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Prime k-Bi-ideals in Γ-Semirings

R.D. Jagatap

Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. In this paper the notions of a k-bi-ideal , prime k-bi-ideal, strongly prime k-bi-ideal , irreducible k-bi-ideal and strongly irreducible k-bi-ideal of a Γ-semiring are introduced. Also the concept of a k-bi-idempotent Γ-semiring is defined. Several characterizations of a k-bi-idempotent Γ-semiring are furnished by using prime, semiprime, strongly prime , irreducible and strongly irreducible k-bi-ideals in a Γ-semiring.

§1.Introduction:

The notion of a Γ-ring was introduced by Nobusawa in [11]. The class of Γ-rings contains not only rings but also ternary rings. As a generalization of a ring , semiring was introduced by Vandiver [17]. The notion of a Γ-semiring was introduced by Rao in [12] as a generalization of a ring, Γ-ring and a semiring.

Ideals play an important role in any abstract algebraic structure. Characterizations of prime ideals in semirings were discussed by Iseki in [5 ,6]. Henriksen in [4] defined more restricted class of ideals in a semiring known as k-ideals. Also several characterizations of k-ideals of a semiring were discussed by Sen and Adhikari in [13,14]. k-ideal in a Γ- semiring was defined by Rao in [12] and in [2] Dutta and Sardar gave some of its properties. Author discussed some properties of k-ideals and full k-ideals of a Γ-semiring in [8]. Prime and semiprime ideals in a Γ-semirings were discussed by Dutta and Sardar in [2].

The notion of a bi-ideal was first introduced for semigroups by Good and Hughes [3].The concept of a bi-ideal for a ring was given by Lajos and Szasz in [9] and they studied bi-ideal for a semigroup in [10]. Shabir, Ali and Batool in [15] gave some properties of bi-ideals in a semiring. Prime bi-ideals in a semigroup was discussed by Shabir and Kanwal in [16].

It is natural to extend the concept of a k-ideal to a k-bi-ideal of a Γ-semiring. Hence in this paper we define a k-bi-ideal as an extension of a k-ideal of a Γ-semiring . Also we define a prime k-bi-ideal, semiprime k-bi-ideal , strongly prime k-bi-ideal , irreducible and strongly irreducible k-bi-ideal of a Γ-semiring. We study some characterizations of irreducible and strongly irreducible k-bi-ideals. Further we introduce the concept of a k-bi-idempotent Γ-semiring . Several characterizations of a k-bi-idempotent Γ-semiring are furnished by using prime, semiprime, strongly prime , irreducible and strongly irreducible k-bi-ideals in a Γ-semiring.

§2. Preliminaries:

First we recall some definitions of the basic concepts of Γ-semirings that we need in sequel. For this we follow Dutta and Sardar [2].

Definition 2.1:- *Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ and $\alpha \in \Gamma$ satisfying the following conditions:*
(i) $a\alpha\,(b + c) = (a\,\alpha b) + (a\,\alpha c)$
(ii) $(b + c)\,\alpha a = (b\,\alpha a) + (c\,\alpha a)$
(iii) $a(\alpha + \beta)c = (a\,\alpha c) + (a\,\beta c)$

(iv) $a\alpha\left(b\beta c\right)=\left(a\alpha b\right)\beta c$; for all $a,b,c\in S$ *and for all* $\alpha,\beta\in\ \Gamma$.

Definition 2.2 :- *An element* 0 in a Γ-semiring S *is said to be an absorbing zero if* $0\alpha a=0=a\alpha 0, a+0=0+a=a$; for all $a\in S$ *and* $\alpha\in\Gamma$.

Definition 2.3:- *A non empty subset T of a* Γ*-semiring S is said to be a sub-* Γ*-semiring of S if (T,+) is a subsemigroup of (S,+) and* $a\alpha b\in\ T$; for all $a,b\in\ T$ and $\alpha\in\Gamma$.

Definition 2.4:- *A nonempty subset T of a* Γ*-semiring S is called a left (respectively right) ideal of S if T is a subsemigroup of (S,+) and* $x\alpha a\in T$ *(respectively* $a\alpha x\in T$), for all $a\in T,\ x\in\ S$ and $\alpha\in\Gamma$.

Definition 2.5 :- *If T is both left and right ideal of a* Γ*-semiring S, then T is known as an ideal of S.*

Definition 2.6:- *A right ideal* I *of a* Γ*-semiring* S *is said to be a right k-ideal if* $a\in\ I$ *and* $x\in\ S$ *such that* $a+x\ \in I$*, then* $x\in\ I$*.*

Similarly we define a left k-ideal of Γ-semiring S*. If an ideal* I *is both right and left k-ideal, then* I *is known as a k-ideal of* S*.*

Example 1:- Let N_0 denotes the set of all positive integers with zero. $S=N_0$ is a semiring and with $\Gamma=S$, S forms a Γ-semiring. A subset $I=3N_0\backslash\{3\}$ of S is an ideal of S but not a k-ideal. Since 6, $9=3+6\in I$ but $3\notin I$.

Example 2:- If $S=\ N$ is the set of all positive integers then (S, max., min.) is a semiring and with $\Gamma=S$, S forms a Γ-semiring. $I_n=\{1,\ 2,3,\cdots\cdots\cdots,n\}$ is a k-ideal for any $n\in I$.

Definition 2.7 :- *For a non empty subset* I *of a* Γ*-semiring S define*
$\overline{I}=\{a\in S|a+x\in I, for some\ x\in I\}$. $\overline{I}$ *is called a k-closure of* I.
Some basic properties of k-closure are given in the following lemma .

Lemma 2.8:- *For non empty subsets* A *and* B *of a* Γ*-semiring* S *we have*,
(1) *If* $A\subseteq B$ *then* $\overline{A}\subseteq\overline{B}$.
(2) $\overline{A}$ *is the smallest (left k-ideal, right k-ideal) k-ideal containing* (left k-ideal, right k-ideal) k-ideal A of S.
(3) $\overline{A}=A$ *if and only if* A *is a (left k-ideal, right k-ideal) k-ideal of* S.
(4) $\overline{\overline{A}}=\overline{A}$,where A is a (left k-ideal, right k-ideal) k-ideal of S.
(5)$\overline{\overline{A}\Gamma\overline{B}}=\overline{A\Gamma B}$, where A and B are (left k-ideals, right k-ideals) k-ideals of S.

Now we give a definition of a bi-ideal.

Definition 2.9 [7]:- *A nonempty subset* B *of* S *is said to be a bi-ideal of* S *if* B *is a sub-*Γ*-semiring of* S *and* $B\Gamma S\Gamma B\subseteq B$.

Example 3:- Let N be the set of natural numbers and let Γ= $2N$. Then N and Γ both are additive commutative semigroup. An image of a mapping $N\times\Gamma\times N\longrightarrow N$ is denoted by $a\alpha b$ and defined as $a\alpha b=$ product of a , α , b; for all $a,b\in N$ and $\alpha\in\Gamma$. Then N forms a Γ-semiring. $B=4N$ is a bi-ideal of N.

Example 4:-Consider a Γ-semiring $S=M_{2X2}(N_0)$,where N_0 denotes the set of natural numbers with zero and $\Gamma=S$. Define $A\alpha B$= usual matrix product of A,α and B; for all $A,\alpha\ ,B\ \in S$. Then
$P=\left\{\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\mid a,b\in N_0\right\}$ is a bi-ideal of a Γ-semiring S.
Lemma 2.8 also holds for a k-bi-ideal similar to left k-ideal, right k-ideal and and k- ideal.
Some results from [5] are stated which are useful for further discussion.

Result 2.10:- *For each nonempty subset X of a* Γ*-semiring S following statements hold.*
(i) $S\Gamma X$ *is a left ideal of S.*
(ii) $X\Gamma S$ *is a right ideal of S.*
(iii) $S\Gamma X\Gamma S$ *is an ideal of S.*
Result 2.11:- *In a* Γ*-semiring* S*, for* $a \in S$ *following statements hold.*
(i) $S\Gamma a$ *is a left ideal of S.*
(ii) $a\Gamma S$ *is a right ideal of S.*
(iii) $S\Gamma a\Gamma S$ *is an ideal of S.*

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

§3. k- Bi-ideal:

We begin with definition of a k-bi-ideal in a Γ-semiring S.

Definition 3.1:- *A nonempty subset* B *of* S *is said to be a k-bi-ideal of* S *if* B *is a sub-*Γ*-semiring of* S *,* $\overline{B\Gamma S\Gamma B} \subseteq B$ and *if* $a \in B$ *and* $x \in S$ *such that* $a + x \in B$*, then* $x \in B$*.*

First we give some concepts in a Γ-semiring that we need in a sequel.
Definition 3.2:- *A k-bi-ideal* B *of* S *is called a prime k-bi-ideal if* $\overline{B_1\Gamma B_2} \subseteq B$ *implies* $B_1 \subseteq B$ *or* $B_2 \subseteq B$*, for any k-bi-ideals* B_1, B_2 *of* S*.*

Definition 3.3:- *A k-bi-ideal* B *of* S *is called a strongly prime k-bi-ideal if* $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B$ *implies* $B_1 \subseteq B$ *or* $B_2 \subseteq B$*, for any k-bi-ideals* B_1, B_2 *of* S*.*

Definition 3.4 :- *A k-bi-ideal* B *of* S *is called a semiprime k-bi-ideal if for any k-bi-ideal* B_1 *of* S*,* $\overline{B_1{}^2}=\overline{B_1\Gamma B_1} \subseteq B$ *implies* $B_1 \subseteq B$*.*

Obviously every strongly prime k-bi-ideal in S is a prime k-bi-ideal and every prime k-bi-ideal in S is a semiprime k-bi-ideal.

Definition 3.5:- *A k-bi-ideal* B *of* S *is called an irreducible k-bi-ideal if* $B_1 \cap B_2 = B$ *implies* $B_1 = B$ or $B_2 = B$*, for any k-bi-ideals* B_1 *and* B_2 *of* S*.*

Definition 3.6 :- *A k-bi-ideal* B *of* S *is called a strongly irreducible k-bi-ideal if for any k-bi-ideals* B_1 *and* B_2 of S , $B_1 \cap B_2 \subseteq B$ *implies* $B_1 \subseteq B$ *or* $B_2 \subseteq B$*.*
Obviously every strongly irreducible k-bi-ideal is an irreducible k-bi-ideal.

Theorem 3.7:- *The intersection of any family of prime k-bi-ideals(or* semiprime *k-bi-ideals) of S is a semiprime k-bi-ideal.*
Proof:- Let $\{P_i | i \in \Lambda\}$ be the family of prime k-bi-ideals of S. For any k-bi-ideal B of S, $\overline{B^2} \subseteq \bigcap_i P_i$ implies $\overline{B^2} \subseteq P_i$ for all $i \in \Lambda$. As P_i are semiprime k-bi-ideals, $B \subseteq P_i$ for all $i \in \Lambda$. Hence $B \subseteq \bigcap_i P_i$. ■

Theorem 3.8 :- *Every strongly irreducible, semiprime k-bi-ideal of* S *is a strongly prime k-bi-ideal .*
Proof:- Let B be a strongly irreducible and semiprime k-bi-ideal of S. For any k-bi-ideals B_1 and B_2 of S, let $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B$. $B_1 \cap B_2$ is a k-bi-ideal of S. Since $\overline{(B_1 \cap B_2)^2} = \overline{(B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2)} \subseteq \overline{B_1\Gamma B_2}$. Similarly we get $\overline{(B_1 \cap B_2)^2} \subseteq \overline{B_2\Gamma B_1}$. Therefore $\overline{(B_1 \cap B_2)^2} \subseteq \overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B$. As B is a semiprime k-bi-ideal, $B_1 \cap B_2 \subseteq B$. But B is a strongly irreducible k-bi-ideal. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime k-bi-ideal of S. ■

Theorem 3.9 :- *If* B *is* a *k-bi-ideal of* S *and* $a \in S$ *such that* $a \notin B$*, then there exists an irreducible k-bi-ideal* I *of* S such that $B \subseteq I$ and $a \notin I$.
Proof :- Let $\mathcal{B}$ be the family of all k-bi-ideals of S which contain B but do not contain an element a. Then $\mathcal{B}$ is a nonempty as $B \in \mathcal{B}$. This family of all k-bi-ideals of S forms a partially ordered set under the inclusion of sets. Hence by Zorn's lemma there exists a maximal k-bi-ideal say I in $\mathcal{B}$. Therefore $B \subseteq I$ and $a \notin I$. Now to show that I is an irreducible k-bi-ideal of S. Let C

and D be any two k-bi-ideals of S such that $C \cap D = I$. Suppose that C and D both contain I properly. But I is a maximal k-bi-ideal in $\mathcal{B}$.Hence we get $a \in C$ and $a \in D$. Therefore $a \in C \cap D = I$ $w$hich is absurd . Thus either $C = I$ or $D = I$. Therefore I is an irreducible k-bi-ideal of S. ■

Theorem 3.10:- *Any proper k-bi-ideal B of S is the intersection of all irreducible k-bi-ideals of S containing B.*
Proof :- Let B be a k-bi-ideal of S and $\{B_i | i \in \Lambda\}$ be the collection of irreducible k-bi-ideals of S containing B, where Λ denotes any indexing set. Then $B \subseteq \bigcap_{i\in\Lambda} B_i$. Suppose that $a \notin B$. Then by Theorem 3.9, there exists an irreducible k-bi-ideal A of S containing B but not a.Therefore $a \notin \bigcap_{i\in\Lambda} B_i$. Thus $\bigcap_{i\in\Lambda} B_i \subseteq B$. Hence $\bigcap_{i\in\Lambda} B_i = B$. ■

Theorem 3.11 :- *Following statements are equivalent in S.*
(1) *The set of k-bi-ideals of S is totally ordered set under inclusion of sets.*
(2) *Each k-bi-ideal of S is strongly irreducible.*
(3) *Each k-bi-ideal of S is irreducible.*
Proof :- **(1)⇒(2)**
Suppose that the set of k-bi-ideals of S is a totally ordered set under inclusion of sets. Let B be any k-bi-ideal of S. To show that B is a strongly irreducible k-bi-ideal of S. Let B_1 and B_2 be any two k-bi-ideals of S such that $B_1 \cap B_2 \subseteq B$. But by the hypothesis , we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Therefore $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \subseteq B$or $B_2 \subseteq B$. Thus B is a strongly irreducible k-bi-ideal of S.
(2) ⇒(3) Suppose that each k-bi-ideal of S is strongly irreducible. Let B be any k-bi-ideal of S such that $B = B_1 \cap B_2$ for any k-bi-ideals B_1 and B_2 of S. But by hypothesis $B_1 \subseteq B$ or $B_2 \subseteq B$. As $B \subseteq B_1$ and $B \subseteq B_2$, we get $B_1 = B$ or $B_2 = B$. Hence B is an irreducible k-bi-ideal of S.
(3) ⇒(1) Suppose that each k-bi-ideal of S is an irreducible k-bi-ideal. Let $B_1$and B_2 be any two k-bi-ideals of S. Then $B_1 \cap B_2$ is also k-bi-ideal of S. Hence $B_1 \cap B_2 = B_1 \cap B_2$ will imply $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$ by assumption. Therefore either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.This shows that the set of k-bi-ideals of S is totally ordered set under inclusion of sets. ■

Theorem 3.12 :- *A prime k-bi-ideal B of S is a prime one sided k-ideal of S.*
Proof :- Let B be a prime k-bi-ideal of S. Suppose B is not a one sided k-ideal of S. Therefore $\overline{B\Gamma S} \nsubseteq B$ and $\overline{S\Gamma B} \nsubseteq B$. As B is a prime k-bi-ideal, $\overline{(B\Gamma S)\,\Gamma(S\Gamma B)} \nsubseteq B$. $\overline{(B\Gamma S)\,\Gamma\,(S\Gamma B)} = \overline{B\Gamma\,(S\Gamma S)\,\Gamma B} \subseteq \overline{B\Gamma S\Gamma B} \subseteq B$, which is a contradiction. Therefore $\overline{B\Gamma S} \subseteq B$ or $\overline{S\Gamma B} \subseteq B$.Thus B is a prime one sided k-ideal of S. ■

Theorem 3.13 :- *A k-bi-ideal B of S is prime if and only if for a right k-ideal R and a left k-ideal L of S, $\overline{R\Gamma L} \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*
Proof :- Suppose that a k-bi-ideal of S is a prime k-bi-ideal of S. Let R be a right k-ideal and L be a left k-ideal of S such that $\overline{R\Gamma L} \subseteq B$. R and L are itself k-bi-ideals of S . Hence $R \subseteq B$ or $L \subseteq B$. Conversely, we have to show that a k-bi-ideal B of S is a prime k-bi-ideal of S. Let A and C be any two k-bi-ideals of S such that $\overline{A\Gamma C} \subseteq B$. For any $a \in A$ and $c \in C$, $\overline{(a)}_r \subseteq A$ and $\overline{(c)}_l \subseteq C$, where$\overline{(a)}_r$ and $\overline{(c)}_l$ denotes the right k-ideal and left k-ideal generated by a and c respectively. Therefore $\overline{\overline{(a)}_r\Gamma\overline{(c)}_l} \subseteq \overline{A\Gamma C} \subseteq B$. Hence by the assumption $\overline{(a)}_r \subseteq B$ or $\overline{(c)}_l \subseteq B$. Therefore $a \in B$ or $c \in B$.Thus $A \subseteq B$ or $C \subseteq B$. Hence B is a prime k-bi-ideal of S. ■

§4 Fully k-Bi-Idempotent Γ-Semiring :

Now we generalize the concept of a fully idempotent semiring introduced by Ahsan in [1] to a fully k-bi-idempotent Γ-semiring . Then we give some characterizations of it.
Definition 4.1:- A Γ- semiring S is said to be k-bi-idempotent if every k-bi-ideal of S is k-idempotent. That is S is said to be k-bi-idempotent if B is a k-bi-ideal of S , then $\overline{B^2} = \overline{B\Gamma B} = B$.

Theorem 4.2:- *In S following statements are equivalent.*
(1) *S is k-bi-idempotent*
(2) $B_1 \cap B_2 = \overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}$ *,for any k-bi-ideals $B_1$and B_2 of S.*
(3) *Each k-bi-ideal of S is semiprime.*

(4) *Each proper k-bi-ideal of S is the intersection of irreducible semiprime k-bi-ideals of S which contain it.*
Proof :- **(1)** $\Rightarrow$ **(2)** Suppose that $\overline{B^2} = B$,for any k-bi-ideal B of S. Let $B_1$and B_2 be any two k-bi-ideals of S. $B_1 \cap B_2$ is also a k-bi-ideal of S. Hence by the assumption $\overline{(B_1 \cap B_2)^2} = B_1 \cap B_2$. Now we have $B_1 \cap B_2 = \overline{(B_1 \cap B_2)^2} = \overline{(B_1 \cap B_2)\Gamma(B_1 \cap B_2)} \subseteq \overline{B_1\Gamma B_2}$. Similarly we get $B_1 \cap B_2 \subseteq \overline{B_2\Gamma B_1}$. Therefore $B_1 \cap B_2 \subseteq \overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}$. As $\overline{B_1\Gamma B_2}$ and $\overline{B_2\Gamma B_1}$ are k-bi-ideals of S, we have $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}$ is a k-bi-ideal of S. Hence $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}$ $=\overline{\left(\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}\right)\Gamma\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}}$. $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq \overline{\overline{(B_1\Gamma B_2)}\Gamma\overline{(B_2\Gamma B_1)}} \subseteq \overline{B_1\Gamma S\Gamma B_1} \subseteq B_1$. Similarly we can show that $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B_2$. Thus $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B_1 \cap B_2$. Hence $B_1 \cap B_2 = \overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)}$.
(2) $\Rightarrow$ **(3)** Let B be any k-bi-ideal of S. Suppose that $\overline{B_1{}^2} = \overline{B_1\Gamma B_1} \subseteq B$, for any k-bi-ideal B_1 of S. By the hypothesis we have, $B_1 = B_1 \cap B_1 = \overline{(B_1\Gamma B_1)} \cap \overline{(B_1\Gamma B_1)} = \overline{B_1\Gamma B_1} \subseteq B$. Hence every k-bi-ideal of S is semiprime.
(3) $\Rightarrow$ **(4)** Let B be a proper k-bi-ideal of S. Hence by Theorem 3.10, B is the intersection of all proper irreducible k-bi-ideals of S which contains B. By assumption every k-bi-ideal of S is semiprime. Hence each proper k-bi-ideal of S is the intersection of irreducible semiprime k-bi-ideals of S which contain it.
(4) $\Rightarrow$ **(1)** Let B be a k-bi-ideal of S. If $B^2 = S$, then clearly result holds. Suppose that $B^2 \neq S$. Then $\overline{B^2}$ is a proper k-bi-ideal of S. Hence by assumption $\overline{B^2}$ is the intersection of irreducible semiprime k-bi-ideals of S which contain it.
$\overline{B^2} = \cap\{B_i | B_i\ is\ irreducible\ semiprime\ k-bi-ideal\ \}$. As each B_i is a semiprime k-bi-ideal, $B \subseteq B_i$, for all i. Therefore $B \subseteq \cap B_i = \overline{B^2}$. $\overline{B^2} \subseteq B$ always. Thus $\overline{B^2} = B$.
Thus we proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) . Hence all the statements are equivalent. ■

Theorem 4.3:-*If S is k-bi-idempotent , then for any k-bi-ideal B of S, B is strongly irreducible if and only if B is strongly prime.*
Proof :- Let S be a k-bi-idempotent Γ-semiring. Suppose that B is a strongly irreducible k-bi-ideal of S. To show that B is a strongly prime k-bi-ideal of S. Let $B_1$and B_2 be any two k-bi-ideals of S such that $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B$. By Theorem 4.2, $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} = B_1 \cap B_2$. Hence $B_1 \cap B_2 \subseteq B$. But B is a strongly irreducible k-bi-ideal of S. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime k-bi-ideal of S.
Conversely, suppose that B is a strongly prime k-bi-ideal of a k-bi-idempotent Γ-semiring S. Let $B_1$and B_2 be any two k-bi-ideals of S such that $B_1 \cap B_2 \subseteq B$. By Theorem 4.2, $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} = B_1 \cap B_2 \subseteq B$. As B is a strongly prime k-bi-ideal, $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly irreducible k-bi-ideal
of S. ■

Theorem 4.4:- *Every k-bi-ideal of S is a strongly prime k-bi-ideal if and only if S is k-bi-idempotent and the set of k-bi-ideals of S is a totally ordered set under the inclusion of sets.*
Proof :- Suppose that every k-bi-ideal of S is a strongly prime k-bi-ideal. Then every k-bi-ideal of S is a semiprime k-bi-ideal. Hence S is k-bi-idempotent by Theorem 4.2. Now to show the set of k-bi-ideals of S is a totally ordered set under inclusion of sets. Let $B_1$and B_2 be any two k-bi-ideals of S from the set of k-bi-ideals of S. $B_1 \cap B_2$ is also a k-bi-ideal of S . Hence by the assumption, $B_1 \cap B_2$ is a strongly prime k-bi-ideal of S. By Theorem 4.2 , $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} = B_1 \cap B_2 \subseteq B_1 \cap B_2$. Then $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus either $B_1 \subseteq B_2$or $B_2 \subseteq B_1$. This shows that the set of k-bi-ideals of S is totally ordered set under inclusion of sets.
Conversely, suppose that S is k-bi-idempotent and the set of k-bi-ideals of S is a totally ordered set under inclusion of sets. Let B be any k-bi-ideal of S. $B_1$and B_2 be any two k-bi-ideals of S such that $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} \subseteq B$. By Theorem 4.2, we have $\overline{(B_1\Gamma B_2)} \cap \overline{(B_2\Gamma B_1)} = B_1 \cap B_2$. Therefore $B_1 \cap B_2 \subseteq B$.But by assumption either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Thus $B_1 \subseteq B$ or $B_2 \subseteq B$. Therefore B is a strongly prime k-bi-ideal of S. ■

Theorem 4.5:- *If the set of k-bi-ideals of S is a totally ordered set under inclusion of sets, then every k-bi-ideal of S is strongly prime if and only if it is prime.*
Proof :- Let the set of k-bi-ideals of S be a totally ordered set under inclusion of sets. As every

strongly prime k-bi-ideal of S is prime , the proof of only if part is obvious. Conversely, suppose that every k-bi-ideal of S is prime. Then every k-bi-ideal of S is semiprime. Hence by Theorem 4.2 , S is k-bi-idempotent. Then by Theorem 4.4 , every k-bi-ideal of S is strongly prime. ■

Theorem 4.6 :- *If the set of bi-ideals of S is a totally ordered set under inclusion of sets, then S is k-bi-idempotent if and only if each k-bi-ideal of S is prime.*
Proof :- Let the set of all k-bi-ideals of S is a totally ordered set under inclusion of sets. Suppose S is k-bi-idempotent. Let B be any k-bi-ideal of S. For any k-bi-ideals B_1 and B_2 of S, $\overline{B_1\Gamma B_2} \subseteq B$. By the assumption we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Assume $B_1 \subseteq B_2$. Then $\overline{B_1\Gamma B_1} \subseteq \overline{B_1\Gamma B_2} \subseteq B$. By Theorem 4.2 , B is a semiprime k-bi-ideal of S. Therefore $B_1 \subseteq B$. Hence B is a prime k-bi-ideal of S.
Conversely, suppose that every k-bi-ideal of S is prime. Hence every k-bi-ideal of S is semiprime. Therefore S is k-bi-idempotent by Theorem 4.2. ■

Theorem 4.7:- *If* S is k-bi-idempotent and B *is a strongly irreducible k-bi-ideal of* S, *then* B *is a prime k-bi-ideal.*
Proof :- Let B be a strongly irreducible k-bi-ideal of a k-bi-idempotent Γ-semiring S. Let B_1 and B_2 be any two k-bi-ideals of S such that $\overline{B_1\Gamma B_2} \subseteq B$. $B_1 \cap B_2$ is also a k-bi-ideal of S. Therefore by Theorem 4.2, $\overline{(B_1 \cap B_2)^2} = (B_1 \cap B_2)$. $B_1 \cap B_2 = \overline{(B_1 \cap B_2)^2} = \overline{(B_1 \cap B_2)\,\Gamma\,(B_1 \cap B_2)} \subseteq \overline{B_1\Gamma B_2} \subseteq B$. As B is strongly irreducible k-bi-ideal of S, then $B_1 \subseteq B$ or $B_2 \subseteq B$. Hence B is a prime k-bi-ideal of S. ■

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Author information

R.D. Jagatap, Y.C. College of Science, Karad, India.
E-mail: ravindrajagatap@yahoo.co.in

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