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PERIODIC SEQUENCES OF NUMBERS IN GENERALIZED ARITHMETIC AND GEOMETRIC ALTERNATE PROGRESSIONS

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Abstract The paper provides a generalization of the arithmetic-geometric alternate sequence introduced recently by Rabago [2].

1 Introduction

The *natural numbers*, usually denoted by $\mathbb{N}$, is given by the sequence $1, 2, 3, 4, 5, 6, 7, \ldots$. This type of number sequence is an example of what we call *arithmetic sequence*. An arithmetic sequence is a number sequence in which every term except the first is obtained by adding a fixed number, called the *common difference*, to the preceeding term. Another example is the sequence $1, 3, 5, 7, 9, 11, \ldots$ whose common difference is 2. Denote the n^{th} term of the arithmetic sequence with first term a and common difference d as a_n and the sum of the first n terms of the sequence as S_n. Then, a_n is define recursively as

$$a_1 = a, \quad a_n = a_{n-1} + d, \quad (n \geq 2).$$

An explicit formula for a_n is given by

$$a_n = a + (n-1)d, \quad (n \geq 2).$$

The sum S_n is given by

$$S_n = \frac{n}{2}[2a + (n-1)d], \quad (n \geq 1).$$

Another type of sequence of numbers is the so-called *geometric sequence*. A geometric sequence is a number sequence in which every term except the first is obtained by multiplying the previous term by a constant, called the *common ratio.* For example, $2, 4, 8, 16, \ldots$ is a geometric sequence with common ratio 2. Let a_n denote the n^{th} term of the geometric sequence with first term a and common ratio r. Then, a_n is define recursively as

$$a_1 = a, \quad a_n = a_{n-1} \cdot r, \quad (n \geq 2).$$

An explicit formula for a_n is given by

$$a_n = a \cdot r^{n-1}, \quad (n \geq 2).$$

The sum S_n is given by

$$S_n = a\frac{r^n - 1}{r - 1}, \; r \neq 1 \quad (n \geq 1).$$

In a recent paper, Rabago [2] introduced the concept of arithmetic-geometric alternate sequence of numbers as follows:

Definition 1.1. A sequence of numbers $\{a_n\}$ is called an arithmetic-geometric alternate sequence of numbers if the following conditions are satisfied:

(i) for any $k \in \mathbb{N}$, $\dfrac{a_{2k}}{a_{2k-1}} = r$,

(ii) for any $k \in \mathbb{N}$, $a_{2k+1} - a_{2k} = d$,

where r and d are called the common ratio and common difference of the sequence $\{a_n\}$, respectively.

In this study, we present two types of generalization of the arithmetic-geometric alternate sequence [2]. We also present in this work an explicit formula for the n^{th} term of the sequence as well as the sum for the first n terms.

2 Periodic Arithmetic-Geometric Alternate Sequence

We start off with the definition of what we call periodic sequence of numbers with alternate common difference and ratio.

Definition 2.1. A sequence of numbers $\{a_n\}$ is called a periodic sequence of numbers with alternate common difference and ratio if for a fixed natural number m the following conditions are satisfied:

(i) for any $k = 1, 2, \ldots$ and for all natural number $j \leq m-1$,

$$a_{m(k-1)+j+1} - a_{m(k-1)+j} = d,$$

(ii) for any $k = 1, 2, \ldots$,

$$\frac{a_{mk+1}}{a_{mk}} = r.$$

Clearly, the above definition takes the following form:

$$a_1,\ a_1+d,\ a_1+2d,\ \ldots,\ a_1+(m-1)d,\ (a_1+(m-1)d)r,\ (a_1+(m-1)d)r+d,\ \ldots,$$
$$(a_1+(m-1)d)r+(m-1)d,\ ((a_1+(m-1)d)r+(m-1)d)r,\ \ldots \tag{2.1}$$

From the previous definition we may define m as the period of the sequence and the terms $\{a_1, a_2, \ldots, a_m\}$ can be defined as the elements of the 1^{st} interval (or period) of length m, $\{a_{m+1}, a_{m+2}, \ldots, a_{2m}\}$ as the elements of the 2^{nd} interval of length m, and so on, and in general, the terms $\{a_{(k-1)m+1}, a_{(k-1)m+2}, \ldots, a_{km}\}$ can be considered as the elements of the k-th interval of length m. It can be observed easily that for each interval, the terms are in arithmetic progression with d as the common difference.

Throughout in the paper we denote the greatest integer contained in x as $\lfloor x \rfloor$.

Theorem 2.2. *Let d and r be any two real numbers such that $r \neq 1$ and $\{a_n\}$ be a periodic sequence of numbers with alternate common difference d and ratio r. Then, the formula for the n^{th} term of $\{a_n\}$ is given by,*

$$a_n = a_1 r^{e_1} + (m-1)\left(\frac{1-r^{e_1}}{1-r}\right)dr + (n-1-me_1)d, \tag{2.2}$$

where $e_1 = \left\lfloor \frac{n-1}{m} \right\rfloor$.

Proof. The formula is clearly true for $n \leq m$. We only have to show that the formula is valid for $n > m$. To do this, first, we will show that formula (2.2) holds for any fixed natural number $k > 1$. We let k be a fixed natural number and $p = m(k-1) + j$, where j is a natural number less than m. Note that $a_{p+1} = a_p + d$ for all $j \leq m-1$. This implies that,

$$a_p = a_1 r^{e_1} + (m-1)\left(\frac{1-r^{e_1}}{1-r}\right)dr + (p-1-me_1)d + d$$

Here, $e_1 = \left\lfloor \frac{p-1}{m} \right\rfloor$. Replacing p by $m(k-1)+j$, we'll obtain,

$$a_p = a_1 r^{k-1} + (m-1)\left(\frac{1-r^{k-1}}{1-r}\right)dr + jd.$$

Because

$$\left\lfloor \frac{m(k-1)+j-1}{m} \right\rfloor = \left\lfloor \frac{m(k-1)+j}{m} \right\rfloor,$$

for all natural number $j \leq m-1$, then

$$a_{p+1} = a_1 r^{e_0} + (m-1)\left(\frac{1-r^{e_0}}{1-r}\right)dr + ((p+1)-1-me_0)d,$$

where $e_0 = \left\lfloor \frac{(p+1)-1}{m} \right\rfloor$.

Now we need to show that $a_{mk+1} = a_m k \cdot r$ for each interval k. Clearly, $a_{mk+1} = a_{mk} \cdot r$ is true for $k = 1$. So, we assume that $a_{mp+1} = a_{mp} \cdot r$ for some natural number $p > 1$. Hence,

$$\begin{aligned} a_{m(p+1)} \cdot r &= \left(a_1 r^p + (m-1)\left(\frac{1-r^p}{1-r}\right) dr + (m(p+1) - 1 - mp)d \right) \cdot r \\ &= a_1 r^{p+1} + (m-1)\left(\frac{1-r^p}{1-r}\right) dr^2 + (m-1)dr \\ &= a_1 r^{p+1} + (m-1)\left(\frac{1-r^{p+1}}{1-r}\right) dr \\ &= a_{m(p+1)+1}, \end{aligned}$$

proving the theorem. □

Lemma 2.3. *For any integer $m > 0$ and natural number n,*

$$\sum_{i=1}^{n} \left\lfloor \frac{i}{m} \right\rfloor = \left\lfloor \frac{n}{m} \right\rfloor \left(n + 1 - \frac{m}{2} \left\lfloor \frac{n+m}{m} \right\rfloor \right).$$

Lemma 2.4. *For any integer $m > 0$ and natural number n,*

$$\sum_{i=1}^{n} r^{e_i} = m - 1 + rm\left(\frac{1-r^{e_n - 1}}{1-r}\right) + (n + 1 - me_n)\, r^{e_n}, \quad (r \neq 1),$$

where $e_i = \left\lfloor \frac{i}{m} \right\rfloor$.

For the proof of Lemma (2.3) and Lemma (2.4), *see* [2] and [3], respectively.

Theorem 2.5. *The sum of the first n terms of (2.1) is given by*

$$S_n = nM + (a_1 - M)R_n + \frac{n(n-1)d}{2} - mdE_n, \tag{2.3}$$

where

$$\begin{aligned} M &= \frac{(m-1)dr}{1-r} \\ R_n &= m - 1 + rm\left(\frac{1-r^{e_n-1}}{1-r}\right) + (n - me_n)\, r^{e_n}, \\ e_n &= \left\lfloor \frac{n-1}{m} \right\rfloor, \\ E_n &= \left\lfloor \frac{n-1}{m} \right\rfloor \left(n - \frac{m}{2} \left\lfloor \frac{n+m-1}{m} \right\rfloor \right). \end{aligned}$$

Proof. Let $m > 0$ be an integer, r be a real number different from 0 and 1, n a natural number, and $e_i = \left\lfloor \frac{i-1}{m} \right\rfloor$. Let $\{a_n\}$ be a sequence of the form as in (2.1). Then,

$$\begin{aligned} \sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} \left(a_1 r^{e_i} + (m-1)\left(\frac{1-r^{e_i}}{1-r}\right) dr + (i - 1 - me_i)d \right) \\ &= \frac{n(m-1)dr}{1-r} + \left(a_1 - \frac{(m-1)dr}{1-r} \right) \sum_{i=1}^{n} r^{e_i} + \frac{n(n-1)d}{2} - md \sum_{i=1}^{n} e_i \end{aligned}$$

and by Lemma (2.3) and Lemma (2.4), conclusion follows. □

We end this section with the following remark.

Remark 2.6. We note that by letting $m \to \infty$ in (2.2), we'll obtain the explicit formula for the usual arithmetic sequence of numbers with common difference d. Also, one may verify that $R_n \to n$ as $m \to \infty$ and that the formula for the sum of n terms S_n given by (2.3) in Theorem (2.5) will approach $a_1 n + \frac{n(n-1)}{2} d$ as $m \to \infty$.

3 Periodic Geometric-Arithmetic Alternate Sequence

In this section, we present another generalization of arithmetic-geometric sequence with the following definition of a periodic sequence of numbers with alternate common ratio and difference.

Definition 3.1. A sequence of numbers $\{a_n\}$ is called a periodic sequence of numbers with alternate common ratio r and difference d if for a fixed natural number m the following conditions are satisfied:

(i) for any $k = 1, 2, \ldots$ and for all natural number $j \le m-1$,

$$\frac{a_{m(k-1)+j+1}}{a_{m(k-1)+j}} = r,$$

(ii) for any $k = 1, 2, \ldots$, $a_{mk+1} - a_{mk} = d$.

It can be seen easily that the number sequence $\{a_n\}$ has the following form:

$$a_1,\ a_1r,\ a_1r^2,\ \ldots,\ a_1r^{m-1},\ a_1r^{m-1}+d,\ (a_1r^{m-1}+d)r,\ (a_1r^{m-1}+d)r^2,\ \ldots,$$
$$(a_1r^{m-1}+d)r^{m-1}+d,\ ((a_1r^{m-1}+d)r^{m-1}+d)r,\ \ldots \tag{3.1}$$

Here we say that the terms $\{a_1, a_2, \ldots, a_m\}$ belong to the 1^{st} interval of length m, $\{a_{m+1}, a_{m+2}, \ldots, a_{2m}\}$ belong to the 2^{nd} interval of length m, and so on, and in general, the terms $\{a_{(k-1)m+1}, a_{(k-1)m+2}, \ldots, a_{km}\}$ belong to the k-th interval of length m. Note that for each interval, the terms are in geometric progression with r as the common ratio.

Theorem 3.2. *Let d and r be any two real numbers such that $r \neq 1$ and $\{a_n\}$ be a periodic sequence of numbers with alternate common ratio r and difference d. Then, the formula for the n^{th} term of $\{a_n\}$ is given by,*

$$a_n = a_1r^{n-1-e_1} + d\left(\frac{1-(r^{m-1})^{e_1}}{1-r^{m-1}}\right)r^{n-1-me_1}, \tag{3.2}$$

where $e_1 = \left\lfloor \frac{n-1}{m} \right\rfloor$.

Proof. Obviously formula (3.2) is valid for every natural number $n \le m$. We only need to verify the validity of the formula for $n > m$. To do this, we first show that for every interval $k = 1, 2, \ldots$, the formula is true and then, we show that for every k, $a_{mk+1} = a_{mk} + d$.

Now, let $p = m(k-1) + j$ with k fixed then, $a_{p+1} = a_p \cdot r$ for all natural number $j \le m-1$. Hence,

$$a_{p+1} = \left(a_1r^{p-1-e_1} + d\left(\frac{1-(r^{m-1})^{e_1}}{1-r^{m-1}}\right)r^{p-1-me_1}\right)\cdot r,$$

where $e_1 = \left\lfloor \frac{p-1}{m} \right\rfloor$. Simplifying and noting that

$$\left\lfloor \frac{m(k-1)+j-1}{m} \right\rfloor = \left\lfloor \frac{m(k-1)+j}{m} \right\rfloor,$$

for all natural number $j \le m-1$, we obtain

$$a_{p+1} = a_1r^{(p+1)-1-e_0} + d\left(\frac{1-(r^{m-1})^{e_0}}{1-r^{m-1}}\right)r^{(p+1)-1-me_0},$$

where $e_0 = \left\lfloor \frac{(p+1)-1}{m} \right\rfloor$. On the other hand, it can be shown easily that $a_{mk+1} = a_{mk} + d$ is true for $k = 1$. So, we assume that $a_{mp+1} = a_{mp} + d$ for some natural number $p > 1$. This implies that,

$$a_{m(p+1)} + d = a_1r^{m(p+1)-1-e_1} + d\left(\frac{1-(r^{m-1})^{e_1}}{1-r^{m-1}}\right)r^{m(p+1)-1-me_1} + d$$

where $e_1 = \left\lfloor \frac{m(p+1)-1}{m} \right\rfloor$. But, $\left\lfloor \frac{m(p+1)-1}{m} \right\rfloor = p$, then

$$\begin{aligned}
a_{m(p+1)} + d &= a_1r^{m(p+1)-1-p} + d\left(\frac{1-(r^{m-1})^{p}}{1-r^{m-1}}\right)r^{m(p+1)-1-mp} + d \\
&= a_1r^{(m-1)(p+1)} + d\left\{\left(\frac{1-(r^{m-1})^{p}}{1-r^{m-1}}\right)r^{m-1} + 1\right\} \\
&= a_1r^{(m-1)(p+1)} + d\left(\frac{1-(r^{m-1})^{p+1}}{1-r^{m-1}}\right) \\
&= a_{m(p+1)+1}.
\end{aligned}$$

This proves the theorem. □

Similar to what we remarked in the previous section, we can notice easily that formula (3.2) will approach the form $a_1 r^{n-1}$ as $m \to \infty$. That is, we'll obtain the explicit formula for the usual geometric sequence of numbers with common ratio r.

Lemma 3.3. *Let R be the sum*

$$\sum_{i=1}^{n} r^{i-1-e_i}, \quad r \neq 0, 1,$$

where $e_i = \left\lfloor \frac{i-1}{m} \right\rfloor$. Then, for any natural numbers m and n,

$$R = \left(\frac{1-r^m}{1-r}\right)\left(\frac{1-(r^{m-1})^p}{1-r^{m-1}}\right) + \frac{1}{r^p}\left(\frac{1-r^{n-mp}}{1-r}\right), \tag{3.3}$$

where $p = \left\lfloor \frac{n-1}{m} \right\rfloor$.

Proof. Let $m > 0$ be an integer, r be a real number different from 0 and 1, n a natural number, and $p = \left\lfloor \frac{n-1}{m} \right\rfloor$. Then,

$$\begin{aligned}
\sum_{i=1}^{n} r^{i-1-e_i} &= \sum_{i=1}^{n} r^{i-1}\left(\frac{1}{r}\right)^{\left\lfloor \frac{i-1}{m} \right\rfloor} \\
&= \left\{\sum_{i=1}^{m} r^{i-1} + \left(\frac{1}{r}\right)\sum_{i=m+1}^{2m} r^{i-1} + \left(\frac{1}{r}\right)^2 \sum_{i=2m+1}^{3m} r^{i-1} + \dots \right. \\
&\quad \left. + \left(\frac{1}{r}\right)^{p-1} \sum_{i=(p-1)m+1}^{mp} r^{i-1}\right\} + \left(\frac{1}{r}\right)^p \sum_{i=mp+1}^{n} r^{i-1} \\
&= \left\{\sum_{i=1}^{m} r^{i-1} + \left(r^{m-1}\right)\sum_{i=1}^{m} r^{i-1} + \left(r^{m-1}\right)^2 \sum_{i=1}^{m} r^{i-1} + \dots \right. \\
&\quad \left. + \left(r^{m-1}\right)^{p-1} \sum_{i=1}^{m} r^{i-1}\right\} + \frac{1}{r^p} \sum_{i=1}^{n-mp} r^{i-1} \\
&= \left(\frac{1-r^m}{1-r}\right)\sum_{j=1}^{p} (r^{m-1})^{j-1} + \frac{1}{r^p}\sum_{i=1}^{n-mp} r^{i-1} \\
&= \left(\frac{1-r^m}{1-r}\right)\left(\frac{1-(r^{m-1})^p}{1-r^{m-1}}\right) + \frac{1}{r^p}\left(\frac{1-r^{n-mp}}{1-r}\right).
\end{aligned}$$

□

Lemma 3.4. *Let $\bar{R}$ be the sum*

$$\sum_{i=1}^{n} r^{i-1-me_i}, \quad r \neq 0, 1,$$

where $e_i = \left\lfloor \frac{i-1}{m} \right\rfloor$. Then, for any natural numbers m and n,

$$\bar{R} = \left\lfloor \frac{n-1}{m} \right\rfloor \left(\frac{1-r^m}{1-r}\right) + \left(\frac{1-r^{n-mp}}{1-r}\right), \tag{3.4}$$

where $p = \left\lfloor \frac{n-1}{m} \right\rfloor$.

We omit the proof since it similar on how we prove (3.3).

Theorem 3.5. *The sum of the first n terms of* (3.1) *is given by*

$$S_n = \left(a_1 - \frac{d}{1-r^{m-1}}\right) R + \left(\frac{d}{1-r^{m-1}}\right) \bar{R}, \tag{3.5}$$

where R and $\bar{R}$ are given by equations (3.3) *and* (3.4)*, respectively.*

Proof. Let $e_i = \left\lfloor \frac{i-1}{m} \right\rfloor$

$$\begin{aligned} \sum_{i=1}^{n} a_i &= \sum_{i=1}^{n} \left(a_1 r^{i-1-e_i} + d \left(\frac{1-(r^{m-1})^{e_i}}{1-r^{m-1}} \right) r^{i-1-me_i} \right) \\ &= \left(a_1 - \frac{d}{1-r^{m-1}} \right) \sum_{i=1}^{n} r^{i-1-e_i} + \left(\frac{d}{1-r^{m-1}} \right) \sum_{i=1}^{n} r^{i-1-me_i} \end{aligned}$$

, and by Lemma (3.3) and Lemma (3.4), conclusion follows. □

Note that the formula given by (3.5) will approach the expression of the form $a_1 \left(\frac{1-r^n}{1-r} \right)$ as $m \to \infty$ because $R \to \frac{1-r^n}{1-r}$ as $m \to \infty$.

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