^t_{1_r}(s) \cdot \nabla \mathbf{v}^t_{2_r}(s)$.

Key to this work is the free energy $\psi_F(t)$ introduced in (3.4), that is

$$\psi_F(\breve{\mathbf{I}}^t) = -\frac{1}{4} \int_0^\infty \frac{1}{\mu'(\tau)} \breve{\mathbf{I}}^t_{(1)}(\tau) \cdot \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau, \tag{5.15}$$

with (3.5)-(3.10) together with the Slemrod functional [17]

$$\Psi_{\mathbf{v}}(\mathbf{v}^t_r) = -\frac{1}{2} \int_0^\infty \mu'(s) \mathbf{v}^t_r(s) \cdot \mathbf{v}^t_r(s) ds. \tag{5.16}$$

For the proof of the next theorem, we need to consider the Slemrod inequality, proved in [17]. If $\mathbf{v} \in L^2(\mathbb{R}; \dot{H}^1_0(\Omega))$ and $\mu \in H^2(\mathbb{R}^+)$ is a function which satisfies the conditions (2.9)-(2.10), then for any $t \in \mathbb{R}$ we have

$$\int_\Omega |\mathbf{v}(\mathbf{x}, t)|^2 d\mathbf{x} \le c \left[\int_\Omega |\mathbf{T}_E(\mathbf{x}, t)|^2 d\mathbf{x} - \int_0^\infty \mu'(s) \int_\Omega \left| \mathbf{v}^t_r(\mathbf{x}, s) \right|^2 d\mathbf{x} ds \right], \tag{5.17}$$

where c is a suitable positive constant depending on Ω.

We shall also use the following notation

$$\|\mathbf{v}(t)\|^2 = \int_\Omega |\mathbf{v}(t)|^2 d\mathbf{x},\ \|\theta(t)\|^2 = \int_\Omega \theta^2(t) d\mathbf{x},\ \|\mathbf{T}_E(t)\|^2 = \int_\Omega |\mathbf{T}_E(t)|^2 d\mathbf{x}. \tag{5.18}$$

We can now show the following theorem.

Theorem 5.1. *Under the hypotheses (2.9)-(2.10) and (5.1), the problem (5.2)-(5.7), for any initial conditions $\chi_0 \in \mathcal{H}$, admits a unique solution $\chi(t) \in \mathcal{H}$ such that*

$$\begin{aligned} &\frac{1}{2} \left[\|\mathbf{v}(t)\|^2 + \Pr \|\theta(t)\|^2 \right] + \frac{1}{4} \|\mathbf{T}_E(t)\|^2 + \\ &\qquad + \int_\Omega \Psi_F(\breve{\mathbf{I}}^t) d\mathbf{x} + \int_\Omega \Psi_{\mathbf{v}}(\mathbf{v}^t_r) d\mathbf{x} \\ &\le M e^{-\varepsilon t} \left\{ \frac{1}{2} \left[\|\mathbf{v}(0)\|^2 + \Pr \|\theta(0)\|^2 \right] + \frac{1}{4} \|\mathbf{T}_E(0)\|^2 \right. \\ &\qquad \left. + \int_\Omega \Psi_F(\breve{\mathbf{I}}^{t=0}) d\mathbf{x} + \int_\Omega \Psi_{\mathbf{v}}(\mathbf{v}^{t=0}_r) d\mathbf{x} \right\}, \end{aligned} \tag{5.19}$$

where M and ε are suitable positive constants.

Proof. Let us introduce the functional

$$
\begin{aligned}
E(t) \quad = \quad & \frac{1}{2}\left[\|\mathbf{v}(t)\|^2 + \mathrm{Pr}\,\|\theta(t)\|^2\right] + \frac{1}{4}\,\|\mathbf{T}_E(t)\|^2 \\
& + \int_\Omega \Psi_F(\breve{\mathbf{I}}^t) d\mathbf{x} + \int_\Omega \Psi_{\mathbf{v}}(\mathbf{v}_r^t) d\mathbf{x}, \qquad (5.20)
\end{aligned}
$$

which espresses the total energy of the system.

In order to evaluate $\frac{d}{dt}E(t)$, firstly let us differentiate the free energy (5.15), for which, using (5.6), integrating by parts and taking account of $(2.12)_2$, $(3.8)_2$ and of the equality $\mathbf{T}_E \cdot \nabla\tilde{\mathbf{v}} = \mathbf{T}_E \cdot \nabla\mathbf{v}$, we obtain

$$
\begin{aligned}
& \int_\Omega \frac{\partial}{\partial t}\psi_F(\breve{\mathbf{I}}^t) d\mathbf{x} = -\frac{1}{2}\int_\Omega \int_0^\infty \frac{1}{\mu'(\tau)}\left[\frac{\partial}{\partial \tau}\breve{\mathbf{I}}^t_{(1)}(\tau) - 2\mu'(\tau)\nabla\tilde{\mathbf{v}}(t)\right] \\
& \quad \cdot \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau d\mathbf{x} = \frac{1}{4}\int_\Omega \left\{ \int_0^\infty \frac{d}{d\tau}\left[\frac{1}{\mu'(\tau)}\right]\left[\breve{\mathbf{I}}^t_{(1)}(\tau)\right]^2 d\tau \right. \\
& \quad \left. + \frac{1}{\mu'(0)}\left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 \right\} d\mathbf{x} + \int_\Omega \nabla\tilde{\mathbf{v}}(t) \cdot \left[-\breve{\mathbf{I}}^t(0)\right] d\mathbf{x} \\
& = -\frac{1}{4}\int_\Omega \int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\breve{\mathbf{I}}^t_{(1)}(\tau) \cdot \breve{\mathbf{I}}^t_{(1)}(\tau) d\tau d\mathbf{x} \\
& \quad + \frac{1}{4\mu'(0)}\int_\Omega \breve{\mathbf{I}}^t_{(1)}(0) \cdot \breve{\mathbf{I}}^t_{(1)}(0) d\mathbf{x} + \int_\Omega \mathbf{T}_E(t) \cdot \nabla\mathbf{v}(t) d\mathbf{x}. \qquad (5.21)
\end{aligned}
$$

We note that the first term and the second one in $(5.21)_3$ are non-positive quantities by virtue of the hypotheses $(2.9)_3$, on the second derivative of $\mu(s)$, and $(2.9)_2$-(2.10), on the first derivative of $\mu(s)$.

Moreover, from (5.16), taking account of (5.7) and integrating by parts, we obtain

$$
\begin{aligned}
& \int_\Omega \frac{\partial}{\partial t}\Psi_v(\mathbf{v}_r^t) d\mathbf{x} = -\frac{1}{2}\int_\Omega \left[\int_0^\infty 2\mu'(s)\frac{\partial \mathbf{v}_r^t(s)}{\partial t} \cdot \mathbf{v}_r^t(s) ds\right] d\mathbf{x} \\
& = \int_\Omega \left\{ \int_0^\infty \mu'(s)\left[\frac{\partial \mathbf{v}_r^t(s)}{\partial s} + \frac{\partial \mathbf{v}(t)}{\partial t}\right] \cdot \mathbf{v}_r^t(s) ds \right\} d\mathbf{x} \\
& = -\frac{1}{2}\int_\Omega \int_0^\infty \mu''(s)\mathbf{v}_r^t(s) \cdot \mathbf{v}_r^t(s) ds d\mathbf{x} \\
& \qquad + \int_\Omega \left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s) ds\right] \cdot \frac{\partial \mathbf{v}(t)}{\partial t} d\mathbf{x}. \qquad (5.22)
\end{aligned}
$$

Finally, equation (5.5), by integrating by parts and using $(4.4)_1$, allows us to obtain

$$
\begin{aligned}
\int_\Omega \frac{\partial}{\partial t}\left[\frac{1}{4}\,|\mathbf{T}_E(t)|^2\right] d\mathbf{x} \quad & = \quad \int_\Omega \left[\int_0^\infty \mu'(s)\nabla\tilde{\mathbf{v}}_r^t(s) ds\right] \cdot \mathbf{T}_E\, d\mathbf{x} \\
& = \quad -\int_0^\infty \mu'(s)\left[\int_\Omega \mathbf{v}_r^t(s) \cdot (\nabla \cdot \mathbf{T}_E)\, d\mathbf{x}\right] ds,
\end{aligned}
$$

whence, by eliminating $-\nabla\cdot\mathbf{T}_E$ by means of equation (5.2) and integrating by parts, we get

$$\int_\Omega \frac{\partial}{\partial t}\left[\frac{1}{4}\left|\mathbf{T}_E(t)\right|^2\right]d\mathbf{x} = \int_\Omega \int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds$$
$$\cdot\left(-\nabla p - \frac{\partial \mathbf{v}}{\partial t} + R\theta\mathbf{k}\right)d\mathbf{x} = -\int_\Omega\left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right]$$
$$\cdot\frac{\partial\mathbf{v}(t)}{\partial t}d\mathbf{x} + R\int_\Omega\left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right]\cdot\mathbf{k}\theta(t)\,d\mathbf{x}, \tag{5.23}$$

since the term in which ∇p is involved vanishes by virtue of $(4.4)_1$ and (5.4).

Summing $(5.22)_3$ and $(5.23)_2$, we obtain

$$\int_\Omega \frac{\partial}{\partial t}\left[\Psi_v(\mathbf{v}_r^t) + \frac{1}{4}\left|\mathbf{T}_E(t)\right|^2\right]d\mathbf{x} = -\frac{1}{2}\int_\Omega\int_0^\infty \mu''(s)\mathbf{v}_r^t(s)$$
$$\cdot\mathbf{v}_r^t(s)dsd\mathbf{x} + R\int_\Omega\left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right]\cdot\mathbf{k}\theta(t)\,d\mathbf{x}. \tag{5.24}$$

It remains to evaluate the time derivative of $\frac{1}{2}\|\mathbf{v}(t)\|^2$ and $\frac{\Pr}{2}\|\theta(t)\|^2$.

For this purpose, we consider the scalar product of the differential equation (5.2) by $\mathbf{v}(t)$ and the other equation (5.3) multiplied by $\vartheta(t)$. Thus, from $(5.18)_1$, taking into account the first differential equation (5.2) and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|\mathbf{v}(t)\right\|^2 \equiv \int_\Omega \frac{\partial\mathbf{v}(t)}{\partial t}\cdot\mathbf{v}(t)d\mathbf{x} = -\int_\Omega \mathbf{T}_E(t)\cdot\nabla\mathbf{v}(t)d\mathbf{x}$$
$$+R\int_\Omega \theta(t)\mathbf{k}\cdot\mathbf{v}(t)d\mathbf{x} = -\left(\mathbf{T}_E,\nabla\mathbf{v}\right) + R\left(\theta,\mathbf{k}\cdot\mathbf{v}\right). \tag{5.25}$$

Here, we have used the boundary condition $(4.4)_1$ in evaluating the integral with $\nabla\cdot\mathbf{T}_E$, for which we have deduced that $(\nabla\cdot\mathbf{T}_E,\mathbf{v}) = -(\mathbf{T}_E,\nabla\mathbf{v})$; moreover, as we have already observed for the term with ∇p in (5.23), we have obtained $-(\nabla p,\mathbf{v}) = (-p,\nabla\cdot\mathbf{v}) = 0$ by virtue of (5.4).

Analogously, from $(5.18)_2$, using the second differential equation (5.3) and integrating by parts, we have

$$\frac{1}{2}\Pr\frac{d}{dt}\|\theta(t)\|^2 \equiv (\dot\theta,\Pr\theta) \equiv \Pr\int_\Omega \frac{\partial\vartheta(t)}{\partial t}\vartheta(t)d\mathbf{x}$$
$$= \int_\Omega\left(\nabla^2\theta + R\mathbf{v}\cdot\mathbf{k}\right)\vartheta d\mathbf{x} = -\int_\Omega\left|\nabla\theta\right|^2 d\mathbf{x}$$
$$+R\int_\Omega \vartheta(t)\mathbf{k}\cdot\mathbf{v}(t)d\mathbf{x} = -\|\nabla\theta\|^2 + R\left(\theta,\mathbf{k}\cdot\mathbf{v}\right), \tag{5.26}$$

by virtue of $(4.4)_2$.

Thus, the time derivative of (5.20), taking into account $(5.21)_3$, (5.24), $(5.25)_3$ and $(5.26)_5$, is given by

$$\frac{d}{dt}E(t) = 2R(\vartheta,\mathbf{k}\cdot\mathbf{v}) + R\int_\Omega\left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right]\cdot\mathbf{k}\theta\,d\mathbf{x}$$
$$-\|\nabla\theta\|^2 - \frac{1}{4}\int_\Omega\int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\breve{\mathbf{I}}_{(1)}^t(\tau)\cdot\breve{\mathbf{I}}_{(1)}^t(\tau)d\tau d\mathbf{x}$$
$$+\frac{1}{4\mu'(0)}\int_\Omega \breve{\mathbf{I}}_{(1)}^t(0)\cdot\breve{\mathbf{I}}_{(1)}^t(0)d\mathbf{x} - \frac{1}{2}\int_\Omega\int_0^\infty \mu''(s)\left|\mathbf{v}_r^t(s)\right|^2 dsd\mathbf{x}. \tag{5.27}$$

Let us observe that only the first two terms in the right-hand side of (5.27) are terms which may lead to instability. In the classical Bénard problem the second term obviously is not present. The last three terms yield the viscous dissipation related to the considered two free energies; the internal dissipation related to $\psi_F(\breve{\mathbf{I}}^t)$ is given by (3.10). This is weaker than the term $-\|\nabla\mathbf{v}\|^2$ which is found in the classical Navier-Stokes theory. The hyperbolic nature of viscoelasticity manifests itself this way.

Let us estimate the effects of the first two terms.

For the first term, by introducing an arbitrary positive α, the arithmetic-geometric mean inequality allows us to derive the following inequality

$$2R(\vartheta, \mathbf{k}\cdot\mathbf{v}) \leq \frac{R}{\alpha}\|\theta\|^2 + \alpha R\|\mathbf{v}\cdot\mathbf{k}\|^2 \leq \frac{R}{\alpha}\|\theta\|^2 + R\alpha\|\mathbf{v}\|^2. \tag{5.28}$$

For the last term, following [10], we find

$$\begin{aligned}
R\int_\Omega \left[\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right]\cdot\mathbf{k}\theta\,d\mathbf{x} &\leq R\int_\Omega \left|-\int_0^\infty \mu'(s)\mathbf{v}_r^t(s)ds\right| |\vartheta|\,d\mathbf{x}\\
&\leq R\int_\Omega \left[\int_0^\infty \sqrt{-\mu'(s)}\sqrt{-\mu'(s)\,|\mathbf{v}_r^t(s)|^2}ds\right]|\vartheta|\,d\mathbf{x}\\
&\leq R\int_\Omega \frac{|\vartheta|}{\sqrt{\beta}}\sqrt{\beta}\sqrt{\int_0^\infty -\mu'(s)ds}\sqrt{\int_0^\infty -\mu'(s)\,|\mathbf{v}_r^t(s)|^2\,ds}\,d\mathbf{x}\\
&\leq R\left[\frac{1}{2\beta}\|\theta\|^2 - \frac{\beta}{2}\int_\Omega\int_0^\infty \mu'(s)\left|\mathbf{v}_r^t(s)\right|^2 dsd\mathbf{x}\right],
\end{aligned} \tag{5.29}$$

where (4.3), i.e. $\mu(0) = 1$, has been used together with the arithmetic-geometric mean inequality with another arbitrary $\beta > 0$.

Introducing these two results into (5.27) we have the following inequality

$$\begin{aligned}
\frac{d}{dt}E(t) \leq& -\|\nabla\theta\|^2 - \frac{1}{4}\int_\Omega\int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\breve{\mathbf{I}}_{(1)}^t(\tau)\cdot\breve{\mathbf{I}}_{(1)}^t(\tau)d\tau d\mathbf{x}\\
&+\frac{1}{4\mu'(0)}\int_\Omega \breve{\mathbf{I}}_{(1)}^t(0)\cdot\breve{\mathbf{I}}_{(1)}^t(0)d\mathbf{x} - \frac{1}{2}\int_\Omega\int_0^\infty [\mu''(s)\\
&+\beta R\mu'(s)]\left|\mathbf{v}_r^t(s)\right|^2 dsd\mathbf{x} + \left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\|\theta\|^2 + R\alpha\|\mathbf{v}\|^2.
\end{aligned} \tag{5.30}$$

By virtue of the Poincarè inequality there exists a constant $C_1(\Omega)$, which depends on the domain Ω, such that

$$\int_\Omega \theta^2 d\mathbf{x} \leq C_1(\Omega)\int_\Omega |\nabla\theta|^2\,d\mathbf{x}. \tag{5.31}$$

We can now use (5.31) and Slemrod's inequality (5.17) to modify the last two terms in (5.30). Thus, we have

$$\begin{aligned}
\frac{d}{dt}E(t) \leq& -\left[1 - C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2 - \frac{1}{4}\int_\Omega\int_0^\infty \frac{\mu''(\tau)}{[\mu'(\tau)]^2}\breve{\mathbf{I}}_{(1)}^t(\tau)\\
&\cdot\breve{\mathbf{I}}_{(1)}^t(\tau)d\tau d\mathbf{x} + \frac{1}{4\mu'(0)}\int_\Omega \breve{\mathbf{I}}_{(1)}^t(0)\cdot\breve{\mathbf{I}}_{(1)}^t(0)d\mathbf{x} - \frac{1}{2}\int_\Omega\int_0^\infty [\mu''(s)\\
&+(2\alpha c+\beta)R\mu'(s)]\left[\mathbf{v}_r^t(s)\right]^2 dsd\mathbf{x} + \alpha cR\int_\Omega |\mathbf{T}_E(t)|^2\,d\mathbf{x}.
\end{aligned} \tag{5.32}$$

Since the extra stress $\mathbf{T}_E$ is continuous with respect to the free energy ψ_F, from the expression $(2.12)_1$ and (3.8) it follows that

$$\mathbf{T}_E(t) = -\breve{\mathbf{I}}^t(0) = 2\int_0^\infty \mu'(s)\nabla\tilde{\mathbf{v}}^t(s)ds, \tag{5.33}$$

where the quantity $\breve{\mathbf{I}}^t(0)$ can be written as

$$\breve{\mathbf{I}}^t(0) = -\int_0^\infty \frac{\partial}{\partial\tau}\breve{\mathbf{I}}^t(\tau)d\tau \tag{5.34}$$

and thus can be estimated as follows

$$\begin{aligned}
\left|\breve{\mathbf{I}}^t(0)\right| &\leq \int_0^\infty \left|\breve{\mathbf{I}}^t_{(1)}(\tau)\right| d\tau \leq \int_0^\infty \sqrt{-\mu'(\tau)}\sqrt{\frac{1}{-\mu'(\tau)}\left|\breve{\mathbf{I}}^t_{(1)}(\tau)\right|^2} d\tau \\
&\leq \left[\int_0^\infty -\mu'(\tau)d\tau\right]^{\frac{1}{2}}\left[\int_0^\infty \frac{1}{-\mu'(\tau)}\left|\breve{\mathbf{I}}^t_{(1)}(\tau)\right|^2 d\tau\right]^{\frac{1}{2}} \\
&\leq \left[\int_0^\infty \frac{1}{-\mu'(\tau)}\left|\breve{\mathbf{I}}^t_{(1)}(\tau)\right|^2 d\tau\right]^{\frac{1}{2}},
\end{aligned} \tag{5.35}$$

where we have used (4.3), i.e. $\mu(0) = 1$.

From (5.33) and $(5.35)_4$ we have the following estimate

$$|\mathbf{T}_E|^2 = \left|\breve{\mathbf{I}}^t(0)\right|^2 \leq -\int_0^\infty \frac{1}{\mu'(\tau)}\left|\breve{\mathbf{I}}^t_{(1)}(\tau)\right|^2 d\tau \equiv 4\psi_F(t). \tag{5.36}$$

Thus, by integrating $(5.36)_2$ on Ω, the inequality (5.32) becomes

$$\begin{aligned}
\frac{d}{dt}E(t) \leq &-\left[1 - C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2 - \frac{1}{4}\int_\Omega\int_0^\infty\left\{\frac{\mu''(\tau)}{[\mu'(\tau)]^2}\right. \\
&\left.+\frac{4\alpha cR}{\mu'(\tau)}\right\}\left[\breve{\mathbf{I}}^t_{(1)}(\tau)\right]^2 d\tau d\mathbf{x} + \frac{1}{4\mu'(0)}\int_\Omega\left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 d\mathbf{x} \\
&-\frac{1}{2}\int_\Omega\int_0^\infty\left[\mu''(s) + (2\alpha c+\beta)R\mu'(s)\right]\left[\mathbf{v}^t_r(s)\right]^2 ds d\mathbf{x}.
\end{aligned} \tag{5.37}$$

The right-hand side of this inequality is non-positive if the following conditions

$$\begin{aligned}
1 &\geq C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R, \\
\mu''(s) &\geq -4\alpha cR\mu'(s) > 0, \\
\mu''(s) &\geq -(2\alpha c+\beta)R\mu'(s) > 0,
\end{aligned} \tag{5.38}$$

are satisfied. Using $(5.1)_2$, we see that this occurs if

$$R \leq \frac{2\alpha\beta}{C_1(\Omega)(\alpha+2\beta)} \tag{5.39}$$

and moreover if

$$\mu''(s) \geq -\xi\mu'(s) \geq -4\alpha cR\mu'(s), \tag{5.40}$$

$$\mu''(s) \geq -\xi\mu'(s) \geq -(2\alpha c+\beta)R\mu'(s) > 0, \tag{5.41}$$

which are satisfied by choosing R such that

$$R \leq \frac{\xi}{4\alpha c}, \quad R \leq \frac{\xi}{\beta + 2\alpha c}. \tag{5.42}$$

Therefore, the conditions for stability are characterized by (5.39) and (5.42), that is by the following condition

$$R \leq \min\left\{\frac{2\alpha\beta}{C_1(\Omega)\,(\alpha+2\beta)},\ \frac{\xi}{\beta+2\alpha c},\ \frac{\xi}{4\alpha c}\right\}. \tag{5.43}$$

Then, (5.37), by using $(5.40)_1$, that is $-\mu''(s) \leq \xi\mu'(s)$, can be written as follows

$$\begin{aligned}
\frac{d}{dt}E(t) \leq & -\left[1 - C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2 \\
& + \frac{1}{4}\left(\xi - 4\alpha c R\right)\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\left[\breve{\mathbf{I}}^t_{(1)}(\tau)\right]^2 d\tau d\mathbf{x} + \frac{1}{4\mu'(0)}\int_\Omega \left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 d\mathbf{x} \\
& + \frac{1}{2}\left[\xi - (2\alpha c + \beta)\,R\right]\int_\Omega\int_0^\infty \mu'(s)\left[\mathbf{v}^t_r(s)\right]^2 ds d\mathbf{x} =: -\rho(t),
\end{aligned} \tag{5.44}$$

where we remember that $\mu'(s) < 0$.

Keeping in mind the definitions (5.15) and (5.16), it is easy to obtain the following different but expressive form of this last inequality

$$\begin{aligned}
\frac{d}{dt}E(t) \leq & -\left[1 - C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2 \\
& - \left(\xi - 4\alpha c R\right)\int_\Omega \mathbf{\Psi}_F d\mathbf{x} + \frac{1}{4\mu'(0)}\int_\Omega \left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 d\mathbf{x} \\
& - \left[\xi - (2\alpha c + \beta)\,R\right]\int_\Omega \mathbf{\Psi}_{\mathbf{v}} d\mathbf{x},
\end{aligned} \tag{5.45}$$

in terms of the assumed free energies $\psi_F(\breve{\mathbf{I}}^t)$ and $\Psi_v(\mathbf{v}^t_r)$.

From (5.44), by assuming that R satisfies (5.43), it follows that

$$\frac{d}{dt}E(t) \leq -\rho(t) \leq 0. \tag{5.46}$$

Thus, by integrating on $(0,t)$, we have

$$E(t) - E(0) \leq -\int_0^t \rho(\eta)d\eta. \tag{5.47}$$

Hence, it follows that

$$0 \leq E(t) \leq E(0) - \int_0^t \rho(\eta)d\eta \leq E(0) \tag{5.48}$$

and also

$$E(0) \geq E(t) + \int_0^t \rho(\eta)d\eta \geq \int_0^t \rho(\eta)d\eta \qquad \forall t \in \mathbb{R}^+. \tag{5.49}$$

Thus, by considering the integral on $(0,\infty)$, we obtain

$$\begin{aligned}
E(0) \geq & \int_0^\infty \rho(\eta)d\eta \geq \int_0^\infty \left\{\left[1 - C_1(\Omega)\left(\frac{1}{\alpha}+\frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2\right. \\
& - \frac{1}{4}\left(\xi - 4\alpha c R\right)\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\left[\breve{\mathbf{I}}^t_{(1)}(\tau)\right]^2 d\tau d\mathbf{x} - \frac{1}{4\mu'(0)}\int_\Omega \left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 d\mathbf{x} \\
& \left. - \frac{1}{2}\left[\xi - (2\alpha c + \beta)\,R\right]\int_\Omega\int_0^\infty \mu'(s)\left[\mathbf{v}^t_r(s)\right]^2 ds d\mathbf{x}\right\} dt.
\end{aligned} \tag{5.50}$$

This relation can be rewritten as

$$
\begin{aligned}
E(0) \geq \int_0^\infty \Bigg\{ & \left[1 - C_1(\Omega)\left(\frac{1}{\alpha} + \frac{1}{2\beta}\right) R\right] \|\nabla\theta\|^2 \\
& + (\xi - 4\alpha c R) \int_\Omega \psi_F(\breve{\mathbf{I}}^t) d\mathbf{x} + \frac{1}{2}\left[\xi - (2\alpha c + \beta) R\right] \left\|\mathbf{v}_r^t(s)\right\|_{\mathcal{D}} \\
& - \frac{1}{4\mu'(0)} \int_\Omega \left[\breve{\mathbf{I}}_{(1)}^t(0)\right]^2 d\mathbf{x} \Bigg\} dt \geq 0,
\end{aligned} \tag{5.51}
$$

where $\|\mathbf{v}_r^t(s)\|_{\mathcal{D}} = 2\int_\Omega \Psi_{\mathbf{v}}\, d\mathbf{x}$ denotes the norm related to (5.14).

Finally, by means of Theorem 4.2, the inequality (5.51) and the definition (5.20) of E, we have

$$
\begin{aligned}
\int_0^\infty E(t) dt = \int_0^\infty \Bigg\{ & \frac{1}{2}\left[\|\mathbf{v}(t)\|^2 + \Pr\|\theta(t)\|^2\right] + \frac{1}{4}\|\mathbf{T}_E(t)\|^2 \\
& + \int_\Omega \Psi_F(\breve{\mathbf{I}}^t) d\mathbf{x} + \int_\Omega \Psi_{\mathbf{v}}(\mathbf{v}_r^t) d\mathbf{x} \Bigg\} dt < \infty.
\end{aligned} \tag{5.52}
$$

The system (5.2)-(5.7) can be rewritten in the following form

$$
\dot{\chi} = A\chi, \qquad \chi(0) = \chi_0, \tag{5.53}
$$

where the operator A, given by the right-hand sides of (5.2)-(5.3) and (5.5)-(5.7), is defined on the domain

$$
\begin{aligned}
\mathcal{D}(A) = \Big\{ & \chi = (\mathbf{v}, \theta, \mathbf{T}_E, \breve{\mathbf{I}}_{(1)}^t, \mathbf{v}_r^t) \in \mathcal{H}; \mathbf{v} \in \dot{H}_0^1(\Omega) \cap H^2(\Omega), \\
& \theta \in H_0^1(\Omega) \cap H^2(\Omega), \mathbf{T}_E \in L^2(\Omega), \\
& \breve{\mathbf{I}}_{(1)}^t \in \mathcal{B}(\Omega, \mathbb{R}^+),\ \mathbf{v}_r^t \in \mathcal{D}(\Omega, \mathbb{R}^+) \Big\}.
\end{aligned} \tag{5.54}
$$

With such a notation, taking account of (5.2)-(5.7), it is easy to verify that

$$
\langle \dot{\chi}, \chi \rangle \equiv \frac{dE(t)}{dt}, \tag{5.55}
$$

where $\frac{dE(t)}{dt}$ and $\dot{\chi}$ are given by (5.27) and (5.53). In fact, keeping in mind the inner products (5.13)-(5.14), we define

$$
\langle \dot{\vartheta}, \vartheta \rangle = \int_\Omega \Pr \frac{\partial \vartheta}{\partial t} \vartheta d\mathbf{x}, \quad \left\langle \frac{\partial \mathbf{T}_E}{\partial t}, \mathbf{T}_E \right\rangle = \frac{1}{2}\int_\Omega \frac{\partial \mathbf{T}_E}{\partial t} \cdot \mathbf{T}_E d\mathbf{x}, \tag{5.56}
$$

$$
\left\langle \frac{\partial \mathbf{v}_r^t}{\partial t}, \mathbf{v}_r^t \right\rangle_{\mathcal{D}} = -\int_\Omega \int_0^\infty \mu'(s) \frac{\partial \mathbf{v}_r^t}{\partial t} \cdot \mathbf{v}_r^t ds d\mathbf{x}, \tag{5.57}
$$

$$
\left\langle \frac{\partial \breve{\mathbf{I}}_{(1)}^t}{\partial t}, \breve{\mathbf{I}}_{(1)}^t \right\rangle_{\mathcal{B}} = -\frac{1}{2}\int_\Omega \int_0^\infty \frac{1}{\mu'(\tau)} \frac{\partial \breve{\mathbf{I}}_{(1)}^t(\tau)}{\partial t} \cdot \breve{\mathbf{I}}_{(1)}^t(\tau) d\tau d\mathbf{x}, \tag{5.58}
$$

and hence we have

$$
\begin{aligned}
\langle \dot{\chi}, \chi \rangle \equiv \langle A\chi, \chi \rangle = \int_\Omega \Bigg\{ & \frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{v} + \Pr \frac{\partial \vartheta}{\partial t} \vartheta + \frac{1}{2}\frac{\partial \mathbf{T}_E}{\partial t} \cdot \mathbf{T}_E \\
- \frac{1}{2}\int_0^\infty \frac{1}{\mu'(\tau)} \frac{\partial \breve{\mathbf{I}}_{(1)}^t(\tau)}{\partial t} \cdot \breve{\mathbf{I}}_{(1)}^t(\tau) d\tau & - \int_0^\infty \mu'(s) \frac{\partial \mathbf{v}_r^t}{\partial t} \cdot \mathbf{v}_r^t ds \Bigg\} d\mathbf{x},
\end{aligned} \tag{5.59}
$$

where the derivatives with respect to time must be substituted by the right-hand sides given by (5.2)-(5.3) and (5.5)-(5.7). With some calculations and few integrations by parts, using again (5.2) in the last integral, we obtain the expression (5.27) and hence (5.55) follows.

Since, by virtue of (5.53), (5.55) and (5.44)-(5.46), we have

$$\begin{aligned}\langle A\chi, \chi\rangle \equiv \frac{dE(t)}{dt} &\leq -\left[1 - C_1(\Omega)\left(\frac{1}{\alpha} + \frac{1}{2\beta}\right)R\right]\|\nabla\theta\|^2 \\ &+\frac{1}{4}\left(\xi - 4\alpha c R\right)\int_\Omega\int_0^\infty \frac{1}{\mu'(\tau)}\left[\breve{\mathbf{I}}^t_{(1)}(\tau)\right]^2 d\tau d\mathbf{x} + \frac{1}{4\mu'(0)}\int_\Omega \left[\breve{\mathbf{I}}^t_{(1)}(0)\right]^2 d\mathbf{x} \\ &+\frac{1}{2}\left[\xi - (2\alpha c + \beta)R\right]\int_\Omega\int_0^\infty \mu'(s)\left[\mathbf{v}^t_r(s)\right]^2 ds d\mathbf{x} \leq 0 \end{aligned} \tag{5.60}$$

for any $\chi \in \mathcal{D}(A)$, with a proof similar to one considered in [10], we can prove that the range of $(A - I)$ is $\mathcal{H}$, then $A : \mathcal{D}(A) \to \mathcal{H}$ is a maximal dissipative operator on $\mathcal{H}$ (see [6] and [15]). Therefore, from the Lummer-Phillips theorem, the operator A generates a strongly continuous semigroup of linear contractions $S(t)$ on $\mathcal{H}$, so that the solutions of the system (5.2)-(5.7) can be written as

$$\chi(t) = S(t)\chi_0. \tag{5.61}$$

Moreover, by virtue of the inequality (5.50), we have

$$\int_0^\infty E(t)dt \equiv \frac{1}{2}\int_0^\infty \langle\chi(t), \chi(t)\rangle\, dt = \frac{1}{2}\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle\, dt < \infty \tag{5.62}$$

for any $\chi_0 \in \mathcal{H}$.

Hence, we use the following lemma due to Dakto [7].

Lemma 5.2. *Given a strongly continuous semigroup of linear contractions $S(t)$ on a Hilbert space $\mathcal{K}$, then, with two suitable constants M and ε, we have*

$$\langle S(t)y_0, S(t)y_0\rangle = Me^{-\varepsilon t}\langle y_0, y_0\rangle \qquad \forall y_0 \in \mathcal{K} \tag{5.63}$$

if and only if $\int_0^\infty \langle S(t)y_0, S(t)y_0\rangle\, dt < \infty$.

A necessary and sufficient condition that a strongly continuous semigroup of linear operator $S(t)$ satisfies the inequality

$$\langle S(t)\chi_0, S(t)\chi_0\rangle \leq M\exp(-\varepsilon t)\langle\chi_0, \chi_0\rangle \qquad \forall\chi_0 \in \mathcal{H}, \tag{5.64}$$

where M and ε are two suitable positive constants, is that the integral

$$\int_0^\infty \langle S(t)\chi_0, S(t)\chi_0\rangle\, dt < \infty, \quad \text{with } \chi_0 \in \mathcal{H}. \tag{5.65}$$

Therefore, from (5.62) and by Dakto's Lemma we obtain the inequality (5.19). ∎

References

[1] G. Amendola, Free energies for incompressible viscoelastic fluids,*Quart. Appl. Math.*, LXVIII (2) (2010), 349-374.

[2] G. Amendola, M. Fabrizio, Thermal convenction in a simple fluid with fading memory, *J. Math. Anal. Appl.*, 366 (2010), 444-459.

[3] G. Amendola, M. Fabrizio and J.M. Golden, *Thermodynamics of Materials with Memory*: *Theory and Applications*, Springer, New York, 2012.

[4] S. Breuer, E.T. Onat, On recoverable work in linear viscoelasticity, *Z. Angew. Math. Phys.,* 15 (1964), 13-21.

[5] B.D. Coleman, D.R. Owen, A mathematical foundation for thermodynamics, *Arch. Rational Mech. Anal.,* 54 (1974), 1-104.

[6] C.M. Dafermos, *Contraction Semi-groups and Trend to Equilibrium in Continuous Mechanics*, Proc. IUTAM/IMU. Conference of applications of functional analysis to mechanics. Lectures notes in mathematics, 503, Springer-Verlag, New York, 1975.

[7] R. Datko, Extending a theorem of A.M. Lyapunov to Hilbert space, *J. Math. Anal. Appl.,* 32 (1970), 610-616.

[8] L. Deseri, M. Fabrizio and J.M. Golden, The concept of a minimal state in viscoelasticity: new free energies and applications to PDE$_S$, *Arch. Rational Mech. Anal.*, **181** (1) (2006), 43-96.

[9] C.R. Doering, B. Eckhardt and J. Schumacher, Failure of energy stability in Oldroyd-B fluids at arbitrarily low Reynolds numbers, *J. Non-Newtonian Fluid Mech*., 135 (2006), 92-96.

[10] M. Fabrizio, C. Giorgi and A. Morro, A continuum theory for first-order phase transitions based on the balance of structure order, Math. *Methods Appl. Sci.* 31 (6) (2008), 627Ű653.

[11] M. Fabrizio, C. Giorgi and V. Pata, A new approch to equations with memory, *Arch. Rational Mech. Anal.,* 198 (2010), 189-232.

[12] M. Fabrizio, B. Lazzari, On asymptotic stability for linear viscoelastic fluids, *Diff. Integral Equat.*, 6 (3) (1993), 491-505.

[13] M. Fabrizio, A. Morro, Dissipativity and irreversibility of electromagnetic systems, *Math. Mod. Meth. Appl. Sci.*, 10 (2) (2000), 217-246.

[14] A. Lozinski, R.G. Owens, An energy estimate for the Oldroyd-B model: theory and applications, *J. Non-Newtonian Fluid Mech*, 112 (2003), 161-176.

[15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Lectures notes in mathematics, 10, University of Mariland, 1974.

[16] L. Preziosi, S. Rionero, Energy stability of steady shear flows of a viscoelastic fluid, *Int. J. Eng. Sci.,* 27 (1989), 1167-1181.

[17] M. Slemrod, An energy stability method for simple fluids, *Arch. Rational Mech. Anal*., 68 (1978), 1-18.

[18] B. Straughan, *The Energy Method, Stability, and Non Linear Convection*, 2nd edition, Springer-Verlag, New York, 2004.

Author information

Giovambattista Amendola, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127-Pisa, Italy.
E-mail: amendola@dma.unipi.it

Mauro Fabrizio, Dipartimento di Matematica, Piazza di Porta S. Donato 5, 40127-Bologna, Italy.
E-mail: mauro.fabrizio@unibo.it

Adele Manes, Dipartimento di Matematica, Largo Bruno Pontecorvo 5, 56127-Pisa, Italy.
E-mail: manes@dm.unipi.it

Received: December 29, 2012

Accepted: March 30, 2013