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GORENSTEIN n-FLAT DIMENSION WITH RESPECT TO A SEMIDUALIZING MODULE

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Abstract. In this paper, we study some properties of G_C-n-flat modules, where C is a semidualizing module and we investigate the relation between the m-G_C-yoke with the m-C-yoke of a module as well as the relation between the G_C-n-flat resolution and the n-flat resolution of a module over GF_n-closed rings. We also obtain a criterion for computing the G_C-n-flat dimension of modules.

1 Introduction

Auslander and Bridger introduced in [1] the G-dimension for finitely generated modules over Noetherian rings. Then, Enochs and Jenda introduced in [4] the Gorenstein projective dimension for arbitrary modules over a general ring, which is a generalization of the G-dimension. Foxby [5], Vasconcelos [14] and Golod [6] independently initiated the study of semidualizing modules, which are common generalizations of dualizing modules and finitely generated projective modules of rank one. Christensen [3] defined semidualizing complexes, and studied them in the context of derived categories. Recently, Holm and White [9] extended the definition of the semidualizing module to a pair of arbitrary associative rings. Especially, they defined the so-called C-projective, C-injective and C-flat modules, to characterize the Auslander class $\mathcal{A}_C(R)$ and the Bass class $\mathcal{B}_C(R)$, with respect to a semidualizing module C. The notion of C-projective (C-injective, C-flat) modules is important for the study of the relative homological algebra with respect to semidualizing modules. For example, Holm and Jørgension [8] used these modules to define C-Gorenstein projective (resp., injective, flat) modules and introduced the notions of C-Gorenstein projective (resp., injective, flat) dimensions. Further, White introduced in [15] the G_C-projective modules and gave a functorial description of the G_C-projective dimension of modules with respect to a semidualizing module C over a commutative ring; and in particular, many classical results about the Gorenstein projectivity of modules were generalized in [15]. In this paper, we give a functorial description of the G_C-n-flat dimension of modules with respect to a semidualizing module.

This paper is organized as follows. In Section 2, we recall some notions and definitions which will be needed in the later sections. In Section 3, we introduce the notions of C-n-flat, C-n-absolutely pure R-modules and using these modules, we further introduce G_C-n-flat and G_C-n-absolutely pure R-modules. Also, we establish the relation between the m-G_C-yoke with the m-C-yoke of a module as well as the relation between the G_C-n-flat resolution and the n-flat resolution of a module over a GF_n-closed ring.

In Section 4, we get some properties of G_C-n-flat dimension of modules. In particular, as an application of the results obtained in Section 3, we get a criterion for computing such a dimension. Let R be a GF_n-closed ring and let M be a left R-module and $m \geq 0$. We prove that the G_C-n-flat dimension of M is at most m if and only if for every non-negative integer t such that $0 \leq t \leq m$, there exists an exact sequence $0 \to X_m \to \cdots \to X_1 \to X_0 \to M \to 0$ in R-Mod such that X_t is G_C-n-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$.

2 Preliminaries

Unless stated otherwise, throughout this paper all rings are associative with identity and all modules are if not specified otherwise, left R-modules. Let R be a ring, we denote by R-Mod (resp., Mod-R) the category of left (resp., right) R-modules. A left R-module M is called n-flat [10] if $Tor_1^R(N, M) = 0$ holds for all finitely presented right R-modules N with projective dimension $\leq n$ and a right R-module M is called n-absolutely pure [10] if $Ext_R^1(N, M) = 0$ holds for all finitely presented right R-modules N with projective dimension $\leq n$.

First we recall some notions from [9, 15].

Definition 2.1. [15] A degreewise finite projective (resp., free) resolution of an R-module M is a projective (resp., free) resolution P of M such that each P_i is a finitely generated projective (resp., free). Note that M admits a degreewise finite projective resolution if and only if it admits a degreewise finite free resolution. However, it is possible for a module to admit a bounded degreewise finite projective resolution but not to admit a bounded degreewise finite free resolution. For example, if $R = k_1 \oplus k_2$, where k_1 and k_2 are fields, then $M = k_1 \oplus 0$ is a projective R-module, but it does not admit a bounded free resolution.

Definition 2.2. [9] Let R and S be rings. An (S, R)-bimodule C is called semidualizing if the following conditions are satisfied:

(1) ${}_SC$ admits a degreewise finite S-projective resolution;

(2) C_R admits a degreewise finite R^{op}-projective resolution;

(3) The homothety map ${}_SS_S \to Hom_{R^{op}}(C, C)$ is an isomorphism;

(4) The homothety map ${}_RR_R \to Hom_S(C, C)$ is an isomorphism;

(5) $Ext_S^i(C, C) = 0$ for any $i \geq 1$;

(6) $Ext_{R^{op}}^i(C, C) = 0$ for any $i \geq 1$.

Definition 2.3. [9] Let C be a semidualizing module for a ring R. An R-module is C-projective if it has the form $C \otimes_R P$ for some projective module P. An R-module is called C-injective if it has the form $Hom_R(C, I)$ for some injective module I. Set

$$\mathcal{P}_C(R) = \{C \otimes_R P \,|\, P \, is \; R - projective\},$$

and

$$\mathcal{I}_C(R) = \{Hom_R(C, I) \,|\, I \, is \; R - injective\}.$$

Definition 2.4. [9] An R-module is called C-flat if it has the form $C \otimes_R F$ for some flat module F. Set $\mathcal{F}_C(R) = \{C \otimes_R F \,|\, F$ is R-flat$\}$.

Setting $C = R$ in the above definitions, we see that $\mathcal{P}_C(R)$, $\mathcal{I}_C(R)$ and $\mathcal{F}_C(R)$ are the classes of ordinary projective, injective and flat R-modules, which we usually denote $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ respectively.

Definition 2.5. [16] An R-module is C-FP-injective if it has the form $Hom_R(C, E)$ for some FP-injective module E. Set $\mathcal{FP}_C(R) = \{Hom_R(C, E) \,|\, E$ is R-FP-injective$\}$.

Any semidualizing module defines two important classes of modules, namely the Auslander and Bass classes, with a certain nice duality property.

Definition 2.6. [15] The Auslander class $\mathcal{A}_C(R)$ with respect to C consists of all modules M satisfying:

(A1) $Tor_i^R(C, M) = 0$ for any $i \geq 1$;

(A2) $Ext_R^i(C, C \otimes_R M) = 0$ for any $i \geq 1$; and

(A3) The natural evaluation homomorphism $\mu_M : M \to Hom_R(C, C \otimes_R M)$ is an isomorphism.

The Bass class $\mathcal{B}_C(R)$ with respect to C consists of all modules N satisfying:

(B1) $Ext_R^i(C, N) = 0$ for any $i \geq 1$;

(B2) $Tor_i^R(C, Hom_R(C, N)) = 0$ for any $i \geq 1$; and

(B3) The natural evaluation homomorphism $\nu_N : C \otimes_R Hom_R(C, N) \to N$ is an isomorphism.

Definition 2.7. [13] A left R-module M is said to be Gorenstein n-flat, if there exists an exact sequence of n-flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong Im(F_0 \to F^0)$ and such that $E \otimes_R -$ leaves the sequence exact whenever E is an n-absolutely pure right R-module.

Definition 2.8. [13] A right R-module M is said to be Gorenstein n-absolutely pure, if there exists an exact sequence of n-absolutely pure right R-modules

$$\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$$

such that $M \cong Im(A_0 \to A^0)$ and such that $Hom_R(E, -)$ leaves the sequence exact whenever E is an n-absolutely pure right R-module.

Definition 2.9. [2] Let R be a ring and let $\mathfrak{X}$ be a class of left R-modules.

(1) $\mathfrak{X}$ is closed under extensions: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions A and C are in $\mathfrak{X}$ implies B is in $\mathfrak{X}$.

(2) $\mathfrak{X}$ is closed under kernels of epimorphisms: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions B and C are in $\mathfrak{X}$ implies A is in $\mathfrak{X}$.

(3) $\mathfrak{X}$ is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms. i.e., for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.

Definition 2.10. [12] Let R be a ring. We call R GF_n-closed if the class of Gorenstein n-flat R-modules is closed under extensions.

3 G_C-n-flat modules

In this section, we first introduce C-n-flat and C-n-absolutely pure modules as follows:

Definition 3.1. (1) An R-module is called C-n-flat if it has the form $C \otimes_R F$ for some n-flat module F. Set $\mathcal{F}_C^n(R) = \{C \otimes_R F \mid {}_RF \text{ is } n\text{-flat }\}$.

(2) An R-module is C-n-absolutely pure if it has the form $Hom_R(C, E)$ for some n-absolutely pure module E. Set $\mathcal{A}b_C^n(R) = \{Hom_R(C, E) \mid E \text{ is } n\text{-absolutely pure } R\text{-module }\}$.

Let $M \in R\text{-}Mod$. M^I (resp., $M^{(I)}$) is the direct product (resp., sum) of copies of a module M indexed by a set I. We denote Add_RM (resp., $Prod_RM$) the subclass of $R\text{-}Mod$ consisting of all modules isomorphic to direct summands of direct sums (resp., direct products) of copies of M. We start with the following proposition

Proposition 3.2. $\mathcal{F}_C^n(R) = Add_RC$.

Proof. It is clear that $\mathcal{F}_C^n(R) \subseteq Add_RC$. Now, we show $Add_RC \subseteq \mathcal{F}_C^n(R)$. For any $M \in Add_RC$, there exists $N \in R\text{-}Mod$ such that $M \oplus N \cong C^{(J)}$ for some cardinal J. Note that $\mathcal{B}_C(R)$ is closed under direct sums and direct summands by [9, Proposition 4.2]. Since $C \cong C \otimes_R R \in \mathcal{B}_C(R)$ by [9, Lemma 5.1], both $C^{(J)}$ and M are in $\mathcal{B}_C(R)$. Since $Hom_R(C, M) \oplus Hom_R(C, N) \cong Hom_R(C, C^{(J)}) \cong R^{(J)}$, $Hom_R(C, M) \in R\text{-}Mod$ is n-flat. Thus, $M \in \mathcal{F}_C^n(R)$ by [9, Lemma 5.1]. □

Definition 3.3. A complete $\mathcal{F}\mathcal{F}_C^n$-resolution is a $\mathcal{A}b_C^n(R) \otimes_R -$ exact exact sequence:

$$\mathcal{X} : \cdots \to F_1 \to F_0 \to C \otimes_R F^0 \to C \otimes_R F^1 \to \cdots \tag{1}$$

in $R\text{-}Mod$ with all F_i and F^i n-flat. A module $M \in R\text{-}Mod$ is called G_C-n-flat if there exists a complete $\mathcal{F}\mathcal{F}_C^n$-resolution as in (1) with $M = Coker(F_1 \to F_0)$. Set $\mathcal{G}\mathcal{F}_C^n(R)$ is the class of G_C-n-flat modules in $R\text{-}Mod$.

Definition 3.4. A complete $\mathcal{A}b_C^n$-resolution is a $Hom_R(\mathcal{A}b_C^n(R), -)$ exact exact sequence:

$$\mathcal{Y} : \cdots \to E_1 \to E_0 \to E^0 \to E^1 \to \cdots \tag{2}$$

in $Mod\text{-}R$ with all $E_i \in \mathcal{A}b_C^n(R)$ and E^i n-absolutely pure. A module $M \in Mod\text{-}R$ is called G_C-n-absolutely pure if there exists a complete $\mathcal{A}b_C^n$-resolution as in (2) with $M = Im(E_0 \to E^0)$. Set $\mathcal{G}\mathcal{A}b_C^n(R)$ is the class of G_C-n-absolutely pure modules in $Mod\text{-}R$.

It is trivial that in case ${}_RC_R = {}_RR_R$, the G_C-n-flat modules are just the usual Gorenstein n-flat modules.

Using the definition, we immediately get the following results.

Proposition 3.5. *If $(F_i)_{i\in I}$ is a family of G_C-n-flat modules, then $\bigoplus F_i$ is G_C-n-flat.*

Proposition 3.6. *A module M is G_C-n-flat if and only if $Tor^R_{\geq 1}(Hom_R(C,E),M) = 0$ and M admits a $\mathcal{F}^n_C$-resolution Y with $Hom_R(C,E)\otimes_R Y$ exact for any n-absolutely pure E.*

Theorem 3.7. *If M is a G_C-n-flat R-module then M^+ is a G_C-n-absolutely pure R-module.*

Proof. Let M be a G_C-n-flat R-module, there exists a C-n-flat R-modules F^0, F^1, $\cdots$, together with an exact sequence

$$\mathcal{X} : 0 \to M \to C\otimes_R F^0 \to C\otimes_R F^1 \to \cdots .$$

Then

$$\mathcal{X}^+ : \cdots \to Hom_R(C,F^{1+}) \to Hom_R(C,F^{0+}) \to M^+ \to 0$$

is exact and each F^{i+} is an n-absolutely pure R-module. Let J be any n-absolutely pure R-module. Then

$$Ext^i_R(Hom_R(C,J),M^+) \cong Tor^R_i(Hom_R(C,J),M)^+ = 0 \ \forall\ i \geq 1,$$
$$Hom_R(Hom_R(C,J),\mathcal{X}^+) \cong (Hom_R(C,J)\otimes_R \mathcal{X})^+$$

is exact. Hence M^+ is a G_C-n-absolutely pure R-module. □

The following result is due to [11].

Proposition 3.8. *Let C be a semidualizing R-module. Then the class $\mathcal{GF}^n_C(R)$ is closed under kernels of epimorphisms and extensions.*

Proposition 3.9. *If F is n-flat R-module, then F and $C\otimes_R F$ are G_C-n-flat. Thus, every R-module admits a G_C-n-flat resolution.*

Proof. Follows from [8, Example 2.8(a) and (b)] and since the class of G_C-n-flat modules contains the class of n-flat modules, every R-module admits a G_C-n-flat resolution. □

Definition 3.10. [9] A semidualizing module C is faithfully semidualizing if and only if for any R-module N, the condition $Hom_R(C,N) = 0$ implies $N = 0$.

Theorem 3.11. *Let C be a semidualizing module, then the class $\mathcal{GF}^n_C(R)$ of G_C-n-flat R-modules is projectively resolving and closed under direct summands.*

Proof. Using the dual of Theorem 2.8 in [15] and together with the [11, Lemma 5.2], we see that $\mathcal{GF}^n_C(R)$ is projectively resolving. Now, comparing Proposition 2.5 with Proposition 1.4 in [7], we get $\mathcal{GF}^n_C(R)$ is closed under direct summands. □

Theorem 3.12. *Let R be commutative n-coherent and C is faithfully semidualizing R-module. If $M_0 \to M_1 \to M_2 \to \cdots$ is a sequence of G_C-n-flat R-modules, then the direct limit $\varinjlim M_m$ is again G_C-n-flat.*

Proof. By [9, Proposition 5.3], the class $\mathcal{F}^n_C(R)$ is preenveloping on the category of R-modules. So we pick for each m a co-proper right $\mathcal{F}^n_C$-resolution F_m of M_m, as illustrated in the next diagram.

$$\begin{tikzcd}
F_0 \arrow[r, equal] \arrow[d] & 0 \arrow[r] & M_0 \arrow[r] \arrow[d] & C\otimes_R F_0^0 \arrow[r] \arrow[d] & C\otimes_R F_0^1 \arrow[r] \arrow[d] & \cdots \\
F_1 \arrow[r, equal] \arrow[d] & 0 \arrow[r] & M_1 \arrow[r] \arrow[d] & C\otimes_R F_1^0 \arrow[r] \arrow[d] & C\otimes_R F_1^1 \arrow[r] \arrow[d] & \cdots \\
\vdots & & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By [7, Proposition 1.8], each map $M_m \to M_{m+1}$ can be lifted to a chain map $F_m \to F_{m+1}$ of complexes. Since we are dealing with sequences (and not arbitrary direct systems), each column above is again a direct system. Thus it makes sense to apply the exact functor $\varinjlim$ to the upon exact sequences, and doing so, we obtain an exact complex,

$$F = \varinjlim F_m = 0 \to \varinjlim M_m \to C \otimes_R \varinjlim F_m^0 \to \cdots$$

where each module $C \otimes_R F^k = C \otimes_R \varinjlim F_m^k$, $k = 0, 1, 2, \cdots$ is C-n-flat. When E is n-absolutely pure right R-module, then $Hom_R(C, E) \otimes_R F_m$ is exact since

$$C \otimes_R F \cong C \otimes_R Hom_{\mathbb{Z}}(E, \mathbb{Q}/\mathbb{Z}) \cong Hom_{\mathbb{Z}}(Hom_R(C, E), \mathbb{Q}/\mathbb{Z})$$

is a C-n-flat R-module, while the first isomorphism comes from that R is n-coherent and the second isomorphism holds by [9, Lemma 1.11], we get the exactness of

$$\begin{aligned} Hom_R(F_m, C \otimes_R F) &= Hom_R(F_m, Hom_{\mathbb{Z}}(Hom_R(C, E), \mathbb{Q}/\mathbb{Z})) \\ &= Hom_{\mathbb{Z}}(Hom_R(C, E) \otimes_R F_m, \mathbb{Q}/\mathbb{Z}) \end{aligned}$$

and hence of $Hom_R(C, E) \otimes_R F_m$, since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module. Since $\varinjlim$ commutes with the homology functor, we also get the exactness of $Hom_R(C, E) \otimes_R F = \varinjlim(Hom_R(C, E) \otimes_R F_m)$. Thus, we have constructed the right half, F, of a complete $\mathcal{FF}_C^n$-resolution for lim M_m.
Since M_m is G_C-n-flat, we also have

$$Tor_i^R(Hom_R(C, E), \varinjlim M) \cong \varinjlim Tor_i^R(Hom_R(C, E), M) = 0$$

for $i > 0$, and all n-absolutely pure right modules E. Thus, $\varinjlim M_m$ is G_C-n-flat. □

Proposition 3.13. *Let R be a GF_n-closed ring. Then, every cokernel in a complete $\mathcal{FF}_C^n$-resolution is G_C-n-flat.*

Proof. Follows from Proposition 3.6, Theorem 3.11 and [11, Lemma 5.4]. □

Lemma 3.14. *Let R be a GF_n-closed ring and let $M \in R$-Mod be G_C-n-flat. Then there exists $\mathcal{A}b_C^n(R) \otimes -$ exact sequences:*

$$0 \to M \to G \to N \to 0$$

and

$$0 \to K \to F \to M \to 0$$

in R-Mod with N, K G_C-n-flat, $G \in Add_R C$, and F n-flat.

Proof. It follows from the definition of G_C-n-flat modules and Proposition 3.13 . □

The following result plays a crucial role in this section.

Lemma 3.15. *Let R be a GF_n-closed ring and suppose that*

$$0 \to A \to G_1 \xrightarrow{f} G_0 \to M \to 0$$

is an exact sequence in R-Mod with G_0, G_1 G_C-n-flat. Then, we have the following exact sequences:

$$0 \to A \to C_1 \to G \to M \to 0, \tag{3}$$

and

$$0 \to A \to H \to F \to M \to 0 \tag{4}$$

with $C_1 \in Add_R C$, F n-flat, and G, H G_C-n-flat.

Proof. Since G_1 is G_C-n-flat, there exists an short exact sequence $0 \to G_1 \to C_1 \to G' \to 0$ with $C_1 \in Add_R C$ and G' G_C-n-flat by Lemma 3.14. Then, we have the following pushout diagram:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & Im(f) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & C_1 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}
$$

Consider the following pushout diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Im(f) \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}
$$

Since G_0 and G' are G_C-n-flat, G is also G_C-n-flat by Theorem 3.11. Connecting the middle rows in the above two diagrams, we get the first desired exact sequence (3).

Since G_0 is G_C-n-flat, there exists an exact sequence $0 \to G'' \to F \to G_0 \to 0$ with F n-flat and G'' G_C-n-flat by Lemma 3.14. Then, we have the following pullback diagram:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G'' \arrow[r, equal] \arrow[d] & G'' \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Im(f) \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}
$$

Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G'' \arrow[r, equal] \arrow[d] & G'' \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & H \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & G_1 \arrow[r] \arrow[d] & Im(f) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since G_1 and G'' are G_C-n-flat, H is also G_C-n-flat by Theorem 3.11. Connecting the middle rows in the above two diagrams, we get the second desired exact sequence (4). □

Definition 3.16. Let m be a positive integer. An R-module A is called an m-C-yoke module (of M) if there exists an exact sequence

$$0 \to A \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in R-Mod with all F_i C-n-flat.

Definition 3.17. Let m be a positive integer, a module A is called an m-G_C-yoke module (of M) if there exists an exact sequence

$$0 \to A \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

in R-Mod with all G_i G_C-n-flat.

The following result establishes the relation between the m-G_C-yoke with the m-C-yoke of a module as well as the relation between the G_C-n-flat resolution and the n-flat resolution of a module.

Lemma 3.18. *Let R be a GF_n-closed ring and let $m \geq 1$ and*

$$0 \to A \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

be an exact sequence in R-Mod with all G_i G_C-n-flat. Then, we have the following:

(i) There exists exact sequences:

$$0 \to A \to C_{m-1} \to \cdots \to C_1 \to C_0 \to N \to 0$$

and

$$0 \to M \to N \to G \to 0$$

in R-Mod with all $C_i \in Add_R C$ and G G_C-n-flat.

(ii) There exist exact sequences

$$0 \to B \to F_{m-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

and

$$0 \to H \to B \to A \to 0$$

in R-Mod with all F_i n-flat and H G_C-n-flat.

Proof. We proceed by induction on m.

(i) When $m = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since, we have a $\mathcal{A}b_C^n(R) \otimes_R -$ exact exact sequence $0 \to G_0 \to C_0 \to G \to 0$ in R-Mod with $C_0 \in Add_R C$ and G G_C-n-flat by Lemma 3.14, we have the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & C_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now, assume that $m \geq 2$ and we have an exact sequence $0 \to A \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with all G_i G_C-n-flat. Put $K = Coker(G_{m-1} \to G_{m-2})$. By Lemma 3.15, we get an exact sequence

$$0 \to A \to C_{m-1} \to G'_{m-2} \to K \to 0$$

in R-Mod with $C_{m-1} \in Add_R C$ and G'_{m-2} G_C-n-flat. Put $A' = Im(C_{m-1} \to G'_{m-2})$. Then, we get an exact sequence $0 \to A' \to G'_{m-2} \to G_{m-3} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod. So, by the induction hypothesis, we get the assertion.

(ii) When $m = 1$, we have an exact sequence $0 \to A \to G_0 \to M \to 0$ in R-Mod. Since, we have a $\mathcal{A}b_C^n(R) \otimes_R -$ exact exact sequence $0 \to H \to F_0 \to G_0 \to 0$ in R-Mod with F_0 n-flat and H G_C-n-flat by Lemma 3.14, we have the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

The middle row and the first column in the above diagram are the desired two exact sequences.

Now, assume that $m \geq 2$ and we have an exact sequence $0 \to A \to G_{m-1} \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with all G_i G_C-n-flat. Put $K = Ker(G_1 \to G_0)$. By Lemma 3.15, we get an exact sequence

$$0 \to K \to G'_1 \to F_0 \to M \to 0$$

in R-Mod with F_0 n-flat and G'_1 G_C-n-flat. Put $M' = Im(G'_1 \to P_0)$. Then, we get an exact sequence $0 \to A \to G_{m-1} \to \cdots \to G_2 \to G'_1 \to G_0 \to M \to 0$ in R-Mod. So, by the induction hypothesis, we get the assertion. □

Here is a version of Schannuel's lemma for $\mathcal{FF}_C^n$-resolutions.

Proposition 3.19. *Let M be a left R-module, and consider the two exact sequences of left R-modules,*

$$0 \to G_m \to G_{m-1} \to \cdots \to G_0 \to M \to 0,$$

and

$$0 \to H_m \to H_{m-1} \to \cdots \to H_0 \to M \to 0,$$

where $G_0, \cdots, G_{m-1}$ and $H_0, \cdots, H_{m-1}$ are G_C-n-flat. If R is GF_n-closed, then G_m is G_C-n-flat if and only if H_m is G_C-n-flat.

Proof. It follows from Proposition 3.5 and Proposition 3.11 . □

4 G_C-n-flat dimensions of modules

The class of G_C-n-flat modules can be used to define the G_C-n-flat dimension.

Definition 4.1. For a module $M \in R\text{-}Mod$, the G_C-n-flat dimension of M, denoted by $G_C - f_n d_R(M)$, is defined as $inf\{m|$ there exists an exact sequence $0 \to G_m \to \cdots \to G_1 \to G_0 \to M \to 0$ in $R\text{-}Mod$ with all G_i G_C-n-flat $\}$. We have $G_C - f_n d_R(M) \geq 0$, and we set $G_C - f_n d_R(M) = \infty$ if no such integer exists.

We start with the following standard result.

Lemma 4.2. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $R\text{-}Mod$.*

(i) $G_C - f_n d_R(N) \leq$ max $\{G_C - f_n d_R(M), G_C - f_n d_R(L) + 1\}$, and the equality holds if $G_C - f_n d_R(M) \neq G_C - f_n d_R(L)$.

(ii) $G_C - f_n d_R(L) \leq$ max $\{G_C - f_n d_R(M), G_C - f_n d_R(N) - 1\}$, and the equality holds if $G_C - f_n d_R(M) \neq G_C - f_n d_R(N)$.

(iii) $G_C - f_n d_R(M) \leq$ max $\{G_C - f_n d_R(L), G_C - f_n d_R(N)\}$, and the equality holds if $G_C - f_n d_R(N) \neq G_C - f_n d_R(L) + 1$.

Proof. It is immediate. □

The proof of the following theorem is similar to [7, Theorem 3.15].

Theorem 4.3. *Assume that R is GF_n-closed and C is a semidualizing module. If any two of the modules M, M' or M'' in a short exact sequence $0 \to M' \to M \to M'' \to 0$ have finite G_C-n-flat dimension, then so has the third.*

Next result is a G_C-n-flat version of the corresponding result about n-flat dimension of modules.

Proposition 4.4. *Let $0 \to L \to M \to N \to 0$ be an exact sequence in $R\text{-}Mod$. If $L \neq 0$ and N is G_C-n-flat, then $G_C - f_n d_R(L) = G_C - f_n d_R(M)$.*

Proof. It follows by Lemma 4.2(3). □

We give a criterion for computing the G_C-n-flat dimension of modules as follows. It generalizes [7, Theorem 3.14]. We denote $\overline{Add_R C} = Add_R C \cup Add_R R$.

Proposition 4.5. *Let R be a GF_n-closed ring and the following statements are equivalent for any $M \in R\text{-}Mod$ and $m \geq 0$.*

(i) $G_C - f_n d_R(M) \leq m$.

(ii) For every non-negative integer t such that $0 \leq t \leq m$, there exists an exact sequence $0 \to X_m \to \cdots \to X_1 \to X_0 \to M \to 0$ in $R\text{-}Mod$ such that X_t is G_C-n-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$.

Proof. $(ii) \Rightarrow (i)$. It is trivial.
$(i) \Rightarrow (ii)$. We proceed by induction on m. Suppose $G_C - f_n d_R(M) \leq 1$. Then there exists an exact sequence $0 \to G_1 \to G_0 \to M \to 0$ in $R\text{-}Mod$ with G_0 and G_1 G_C-n-flat. By Lemma 3.15 with $A = 0$, we get the exact sequences $0 \to C_1 \to G_0' \to M \to 0$ and $0 \to G_1' \to F_0 \to M \to 0$ in $R\text{-}Mod$ with $C_1 \in Add_R C$, F_0 n-flat, and G_0', G_1' G_C-n-flat.

Now, suppose $G_C - f_n d_R(M) = m \geq 2$. Then there exists an exact sequence $0 \to G_m \to \cdots \to G_1 \to G_0 \to M \to 0$ in R-Mod with G_i G_C-n-flat for any $0 \leq i \leq m$. Set $A = Coker(G_3 \to G_2)$. By applying Lemma 3.15 to the exact sequence $0 \to A \to G_1 \to G_0 \to M \to 0$, we get an exact sequence $0 \to G_m \to \cdots \to G_2 \to G_1' \to F_0 \to M \to 0$ in R-Mod with G_1' G_C-n-flat and F_0 n-flat. Set $N = Coker(G_2 \to G_1')$. Then, we have $G_C - f_n d_R(N) \leq m-1$. By the induction hypothesis, there exists an exact sequence

$$0 \to X_m \to \cdots \to X_t \to \cdots \to X_1 \to F_0 \to M \to 0$$

in R-Mod such that F_0 is n-flat and X_t is G_C-n-flat and $X_i \in \overline{Add_R C}$ for $i \neq t$ and $1 \leq t \leq m$.

Now, we need only to prove (ii) for $t = 0$. Set $B = Coker(G_2 \to G_1)$. By the induction hypothesis, we get an exact sequence $0 \to X_m \to \cdots \to X_3 \to X_2 \to G_1' \to B \to 0$ in R-Mod with G_1' G_C-n-flat and $X_i \in \overline{Add_R C}$ for any $2 \leq i \leq m$. Set $D = Coker(X_3 \to X_2)$. Then by applying Lemma 3.15 to the exact sequence $0 \to D \to G_1' \to G_0 \to M \to 0$, we get the exact sequence $0 \to D \to C_1 \to G_0' \to M \to 0$ in R-Mod with $C_1 \in Add_R C$ and G_0' G_C-n-flat. Thus, we obtain the desired exact sequence

$$0 \to X_m \to \cdots \to X_2 \to X_1 \to G_0' \to M \to 0$$

in R-Mod with all $X_i \in \overline{Add_R C}$ and G_0' G_C-n-flat. □

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