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A CENTRAL FINITE VOLUME SCHEME FOR BOND PRICING PROBLEMS

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Abstract. In the present work, we propose a new method to numerically approximate partial differential equations of bond pricing problems. Existing finite difference schemes are not always accurate at boundaries since the partial differential equation degenerates into hyperbolic. Furthermore, a different method is required for each bond pricing problem. Our proposed method is a new central black box finite volume scheme to solve bond pricing problems. It is a predictor corrector technique that uses cell averages. We first predict point values from non-oscillatory piecewise-linear reconstruction of cell averages. During the corrector step, we make use of the staggered averaging along with the predicted mid values to realise the evolution of these averages. Finally, numerical experiments are presented for illustrating the performance of our scheme for different bond pricing problems. We also show that approximations are bounded by their initial conditions.

1 Introduction

Bond pricing valuation is a subject of tremendous importance in modern financial theory and practice. A bond is a financial instrument or contract which is paid for up-front and yields a known cash dividend at fixed times during the life of the contract [9]. The cash dividend is usually called a coupon, and is often paid semiannually or annually. Bonds in general carry coupons and there also exists a special kind of bond without coupons known as the zero-coupon bond (ZCB) [3]. A ZCB is a bond which is bought at a lower price than its face value, with the face value repaid at time of maturity. Only, in some rare cases, the analytical solutions of ZCB pricing have been given [16, 6].

With rapidly growing complexity of financial products, various numerical methods have been developed for approximating bond pricing [3]. Wang [20] presented a novel numerical method for a degenerate Black-Scholes partial differential equation. The scheme is based on a fitted finite volume spatial discretization and an implicit time stepping technique. Extensions and other applications of the fitted volume method can be found in [10, 13, 4, 19, 5].

Chernogorova and Valkov [3] have numerically approximated a degenerate parabolic equation with dynamical boundary conditions of zero-coupon bond pricing. They implemented a finite volume method to discretize the differential problem. More on computational methods for problems arising in finance has been mentioned in the books [1, 14].

In this paper, we focus on finite volume methods for approximating bond pricing problem since classical since finite difference methods may fail to give accurate approximations near boundary [3]. Finite volume can be divided into upwind method and central method. The prototype of upwind method is the first order Godunov scheme in which a piecewise constant interpolant, based on previously computed averages, is evolved exactly to the next step according to the conservation law. Central method on the other hand can be viewed as a high extension of the Lax-Friedrich (LxF) scheme. In its staggered version, the LxF scheme is based on constructing a piecewise constant reconstruction which is then evolved exactly in time and finally projected on staggered cell averages [18].

In our work, we propose a new central finite volume (CFV) scheme based on the NT scheme [11] for bond pricing problems that has a black box approach. Our CFV scheme is a predictor-corrector technique where we predict point values using non-oscillatory piecewise-linear re-

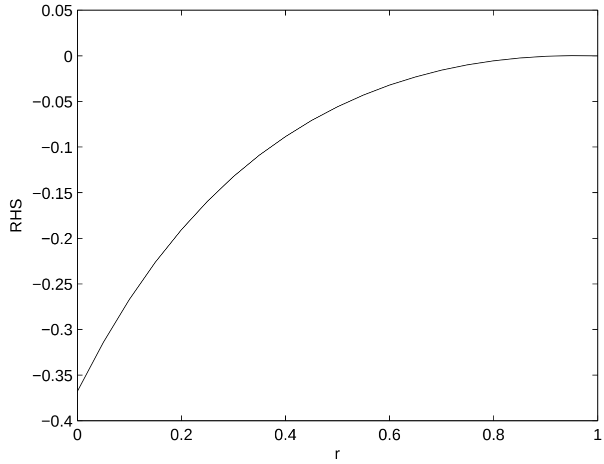

Figure 1. Evaluation of the right hand side of the parabolic equation (2.4).

constructions of given cell averages. During the corrector step, we make use of the staggered averaging along with the predicted mid-values to realise the evolution of these averages. The motivation for the construction of a new black box central finite volume scheme originates from many points.

- The approach of certain existing methods shows result in major contradiction with the bond pricing problem. For example, the model used in [3] was supposed to give a forward parabolic equation (FPE) that always has zero as right hand side. However, as illustrated in Figure 1 this is not the case for ZCB premium.
- Finite difference methods may fail to give accurate approximations near boundaries [3].
- Each different bond pricing problem require a new method.

This paper is organised as follows. In Section 2, we discuss the bond pricing partial differential equation form with basic assumptions followed by some basic definitions. In Section 3, we present the derivation of our proposed central finite volume scheme to solve the bond pricing problems followed by some numerical experiments in Section 4. Finally in Section 5, we make a concise conclusion.

2 Bond Pricing Partial Differential Equation

To fix ideas and notation, we consider a single factor model [3]. This factor is the instantaneous risk free interest rate r which is assumed to follow a stochastic process of the form

$$\mathrm{d}r = \theta(r)\mathrm{d}t + \omega(r)\mathrm{d}z, \tag{2.1}$$

where $\theta(r)$ is the instantaneous drift, $\omega(r)$ is the instantaneous volatility and dz is the increment of a Wiener process [8]. Since the spot rate, in practice, is never greater than a certain number, which is assumed R, and never less than or equal to zero, we suppose that $r \in [0, R]$. We also mention the following assumptions [3]:

Assumption 2.1. $\theta(r)$ is a Lipschitz function, satisfying

$$\theta(0) \geq 0, \quad \theta(R) \leq 0. \tag{2.2}$$

Assumption 2.2. $\omega(r)$ is a non-negative and smooth bounded function, satisfying

$$\omega(0) = \omega(R) = 0, \quad \omega(r) \geq 0 \quad r \in (0, R). \tag{2.3}$$

The degenerate parabolic equations are used in producing several models of mathematical finance [2, 7, 15]. The ZCB satisfying the following backward parabolic equation is taken into consideration [15]:

$$P_t + \frac{1}{2}\omega^2(r)P_{rr} + (\theta(r) + \lambda(t) + \omega(r))P_r - rP = 0, \quad (r,t) \in Q \equiv [0,R] \times [0,T), \tag{2.4}$$

$$P(r,T) = Z, \tag{2.5}$$

where T is the maturity and Z is a fixed constant. Function $\lambda(t)$ in (2.4) is called the market price risk. For the given functions θ, ω and λ, the problem of ZCB pricing consists of the determination of the solution $P(r,t)$ from equation (2.4), which is often referred to as a direct problem.

Being different from the classical parabolic equation in which the principal coefficient is assumed to be strictly positive, the parabolic equation (2.4) goes with the second order differential equations with non negative characteristic form. The main difficulty in such type of problems is the degeneracy. Without any difficulty it can be pointed out that at $r = 0$ and $r = R$, equation (2.4) degenerates into hyperbolic equation with positive and negative characteristics respectively

$$\frac{\partial P}{\partial t} + \theta(0)\frac{\partial P}{\partial r} = 0, \tag{2.6}$$

$$\frac{\partial P}{\partial t} + \theta(R)\frac{\partial P}{\partial r} = RP. \tag{2.7}$$

By the Fichera's theory [12] for degenerate parabolic equations, we have that at the degenerate boundaries $r = 0$ and $r = R$ boundary conditions should not be given. Therefore, the maturity data $P(r,T)$ finalises the solution $P(r,t)$ of problem (2.4) and (2.5) in an exceptional way.

By making the change of variable $\hat{t} = T - t$ and letting $\hat{\lambda}(t) = \lambda(T-t)$, when coming back to t, the function P satisfies the following parabolic equation

$$\frac{\partial P}{\partial t} - \frac{\omega^2(r)}{2}\frac{\partial^2 P}{\partial r^2} - (\theta(r) + \lambda(t) + \omega(r))\frac{\partial P}{\partial r} + rP = 0, \quad (r,t) \in Q \tag{2.8}$$

with initial condition

$$P(r,0) = P_0(r). \tag{2.9}$$

For the concrete model (2.4) and (2.5), we consider $P_0(r) = Z$.

If the functions θ and ω satisfy Assumptions 2.1 and 2.2 and the initial data $P_0(r)$ is a continuous function then there exists a classical solution of the problem (2.8) and (2.9); P has continuous first derivative with respect to t and second derivative with respect to r up to the boundary ∂Q and satisfies Equation (2.8) [12, 7]. In Section 4, we show that our scheme is bounded by the following lemma [3]:

Lemma 2.3. *Let Assumptions 2.1 and 2.2 hold. Then,*

$$0 \le P(r,t) \le P_0(r). \tag{2.10}$$

The Dirichlet problem on the domain $(0,X) \times (0,T)$, $0 < X < \infty$ for the Black-Scholes equation [2, 17] has the form (2.4) with coefficients:

$$\frac{1}{2}\sigma^2 r^2 \quad \text{at} \quad \frac{\partial^2 P}{\partial r^2} \quad \text{and} \quad (d(t) - D(x,t))r \quad \text{at} \quad \frac{\partial P}{\partial r}. \tag{2.11}$$

In [17], P denotes the value of a European call or put option, $\sigma = \text{const} > 0$ denotes the volatility of the asset, the interest rate are denoted by r and D are the dividends. It is assumed that $r > D$.

Later on, in this paper, we will focus on the following fully conservative form of equation (2.4) [3]:

$$\frac{\partial P}{\partial t} - \frac{\partial}{\partial r}\left(\frac{\omega^2(r)}{2}\frac{\partial P}{\partial r} + (\theta(r) + (\lambda(t) - \omega')\omega)P\right) + (r + \theta' + \lambda(t)\omega' - (\omega\omega')')P = 0. \tag{2.12}$$

2.1 Notations and Basic Definitions

We consider a uniform spatial grid where the cell $I_j = [r_{j-\frac{1}{2}}, r_{j+\frac{1}{2}}]$ has width h and let $r_j = \frac{1}{2}(r_{j-\frac{1}{2}} + r_{j+\frac{1}{2}})$ be the mid-cell grid point of I_j. Also let $\Delta t = t^{n+1} - t^n$ where t^n is the n^{th} time level and denote $P_j^n \approx P(r_j, t^n)$. Let the approximation to the cell average of P over I_j be given by

$$\bar{P}_j^n = \frac{1}{h}\int_{I_j} P(r_j, t^n)\,\mathrm{d}r.$$

3 Central Finite Volume Scheme

In this section, we discuss about the construction of our new central finite volume (CFV) scheme for bond pricing problems. We consider the equation (2.12) where for the given functions θ, ω and λ, the problem of bond pricing consists of the determination of the solution $P(r,t)$ [3]. To approximate solutions of (2.12), we introduce a piecewise-linear approximate solution at the discrete time levels, $t^n = n\Delta t$, based on the linear functions $S_j(r, t^n)$ which are supported at the cells I_j,

$$\begin{aligned} P(r,t)|_{t=t^n} =& \sum_j S_j(r,t^n)\chi_j(r), \\ =& \sum_j \left[\bar{P}_j^n + P_j'\left(\frac{r-r_j}{\Delta r}\right)\right]\chi_j(r), \end{aligned} \tag{3.1}$$

where $\chi_j(r)$ is a characteristic function of the cell and $P_j' \sim h\cdot\partial_r P(r_j, t^n) + \mathcal{O}(h^2)$.

Integrating (2.12) over $[(r_j, r_{j+1}) \times (t^n, t^{n+1})]$, we obtain

$$\begin{aligned} &\int_{r_j}^{r_{j+1}} P(r,t^{n+1})\,\mathrm{d}r - \int_{r_j}^{r_{j+1}} P(r,t^n)\,\mathrm{d}r \\ &- \int_{t^n}^{t^{n+1}} \left[\left(\frac{\omega^2(r)}{2}\frac{\partial P}{\partial r} + (\theta(r) + (\lambda(t) - \omega')\omega)P\right)\right]_{r_j}^{r_{j+1}} \mathrm{d}t \\ &+ \int_{t^n}^{t^{n+1}}\int_{r_j}^{r_{j+1}} (r + \theta' + \lambda(t)\omega' - (\omega'\omega)')P\ \mathrm{d}r\mathrm{d}t = 0. \end{aligned} \tag{3.2}$$

We let $\int_{r_j}^{r^{j+1}} P(r,t^{n+1})\,\mathrm{d}r = h\bar{P}_{j+\frac{1}{2}}^{n+1}$, and integrate the second part of (3.2):

$$\begin{aligned} \int_{r_j}^{r_{j+1}} P(r,t^n)\,\mathrm{d}r =& \int_{r_j}^{r_{j+\frac{1}{2}}} \left(\bar{P}_j^n + (r-r_j)\frac{P_j'}{h}\right)\mathrm{d}r + \int_{r_{j+\frac{1}{2}}}^{r_{j+1}} \left(\bar{P}_{j+1}^n + (r-r_{j+1})\frac{P_{j+1}'}{h}\right)\mathrm{d}r, \\ =& h\left(\frac{1}{2}(\bar{P}_j^n + \bar{P}_{j+1}^n) + \frac{1}{8}(P_j' - P_{j+1}')\right). \end{aligned} \tag{3.3}$$

We subsequently use the mid point rule for the time integration of equation (3.2),

$$\int_{t^n}^{t^{n+1}} \frac{\omega_j^2}{2}P_j' + (\theta_j + (\lambda - \omega_j')\omega_j)P_j\,\mathrm{d}t \approx \frac{\Delta t}{2}\left(\frac{\omega_j^2}{2}(P_j^{n+\frac{1}{2}})' + (\theta_j + (\lambda^{n+\frac{1}{2}} - \omega_j')\omega_j)P_j^{n+\frac{1}{2}}\right). \tag{3.4}$$

Finally, we replace P by $\left(\bar{P}_j + (r-r_j)\frac{P_j'}{h}\right)$ in the last part of equation (3.2) and on integrating we obtain,

$$\begin{aligned}
&\int_{t^n}^{t^{n+1}}\int_{r_j}^{r_{j+1}} (r+\theta'+\lambda\omega'-(\omega'\omega)')P\ \mathrm{d}r\mathrm{d}t \\
&\qquad\approx \int_{t^n}^{t^{n+1}}\int_{r_j}^{r_{j+\frac{1}{2}}} (r+\theta'+\lambda\omega'-(\omega'\omega)')\left(\bar{P}_j+(r-r_j)\frac{P'_j}{h}\right)\mathrm{d}r\mathrm{d}t \\
&\qquad\quad+\int_{t^n}^{t^{n+1}}\int_{r_{j+\frac{1}{2}}}^{r_{j+1}} (r+\theta'+\lambda\omega'-(\omega'\omega)')\left(\bar{P}_{j+1}+(r-r_{j+1})\frac{P'_{j+1}}{h}\right)\mathrm{d}r\mathrm{d}t.
\end{aligned}\tag{3.5}$$

Simplifying equation (3.5), we get

$$\begin{aligned}
&\frac{\Delta t}{2}\left[\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)\left(\bar{P}_j^{n+\frac{1}{2}}+\frac{1}{2}(P_j^{n+\frac{1}{2}})'\right)\right] \\
&-\frac{\Delta t}{2}\left[\left(\frac{r_j^2}{2}+\theta_j+(\lambda^{n+\frac{1}{2}}-\omega'_j)\omega_j\right)\bar{P}_j^{n+\frac{1}{2}}-\frac{(P_j^{n+\frac{1}{2}})'}{h}\int_{r_j}^{r_{j+\frac{1}{2}}}\left(\frac{r^2}{2}+\theta+(\lambda^{n+\frac{1}{2}}-\omega')\omega\right)\mathrm{d}r\right] \\
&+\frac{\Delta t}{2}\left[\left(\frac{r^2_{j+1}}{2}+\theta_{j+1}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+1})\omega_{j+1}\right)\bar{P}_{j+1}^{n+\frac{1}{2}}\right] \\
&-\frac{\Delta t}{2}\left[\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)\left(\bar{P}_{j+1}^{n+\frac{1}{2}}-\frac{1}{2}(P_{j+1}^{n+\frac{1}{2}})'\right)\right] \\
&-\frac{\Delta t}{2}\left[\frac{(P_{j+1}^{n+\frac{1}{2}})'}{h}\int_{r_{j+\frac{1}{2}}}^{r_{j+1}}\left(\frac{r^2}{2}+\theta+(\lambda^{n+\frac{1}{2}}-\omega')\omega\right)\mathrm{d}r\right].
\end{aligned}$$

Hence, when adding simplified terms of (3.2) together, we obtain an expression as shown below:

$$\bar{P}_{j+1}^{n+\frac{1}{2}}=\frac{1}{2}(\bar{P}_j^n+\bar{P}_{j+1}^n)-\frac{1}{8}(P'_{j+1}-P'_j)+\frac{\Delta t}{2h}A,$$

where,

$$\begin{aligned}
A=&(P_{j+1}^{n+\frac{1}{2}})'\left[\left(\frac{\omega^2_{j+1}}{2h}-\frac{1}{2}\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)\right)\right] \\
&+(P_{j+1}^{n+\frac{1}{2}})'\left[\frac{1}{h}\int_{r_{j+\frac{1}{2}}}^{r_{j+1}}\left(\frac{r^2}{2}+\theta+(\lambda^{n+\frac{1}{2}}-\omega')\omega\right)\mathrm{d}r\right] \\
&+(P_j^{n+\frac{1}{2}})'\left[-\frac{\omega_j^2}{2h}-\frac{1}{2}\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)\right] \\
&+(P_j^{n+\frac{1}{2}})'\left[\frac{1}{h}\int_{r_j}^{r_{j+\frac{1}{2}}}\left(\frac{r^2}{2}+\theta+(\lambda^{n+\frac{1}{2}}-\omega')\omega\right)\mathrm{d}r\right] \\
&+\bar{P}_j^{n+\frac{1}{2}}\left[-\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)+\frac{r_j^2}{2}\right] \\
&+\bar{P}_{j+1}^{n+\frac{1}{2}}\left[-\left(\frac{r^2_{j+\frac{1}{2}}}{2}+\theta_{j+\frac{1}{2}}+(\lambda^{n+\frac{1}{2}}-\omega'_{j+\frac{1}{2}})\omega_{j+\frac{1}{2}}\right)+\frac{r^2_{j+1}}{2}\right].
\end{aligned}$$

We let $\phi(r,t)=\theta(r)+(\lambda(t)-\omega'(r))\omega(r)$ and $\Phi=\int\phi\mathrm{d}r$. We note that Φ can be evaluated exactly for the examples considered, and for the sake of simplicity there is no need to find any constant of integration. On simplifying the expression for A we obtain,

$$\begin{aligned}A = &-\frac{h}{2}r_{j+1}\overline{P}_{j+1}^{n+\frac{1}{2}} - \frac{h}{2}r_j\overline{P}_j^{n+\frac{1}{2}} + (\overline{P}_{j+1}^{n+\frac{1}{2}} - \overline{P}_j^{n+\frac{1}{2}})\left[\frac{h^2}{8} - \phi_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right]\\ &+ \frac{1}{8h}(P_{j+1}^{n+\frac{1}{2}})'\left(4\omega_{j+1}^2 + h^2 r_{j+1} + 8\Phi_{j+1}^{n+\frac{1}{2}}\right) - \frac{1}{8h}(P_j^{n+\frac{1}{2}})'\left(4\omega_j^2 + h^2 r_j + 8\Phi_j^{n+\frac{1}{2}}\right)\\ &+ \left((P_{j+1}^{n+\frac{1}{2}})' + (P_j^{n+\frac{1}{2}})'\right)\left[-\frac{\phi_{j+\frac{1}{2}}^{n+\frac{1}{2}}}{2} - \frac{h^2}{24} - \frac{1}{h}\Phi_{j+\frac{1}{2}}^{n+\frac{1}{2}}\right].\end{aligned}$$

In general, we obtain the central scheme as shown below:

$$\overline{P}_{j+\frac{1}{2}}^{n+1} = (F_j + F_{j+1}) - (G_{j+1} - G_j) + Q_{j+\frac{1}{2}},$$

where

$$\begin{aligned}F_j =&\frac{\overline{P}_j^n}{2} - \frac{\Delta t}{24}\left(12 r_j \overline{P}_j^{n+\frac{1}{2}} + h\overline{P}_j^{n+\frac{1}{2}}\right),\\ G_j =&\frac{(P_j^n)'}{8} - \frac{\Delta t h}{8}(\overline{P}_j^{n+\frac{1}{2}}) - \frac{\Delta t}{8h^2}(\overline{P}_j^{n+\frac{1}{2}})'\left(4\omega_j^2 + h^2 r_j + 8\Phi_j^{n+\frac{1}{2}}\right),\\ Q_{j+\frac{1}{2}} =&\frac{\Delta t}{h^2}\left((\overline{P}_j^{n+\frac{1}{2}})' - (\overline{P}_{j+1}^{n+\frac{1}{2}})'\right)\Phi_{j+\frac{1}{2}}^{n+\frac{1}{2}}\\ &- \frac{\Delta t}{h}\left((\overline{P}_j^{n+\frac{1}{2}}) - (\overline{P}_{j+1}^{n+\frac{1}{2}}) + \frac{(\overline{P}_j^{n+\frac{1}{2}})' + (\overline{P}_{j+1}^{n+\frac{1}{2}})'}{2}\right)\phi_{j+\frac{1}{2}}^{n+\frac{1}{2}}.\end{aligned}$$

Time integrals are computed by second order accurate mid point quadrature rule. Taylor expansion is used to predict the required mid values of P:

$$P_j^{n+\frac{1}{2}} \approx \overline{P}_j^n + \frac{\Delta t}{2}(P_j^n)_t,$$

and we approximate $(P_j^n)_t$ from (2.4):

$$\begin{aligned}P_j^{n+\frac{1}{2}} \approx&\overline{P}_j^n + \frac{\Delta t}{2}\left(\frac{\omega^2}{2}P_{rr} + (\theta + \lambda\omega)P_r - rP\right),\\ \approx&P_j^n + \frac{\Delta t}{2}\left(\frac{\omega_j^2}{2h^2}(P_j^n)'' + (\theta_j + \lambda^n\omega_j)\frac{P_j'}{h} - r_j P_j\right),\end{aligned}$$

where

$$\begin{aligned}P_j' =&\frac{(P_{j+1} - P_{j-1})}{2},\\ P_j'' =&(P_{j+1} - 2P_j + P_{j-1}).\end{aligned}$$

In our present work, we use the second order interpolation at the boundaries to calculate fictitious points, which are points located outside the boundary.

4 Numerical Experiments

Experiment 1: We perform numerical experiment for our scheme with the following coefficients:

$$\theta(r) = \alpha + \beta r, \quad \omega(r) = \sigma r^\gamma \quad \text{and} \quad \lambda = 0,$$

where $\alpha = 0.02$, $\beta = -1$, $\gamma = 1$ and $\sigma = 0.35$. We take $R = 0.2$; as a sufficiently high level of the short rate; it corresponds to 20%; and $T = 1$; the approximate analytical solution, which we

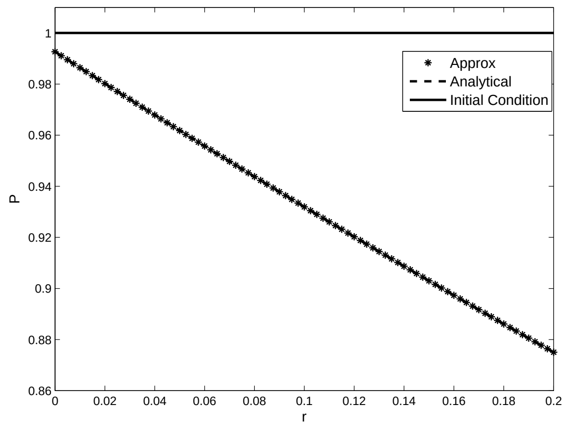

Figure 2. Analytical solution $U = \exp(\ln P^{ap})$, numerical solution for $N = 80$, $T = 1$.

use at the right boundary condition, as well as for the comparison, has the accuracy derived for $t \to 0^+$; hence T cannot be too large. We approximately solve the model problem with known analytical solution given by [16] with initial condition $P = 1$:

$$U = \exp(\ln P^{ap}),$$

where

$$\ln P^{ap} = -rB + \frac{\alpha}{\beta}(t - B) + (r^{2\gamma} + qt)\frac{\sigma^2}{4\beta}\left(B^2 + \frac{2}{\beta}(t - B)\right)$$
$$-q\frac{\sigma^2}{8\beta^2}\left(B^2(2\beta t - 1) - 2B\left(2t - \frac{3}{\beta}\right) + 2t^2 - \frac{6t}{\beta}\right),$$

and

$$B = (\exp(\beta t) - 1)/\beta, \qquad q(r) = \gamma(2\gamma - 1)\sigma^2 r^{2(2\gamma-1)} + 2\gamma r^{2\gamma-1}(\alpha + \beta r).$$

The calculations are performed with constant time step, $dt = 0.00001$. From Figure 2, we clearly see that the approximated solution are near the analytical solution.

Experiment 2: We perform numerical experiment for our scheme with the following coefficients [3]:

$$\omega(r) = r(R - r), \quad \theta(r) = r(R - r) \quad \text{and} \quad \lambda(t) = 0.25(1 + t^2)^{-1}.$$

We approximate the model problem with initial condition $P = \exp(-r)$. We take $R = 1$ and $T = 1$. The calculations are performed with constant time step $dt = 0.00001$.

Experiment 3: We solve another problem same as experiment 2 but with coefficients taken as follows:

$$\omega(r) = r(R - r), \quad \theta(r) = r(R - r)(0.5R - r) \quad \text{and} \quad \lambda(t) = 0.25(1 + t^2)^{-1}.$$

Experiment 4: We solve another problem same as experiment 2, which according to [3] is considered as a harder case. The coefficients taken are:

$$\omega(r) = r(R - r), \quad \theta(r) = (R/2 - r) \quad \text{and} \quad \lambda(t) = 0.25(1 + t^2)^{-1}.$$

From Figures 3, 4 and 5, we clearly observe that the lemma 2.3 holds.

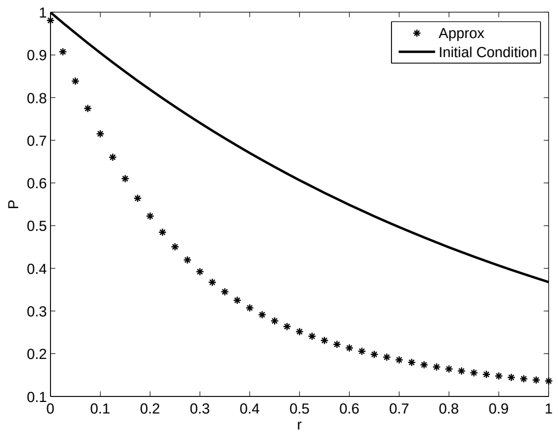

Figure 3. Experiment 2: Numerical solution for $N = 40$ and $T = 1$ with initial condition $P = \exp(-r)$.

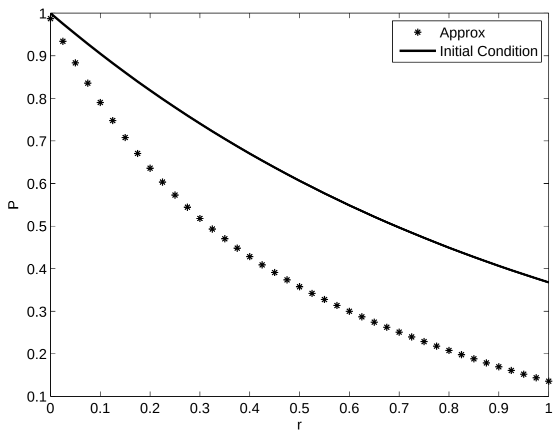

Figure 4. Experiment 3: Numerical solution for $N = 40$ and $T = 1$ with initial condition $P = \exp(-r)$.

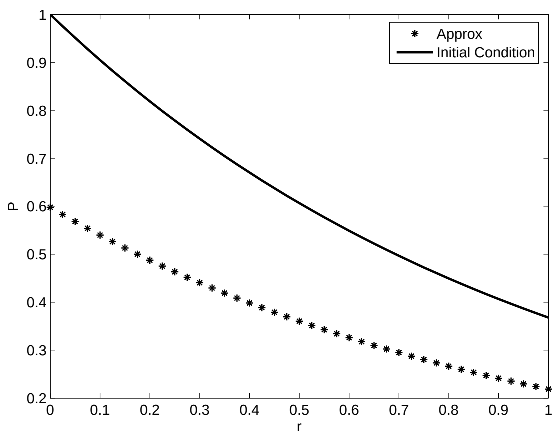

Figure 5. Experiment 4: Numerical solution for $N = 40$ and $T = 1$ with initial condition $P = \exp(-r)$.

5 Conclusion

In this paper, we presented a new central finite volume method for approximating zero coupon bond pricing problems. The scheme is constructed in a similar way as the NT scheme [11]. The strategy of the our proposed scheme is a predictor corrector technique. We first predict point values which are based on non oscillatory piecewise-linear reconstruction from cell averages. For the corrector step, we make use of the staggered averaging along with the predicted mid values to realise the evolution of these averages. We performed numerical experiments for a meaningful set of parameters. We observed that the approximation values are near the analytical solution from our first experiment. Our central finite volume scheme is bounded by the initial condition. This work can be extended in the sense that the stability of our scheme needs to be studied as we have to use a relatively small time step to solve the bond pricing problems.

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