Palestine Journal of Mathematics

Vol. 5(1) (2016), 154–158

IRREDUCIBLE ELEMENTS IN ADL'S

B. Venkateswarlu, Ch. Santhi Sundar Raj and R. Vasu Babu

Communicated by Ayman Badawi

MSC 2010 Classifications: 06D99.

Keywords and phrases: Distributive Lattice, Almost Distributive Lattice, $\wedge-$ irreducible , $\vee-$ irreducible, Boolean algebra, Boolean ring.

The authors thank Professor U. M. Swamy for his help in preparing this paper.

Abstract. The notion of an Almost Distributive Lattice (ADL) is a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras and Boolean rings. In this paper, we introduce the concepts of $\wedge-$ irreducible (strongly $\wedge-$ irreducible) and $\vee-$ irreducible (strongly $\vee-$ irreducible) elements in an ADL and prove certain properties of these. Unlike the case of distributive lattices, $\wedge-$ irreducibility and strongly $\wedge-$ irreducibility are not equivalent. However, it is proved here that an element in ADL is $\vee-$ irreducible if and only if it is strongly $\vee-$ irreducible.

1 Introduction

The axiomatization of Boole's propositional two valued logic led to the concept of Boolean algebra, which is a complemented distributive lattice. M. H. Stone [4] has proved that any Boolean algebra can made into a Boolean ring (a ring with unity in which every element is an idempotent) and vice-versa. The concept of an Almost Distributive Lattice (ADL) was introduced by U. M. Swamy and G. C. Rao [5] as a common abstraction of several lattice theoretic and ring the theoretic generalizations of Boolean algebra and Boolean rings. In this paper, we introduce the notions of $\wedge$ and $\vee$ irreducibilities and strongly $\wedge$ and $\vee$ irreducibilities among elements of an Almost Distributive Lattice (ADL) and prove certain properties of these. Unlike the case of distributive lattices, $\wedge-$ irreducibility and strongly $\wedge-$ irreducibility are not equivalent for the elements of an ADL. However, an element in an ADL is $\vee-$ irreducible if and only if it is strongly $\vee-$ irreducible.

2 Preliminaries

In this section, we recall from [5] and [3] certain preliminary concepts and results concerning ADL's . We refer to [1] and [2] for the elementary notions and results regarding partially ordered sets, lattices and distributive lattices. Let us begin with the following fundamental definition [5].

Definition 2.1. An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following independent identities.

(1). $0 \wedge a \approx 0$

(2). $a \vee 0 \approx a$

(3). $a \wedge (b \vee c) \approx (a \wedge b) \vee (a \wedge c)$

(4). $(a \vee b) \wedge c \approx (a \wedge c) \vee (b \wedge c)$

(5). $a \vee (b \wedge c) \approx (a \vee b) \wedge (a \vee c)$

(6). $(a \vee b) \wedge b \approx b$.

The identities given above are independent, in the sense that no one of them is a consequence of others. The element 0 is called the zero element of the ADL.

Example 2.2. Any distributive lattice bounded below is an ADL where the zero element is the smallest element .

Example 2.3. Let $(R,+,.,0,1)$ be a commutative regular ring with unity 1. For any a and $b \in R$, define
$$a \wedge b = a_0 b \ \text{ and } \ a \vee b = a + b - a_0 b,$$
where a_0 is the unique idempotent in R such that $aR = a_0R$. Then $(R, \wedge, \vee, 0)$ is an ADL, where 0 is the additive identity in R.

Example 2.4. Let X be any non empty set and 0 be an arbitrarily fixed element in X. For any a and $b \in X$, defining
$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$
Then $(X, \wedge, \vee, 0)$ is an ADL and is called a discrete ADL.

Definition 2.5. Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define
$$a \leq b \ \text{ if and only if } \ a = a \wedge b \ (\Leftrightarrow a \vee b = b).$$
Then $\leq$ is a partial order on A

An ADL $A = (A, \wedge, \vee, 0)$ is called an associative ADL if the operation $\vee$ is associative; that is $(a \vee b) \vee c = a \vee (b \vee c)$ for all a, b and $c \in A$. Throughout this paper, by an ADL we mean an associative ADL only. We recall the following properties of ADL's.

Theorem 2.6. *The following hold for any elements a, b and c in an ADL $A = (A, \wedge, \vee, 0)$.*

(1). $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$

(2). $a \wedge a = a = a \vee a$

(3). $a \wedge b \leq b \leq b \vee a$

(4). $a \wedge b = a \Leftrightarrow a \vee b = b$ *and* $a \wedge b = b \Leftrightarrow a \vee b = a$

(5). $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(6). $a \vee (b \vee a) = a \vee b$

(7). $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(8). $a \wedge b = b \wedge a \Leftrightarrow glb\{a, b\} = a \wedge b \Leftrightarrow lub\{a, b\} = a \vee b$

(9). If $a \leq b$, *then* $a \wedge b = a = b \wedge a$ *and* $a \vee b = b = b \vee a$

(10). $(a \wedge b) \wedge c = (b \wedge a) \wedge c$ *and* $(a \vee b) \wedge c = (b \vee a) \wedge c$

Definition 2.7. Let I be a non empty subset of an ADL A. Then I is called

(1) an ideal of A if $a \vee b \in I$ for all a and $b \in I$ and $x \wedge a \in I$ for all $x \in I$ and $a \in A$.

(2) a filter of A if $a \wedge b \in I$ for all a and $b \in I$ and $a \vee x \in I$ for all $x \in I$ and $a \in A$.

As consequence, for any ideal I of A, $a \wedge x \in I$ for all $x \in I$ and $a \in A$ and for any filter F of A, $x \vee a \in F$ for all $x \in F$ and $a \in A$. For any $X \subseteq A$, the smallest ideal (filter) of A containing X is called the ideal (filter) generated by X in A and is denoted by $< X]$ and $[X >$ respectively. It is known that
$$< X] = \Big\{ \big(\bigvee_{i=1}^{n} x_i \big) \wedge a \mid n \geq 0, \ x_i \in X \text{ and } a \in A \Big\}$$
$$\text{and} \quad [X > = \Big\{ a \vee \big(\bigwedge_{i=1}^{n} x_i \big) \mid n \geq 0, \ x_i \in X \text{ and } a \in A \Big\}.$$
When $X = \{x\}$, we write $< x]$ for $< \{x\}]$ and $[x >$ for $[\{x\} >$. Note that $< x] = \{x \wedge a \mid a \in A\}$ and $[x >= \{a \vee x \mid a \in A\}$.

Definition 2.8. Two elements a and b of an ADL A are said to be associates to each other if $a \wedge b = b$ and $b \wedge a = a$ (this is equivalent or saying that $< a] =< b]$ or $[a >= [b >$ or, $a \vee b = a$ and $b \vee a = b$). This situation is denoted by $a \sim b$.

$\sim$ becomes a congruence relation on the algebra $(A, \wedge, \vee, 0)$. It is known that $\sim$ is the smallest congruence on an ADL A such that the quotient $A/\sim$ is a lattice (and hence a distributive lattice).

3 $\wedge-$ Irreducible elements

For any elements a and b in an ADL A, we have from 2.6 (7 and 8) that $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a \Leftrightarrow glb\{a, b\}$ exists and equals to $a \wedge b \Leftrightarrow lub\{a, b\}$ exists and equals to $a \vee b$. Now, we introduce the following.

Definition 3.1. Let $A = (A, \wedge, \vee, 0)$ be an ADL and $p \in A$. Then p is said to be $\wedge-$ irreducible if, for any a and $b \in A$,

$$p = a \wedge b = b \wedge a \Rightarrow \text{ either } p = a \text{ or } p = b.$$

Example 3.2.

(1). In a discrete ADL X (as given in 2.4), every element is $\wedge-$ irreducible; for, $a \wedge b = b \wedge a$ implies that either $a = b$ or $a = 0$ or $b = 0$.

(2). Let A_1 and A_2 be two non trivial ADL's and $A = A_1 \times A_2$. Then A is an ADL under the coordinate wise operations. Choose $0 \neq a_1 \in A_1$ and $0 \neq b_2 \in A_2$. Then $(0, 0) = (a_1, 0) \wedge (0, b_2) = (0, b_2) \wedge (a_1, 0)$. Therefore $(0, 0)$ is not $\wedge-$ irreducible in A.

Definition 3.3. An element p in and ADL A is said to be strongly $\wedge-$ irreducible if, for any a and $b \in A$,

$$a \wedge b = b \wedge a \leq p \;\Rightarrow\; \text{either } \; a \leq p \text{ or } b \leq p.$$

In a distributive lattice, $a \wedge b \leq p$ is equivalent to saying that $p = (a \vee p) \wedge (b \vee p)$ and therefore an element in a distributive lattice is $\wedge-$ irreducible if and only if it is strongly $\wedge-$ irreducible. However, it is different in the case of ADL's.

The following is an easy verification, for we always have $a \wedge b \leq b$ and $b \wedge a \leq a$ for all a and b in an ADL A.

Theorem 3.4. *In any ADL, every strongly $\wedge-$ irreducible element is $\wedge-$ irreducible. The converse is not true.*

Example 3.5. Let A_1 and A_2 be two discrete ADL's, each with at least three elements and $A = A_1 \times A_2$, the product ADL. Choose $0 \neq p_1 \in A_1$ and $0 \neq p_2 \in A_2$ and let $p = (p_1, p_2)$. Then it can be easily verified that p is $\wedge-$ irreducible in A. However, p is not strongly $\wedge-$ irreducible; for, choose $a_i \in A_i - \{0, p_i\}$. Then

$$(0, a_2) \wedge (a_1, 0) = (a_1, 0) \wedge (0, a_2) = (0, 0) \leq p$$
$$\text{and } (0, a_2) \nleq p \;\text{ and }\; (a_1, 0) \nleq p.$$

Theorem 3.6. *Let p and q be associates to each other in an ADL A. Then p is $\wedge-$ irreducible if and only if q is $\wedge-$ irreducible.*

Proof. Since $p \sim q$, we have $p \wedge q = q,\; q \wedge p = p,\; p \vee q = p$ and $q \vee p = q$. Suppose that p is $\wedge-$ irreducible. Let a and $b \in A$ such that $q = a \wedge b = b \wedge a$. Then

$$p = p \vee q = p \vee (a \wedge b) = p \vee (b \wedge a) \text{ and therefore}$$
$$p = (p \vee a) \wedge (p \vee b) = (p \vee b) \wedge (p \vee a).$$

Since p is $\wedge-$ irreducible, we get that either $p = p \vee a$ or $p = p \vee b$. If $p = p \vee a$, then $a = p \wedge a = (q \wedge p) \wedge a = (p \wedge q) \wedge a = q \wedge a = (b \wedge a) \wedge a = b \wedge a = q$. Similarly, if $p = p \vee b$, then $b = q$. Therefore q is $\wedge-$ irreducible. The converse follows form the symmetricity of $\sim$. □

The validity of the above theorem is not known for the strongly $\wedge-$ irreducible elements.

4 $\vee-$ Irreducible elements

Unlike the case of lattices, we do not get an ADL again by interchanging the operations $\wedge$ and $\vee$ in a given ADL, for the simple reason that $\wedge$ distributes over $\vee$ both from left and right, while $\vee$ distributes over $\wedge$ from left alone. In fact, it is known that an ADL becomes a (distributive) lattice if and only if $\vee$ distributes over $\wedge$ from right also. This necessitates the study of $\vee-$ irreducible elements respectively.

Definition 4.1. Let $A = (A, \wedge, \vee, 0)$ be an ADL and $p \in A$. Then p is said to be $\vee-$ irreducible if, for any a and $b \in A$,

$$p = a \vee b = b \vee a \Longrightarrow \text{ either } p = a \text{ or } p = b.$$

Example 4.2.

(1). The zero element 0 in any ADL is always $\vee-$ irreducible, since $a \leq a \vee b$ for all a and b.

(2). Every element in a discrete ADL is $\vee-$ irreducible.

(3). Let A_1 and A_2 be two discrete ADL's and $A = A_1 \times A_2$. Let $0 \neq a_1 \in A_1$ and $0 \neq a_2 \in A_2$ and $a = (a_1, a_2)$. Then

$$a = (0, a_2) \vee (a_1, 0) = (a_1, 0) \vee (0, a_2)$$

and therefore a is not $\vee-$ irreducible in A.

The following is a straight forward verification.

Theorem 4.3. *Let A_1 and A_2 be two ADL's and $A = A_1 \times A_2$. An element (p_1, p_2) is $\vee-$ irreducible in A if and only if $p_1 = 0$ and p_2 is $\vee-$ irreducible in A_2, or $p_2 = 0$ and p_1 is $\vee-$ irreducible in A_1.*

Definition 4.4. An element p in an ADL A is said to be strongly $\vee-$ irreducible if, for any a and $b \in A$,

$$p \leq a \vee b = b \vee a \Longrightarrow \text{ either } p \leq a \text{ or } p \leq b.$$

We have proved in the previous section that a $\wedge-$ irreducible element in an ADL need not be strongly $\wedge-$ irreducible. This is not the case with $\vee-$ irreducibility.

Theorem 4.5. *Let A be an ADL and $p \in A$. Then p is $\vee-$ irreducible if and only if p is strongly $\vee-$ irreducible.*

Proof. Suppose that p is $\vee-$ irreducible. Let a and $b \in A$ such that $p \leq a \vee b = b \vee a$. Then $p = p \wedge (a \vee b) = p \wedge (b \vee a)$ and therefore

$$p = (p \wedge a) \vee (p \wedge b) = (p \wedge b) \vee (p \wedge a).$$

Since p is $\vee-$ irreducible, either $p = p \wedge a$ or $p = p \wedge b$ and therefore $p \leq a$ or $p \leq b$. Thus p is strongly $\vee-$ irreducible. Let a and $b \in A$ such that $p = a \vee b = b \vee a$. Then either $p \leq a$ or $p \leq b$.

$$p \leq a \Longrightarrow p = a \quad (\text{ since } a \leq a \vee b = p)$$
$$p \leq b \Longrightarrow p = p \wedge b = (a \vee b) \wedge b = b.$$

Thus p is $\vee-$ irreducible. □

Theorem 4.6. *Let p and q be associates to each other in an ADL A. Then p is (strongly) $\vee-$ irreducible if and only if so is q.*

Proof. Since $p \sim q$, we have $p \wedge q = q, q \wedge p = p, p \vee q = p$ and $q \vee p = q$. Suppose that p is $\vee-$ irreducible. Let a and $b \in A$ such that $q = a \vee b = b \vee a$. Then

$$p = q \wedge p = (a \vee b) \wedge p = (a \wedge p) \vee (b \wedge p)$$
$$\text{and } \; p = q \wedge p = (b \vee a) \wedge p = (b \wedge p) \vee (a \wedge p).$$

Since p is $\vee-$ irreducible, we get that either $p = a \wedge p$ or $p = b \wedge p$. Now

$$p = a \wedge p \Longrightarrow q = p \wedge q = a \wedge p \wedge q = a \wedge q = a \wedge (a \vee b) = a$$
$$\text{and } \; p = b \wedge p \Longrightarrow q = p \wedge q = b \wedge p \wedge q = b \wedge q = b \wedge (b \vee a) = b.$$

Thus q is $\vee-$ irreducible. The converse follows from the symmetricity of $\sim$. Now, Theorem 4.6 completes the proof.

□

References

[1] S. Burris and H. P. Sankappanavar, *A course in Universal algebra ,* Springer - Verlag, Newyork, 1980.

[2] G. Gratzer, *General Lattice Theory ,* Academic Press, New York, 1978.

[3] G. C. Rao and M. S. Rao, $\alpha-$ ideals and prime ideals of Almost Distributive Lattices, *Inter. J. of Algebra,* **3,** 221 – 229 (2009).

[4] M. H. Stone, The thoery of representations for Boolean Algebras, *Trans. Amer. Math. Soc.,* **40**, 37 – 111 (1936).

[5] U. M. Swamy and G. C. Rao, Almost Distributive Lattices, *J. Australian Math. Soc., (series A),* **31**, 77 – 91 (1981).

[6] U. M. Swamy and B. Venkateswarlu, Irreducible elements in Algebraic lattices, *Int. J. of Algebra and Computation,* **20(8)** , 969 – 975 (2010) .

[7] B. Venkateswarlu, B. Vasu Babu and E. Tigist, Irreducible ideals in rings, *Italian J. of pure and applied Math.,* **32**, 461 – 466 (2014).

[8] B. Venkateswarlu and U. M. Swamy, A characterization of irreducible elements in lattices, *Afr. Mat.* **26**, 619–626 (2015).

Author information

B. Venkateswarlu, Department of Mathematics, GIT, GITAM University, Visakhapatnam- 530 045, A.P., India.
E-mail: bvlmaths@gmail.com (Corresponding author)

Ch. Santhi Sundar Raj, Department of Engineering Mathematics, Andhra University, Visakhapatnam- 530 003, A.P., India.
E-mail: santhisundarraj@yahoo.com

R. Vasu Babu, Department of Mathematics, Shri Vishnu Engineering college for Women, Bhimavaram -534 201, W. G. Dist., A.P., India.
E-mail: vasukgr@gmail.

Received: July 22, 2014.

Accepted: February 21, 2015