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PRIME LIE RINGS OF GENERALIZED DERIVATIONS OF COMMUTATIVE RINGS

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Abstract Let R be a commutative ring with identity. By a Breŝar generalized derivation of R we mean an additive map $g : R \to R$ such that $g\,(xy) = g\,(x)\,y + xd\,(y)$ for all $x, y \in R$, where d is a derivation of R. And an additive mapping $f : R \to R$ is called a generalized derivation in the sense of Nakajima if it satisfies $f(xy) = f(x)y + xf(y) - xf(1)y$ for all $x, y \in R$. In this paper we extend some results of Chebotar and Lee [2] and, Liu and Passman [5] to generalized derivations of Nakajima and Breŝar. The main aim of this note is to give some properties of the Lie ring Rg which is the set of all Breŝar generalized derivations of R of the form rg with $r \in R$ and is to apply similar results to generalized derivations of Nakajima.

1 Introduction

In [3], C. R. Jordan and D. A. Jordan investigated how the ideal structure of the Lie ring of derivations of R is determined by the ideal structure of R. They proved that if R is a prime ring of characteristic not 2, then the Lie ring $D(R)$ of all derivations of R is a prime Lie ring. For this theorem, different proofs were given for the case where R is commutative and the case where R is noncommutative. In commutative case, the authors studied the structure of the Lie ring Rd of all derivations of R of the form $rd : x \mapsto rd(x)$, $r \in R$, which is a Lie subring of $D(R)$. And they proved that if R is a commutative domain of characteristic not 2 with identity, then Rd is a prime Lie ring. Later in [7], some conditions on R were weakened by A. Nowicki. Recently M. A. Chebotar and P. H. Lee, in [2], investigated the structure of the Lie ring Rd and established a relationship between the d-ideals of R and the ideals of Rd.

In this paper, we first give some definitions such as g-invariant subset, g-ideal and g-prime ring where g is a generalized derivation which was introduced by M. Breŝar [1]. Furthermore we introduce the Lie ring of all Breŝar generalized derivations of R and denote this ring as a left R-module. Then we extend some results of [2] to Breŝar generalized derivations and study the ideal structure of the Lie ring Rg. Finally, we establish similar relationships for generalized derivations in the sense of Nakajima.

2 Results

Let R be a ring. An additive mapping $d : R \to R$ is called a *derivation* if $d\,(xy) = d\,(x)\,y + xd\,(y)$ holds for all $x, y \in R$. The study of derivations of prime rings was initiated by Posner [8]. In [1], Breŝar defined the following notation. An additive mapping $g : R \to R$ satisfying

$$g\,(xy) = g\,(x)\,y + xd\,(y)$$

for all $x, y \in R$, where d is a derivation on R is said to be a *Breŝar generalized derivation* or *generalized d-derivation* of R and denoted by (g, d). In particular, if $1 \in R$, then $g(y) = g(1)y + d(y)$ for all $y \in R$. We denote by $BDer(R)$ the set of Breŝar generalized derivations of R.

A similar notion was introduced in [6] by Nakajima who gave some categorical properties of that generalized derivations without using derivations. Now we give the following which is defined on an unital ring R to investigate some of their properties on a Lie ring in the last section. An additive mapping $f : R \to R$ is called a generalized derivation in the sense of Nakajima if it satisfies

$$f(xy) = f(x)y + xf(y) - xf(1)y$$

for all $x, y \in R$. We denote the set of this type of generalized derivations by $gDer(R)$.

Let R be a ring with identity element 1. If $f : R \to R$ is a generalized derivation in the sense of Nakajima, then $d = f - f(1)_l$ is a derivation of R, where $f(1)_l$ is the left multiplication by $f(1)$, and we see that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. So it means that the notions of generalized derivations of Nakajima and Brešar coincide when R is an unital ring.

Now we remind some definitions and lemmas from [2], [3] and [7]:

A nonempty subset T of the ring R is called *d-invariant* if $d(T) \subseteq T$. An ideal of R is called a *d-ideal* if it is d-invariant.

For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$.

Let $L(R)$ be the Lie ring of R; thus the elements of $L(R)$ are the elements of R and the product in $L(R)$ is given by $[x, y]$ for all $x, y \in R$.

An ideal I of R is said to be a *d-prime ideal* of R if $I \neq R$, $d(I) \subseteq I$ and for all d-ideals A, B of R, the inclusion $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. The ring R is said to be *d-prime* if 0 is a d-prime ideal of R.

For Lie rings, we have analogous definition. A Lie ring $L(R)$ is said to be *(Lie) prime* if the product of any nonzero ideals of $L(R)$ is always nonzero.

Let S be a nonempty subset of R. The left annihilator $l(S)$ of S is the set of all elements a in R such that for each s in S, $as = 0$; that is, $l(S) = \{a \in R \mid aS = 0\}$. The right annihilator $r(S)$ of S in R is similarly defined. If R is a commutative ring, then there is no distinction between the left annihilator and the right annihilator of a nonempty subset S of R. In this case, we just call it the annihilator of S in R and denote it by $\mathrm{Ann}(S)$.

Lemma 2.1. *[2, Lemma 2.2] Let R be a commutative ring, and d a nonzero derivation of R such that R is d-prime. Then;*
(1) if S is a nonzero d-invariant subset of R, then $\mathrm{Ann}(S) = 0$.
(2) if $rd = 0$ for some $r \in R$, then $r = 0$.

2.1 Preliminary considerations

Now we give some new definitions and make some preliminary remarks we need later.

Let R be a ring and the pair (g, d) a Brešar generalized derivation of R. A nonempty subset T of R is called *g-invariant* if $g(T) \subseteq T$ and $d(T) \subseteq T$. An ideal of R is called a *g-ideal* if it is g-invariant.

We say that a ring R is *g-prime* if the product of any two nonzero g-ideals of R is nonzero.

The set of all Brešar generalized derivations of R, which we denoted by $BDer(R)$, is defined as a Lie ring whose elements are the Brešar generalized derivations of R where the additive structure of $(g, d), (\alpha, d') \in BDer(R)$ is given by pointwise addition

$$(g + \alpha)(x) = g(x) + \alpha(x) \quad (x \in R),$$

and the product is given by the commutator of the composites

$$[g, \alpha](x) = g(\alpha(x)) - \alpha(g(x)) \quad (x \in R).$$

If R is a commutative ring, then $BDer(R)$ can be viewed as a left R-module in a natural way: For $g \in BDer(R)$ and $s \in R$, the pair (sg, sd) is the Brešar generalized derivation of R which maps an element x of R to $sg(x)$.

Lemma 2.2. *Let R be a ring, S be a nonempty subset of R and (g, d) a nonzero Brešar generalized derivation of R. If S is g-invariant, so is the left annihilator $l(S)$ of S in R.*

Proof. Let $s \in S$. Since S is g-invariant, it follows that $g(s) \in S$ and also $d(s) \in S$.
Now let $a \in l(S)$. Using the definition of the left annihilator $l(S)$ of S in R, we get $aS = 0$. That is, $as = 0$ for all $s \in S$. Since $0 = g(as) = g(a)s + ad(s)$ for all $a \in l(S), s \in S$, we have $g(a)s = 0$ and then

$$g(a) \in l(S) \qquad \text{for all } a \in l(S). \tag{2.1}$$

Moreover since $0 = d(as) = d(a)s + ad(s)$ for all $a \in l(S), s \in S$, we get $d(a)s = 0$, so

$$d(a) \in l(S) \qquad \text{for all } a \in l(S). \tag{2.2}$$

From (2.1) and (2.2), it follows that $g(l(S)) \subseteq l(S)$ and $d(l(S)) \subseteq l(S)$. So this means that $l(S)$ is g-invariant. □

Remark 2.3. If R is a commutative ring, then the right annihilator $r(S)$ of S in R will be g-invariant by a similar proof of preceding lemma.

Corollary 2.4. *If a ring R is g-prime then the left annihilator (or the right annihilator) of any nonzero g-ideal of R is zero.*

Lemma 2.5. *Let R be a commutative ring, and (g,d) a Brešar generalized derivation of R. If R is g-prime and K is a nonzero g-invariant subset of R, then* $\mathrm{Ann}(K)=0$.

Proof. Let I be the ideal of R generated by the subset K of R, that is, $I=K+KR$. Clearly, I is nonzero and $Ann(K)=\mathrm{Ann}(I)$. Since K is g-invariant, then I is g-invariant and therefore I is a g-ideal of R. By the g-primeness of the ring R, Corollary 2.4 implies that the annihilator of any nonzero g-ideal of R is zero. That is $\mathrm{Ann}(I)=0$ and so we have $\mathrm{Ann}(K)=0$. □

Lemma 2.6. *Let R be a commutative ring, and (g,d) a nonzero Brešar generalized derivation of R where d is a nonzero derivation of R. If R is g-prime and m is an element in R such that $mg=0$, then $m=0$.*

Proof. If $mg=0$, that is, $mg(x)=0$ for all $x\in R$, then we obtain

$$0=mg(rx)=mg(r)x+mrd(x) \qquad \text{for all } x,r\in R.$$

Since $mg=0$, we get $mrd(x)=0$ for all $x,r\in R$. From Lemma 2.1(ii), we have $mr=0$ for all $r\in R$. Since $d(R)\subseteq R$, we obtain that $md(R)=0$. By using Lemma 2.1(ii) again, we have $m=0$. □

2.2 The structure of the Lie ring Rg

Let R be a commutative ring. The set, Rg, of all Brešar generalized derivations of R of the form $rg: x\mapsto rg(x)$ with $r\in R$, is a subring of the Lie ring $BDer(R)$ of Brešar generalized derivations of R. In this paper we shall not view Rg as a Lie subring of $BDer(R)$ but rather as a Lie ring whose elements are the elements of R and whose product is as follows:

$$[ag,bg]=(ag(b)-bg(a))\,g \qquad \text{for all } a,b\in R.$$

Let (g,d) be a Brešar generalized derivation of R and A be an ideal of the Lie ring Rg. Then we set

$$\gamma(A)=\{a\in R \mid ag\in A \text{ and } ad(x)\,g\in A \text{ for all } x\in R\}.$$

It's easy to see that $\gamma(A)$ is an additive subgroup of R. And if R contains an identity and $g(1)\gamma(A)\subseteq\gamma(A)$, then $\gamma(A)$ is g-invariant; for if $a\in\gamma(A)$ then $d(a)g=[g,ag]\in A$ and $d(a)d(x)g=[g,ad(x)g]-ad^2(x)\,g\in A$ so we get $d(a)\in\gamma(A)$; and also $g(a)g=g(1)ag+d(a)g\in A$ and $g(a)d(x)g=g(1)ad(x)g+d(a)d(x)g\in A$ since $g(1)a\in\gamma(A)$ so we have $g(a)\in\gamma(A)$. To prove all these identities we use the definition of Brešar generalized derivation where $1\in R$.

From now on we shall assume that if R contains an identity and if there exists an ideal A of the Lie ring Rg, then $g(1)\gamma(A)\subseteq\gamma(A)$.

Theorem 2.7. *Let R be a 2-torsion free commutative ring with identity and (g,d) a nonzero Brešar generalized derivation of R where d is a nonzero derivation of R. If R is g-prime, then Rg is a prime Lie ring.*

Proof. Suppose that A and B are two ideals of the Lie ring Rg such that $[A,B]=0$. Let $a\in\gamma(A)$ and $b\in\gamma(B)$. That is, $ag\in A$, $bg\in B$, $ad(x)\,g\in A$ and $bd(x)\,g\in B$ for all $x\in R$. Then we have $[ag,bg]=0$, or equivalently, $(ad(b)-bd(a))\,g=0$ by using the definition of Brešar generalized derivation where R contains an identity. By Lemma 2.6 we get

$$ad(b)=bd(a) \qquad \text{for all } a\in\gamma(A),\ b\in\gamma(B). \tag{2.3}$$

Moreover, we obtain $[ag,bd(x)\,g]=0$, or equivalently, $abd^2(x)\,g=0$ by using (2.3). Also applying Lemma 2.6 to the last equation, we have

$$abd^2(x)=0 \qquad \text{for all } a\in\gamma(A),\ b\in\gamma(B), x\in R. \tag{2.4}$$

Now suppose that $A \neq 0$, $B \neq 0$; then $\gamma(A) \neq 0$ and $\gamma(B) \neq 0$. Recall that both $\gamma(A)$ and $\gamma(B)$ are g-invariant. Applying Lemma 2.5 twice to (2.4), we have $d^2(x) = 0$ for all $x \in R$. When we replace x by x^2 in the last equation, we conclude from that

$$2d(x)^2 = 0 \qquad \text{for all } x \in R.$$

Since R is a 2-torsion free ring, then we get $d(x)^2 = 0$ for all $x \in R$. By linearizing the last identity, we have $2d(x)d(y) = 0$ for all $x, y \in R$. So we use the 2-torsion freeness of R again, and we find that

$$d(x) = 0 \qquad \text{for all } x \in R \tag{2.5}$$

from Lemma 2.1(ii), giving a contradiction. This proves the theorem. □

Theorem 2.8. *Let R be a commutative ring with identity, and (g, d) a nonzero Bresãr generalized derivation of R where d is a nonzero derivation of R such that R is g-prime. If J is a g-ideal of R, then Jg and $Jd(x)g$ are ideals of the Lie ring Rg for all $x \in R$. Conversely, any nonzero ideal of the Lie ring Rg contains ideals of the form Jg and $Jd(x)g$ for all $x \in R$ for some nonzero ideal J of the ring R which satisfies $g(J) \subseteq J$.*

Proof. If J is a g-ideal of R, then $[ag, sg] = (ad(s) - sd(a))g \in Jg$ and $[ad(x)g, sg] = (ad(s) - sd(a))d(x)g - asd^2(x)g \in Jd(x)g$ for any $a \in J$ and $s, x \in R$. Therefore, Jg and $Jd(x)g$ are ideals of the Lie ring Rg.

Conversely, let A be a nonzero ideal of Rg. It follows by Theorem 2.7 that Rg is prime. Hence $[A, A] \neq 0$. Then, since Rg is prime, there exists $a, b \in \gamma(A)$ such that $[ag, bg] = (ad(b) - bd(a))g \neq 0$.

Let $s \in R$. Then $[ag, bsg] + [asg, bg] \in A$, that is,

$$(ad(bs) - bsd(a) + asd(b) - bd(as))g \in A.$$

Moreover, $[ad(x)g, bsg] + [asg, bd(x)g] \in A$, that is,

$$(ad(bs) - bsd(a) + asd(b) - bd(as))d(x)g \in A.$$

So we obtain

$$ad(bs) - bsd(a) + asd(b) - bd(as) \in \gamma(A).$$

By extending the last relation, we get

$$2(ad(b) - bd(a))s \in \gamma(A) \qquad \text{for all } s \in R.$$

Thus there exists a nonzero ideal $I = 2(ad(b) - bd(a))R$ of R contained in $\gamma(A)$. Since $\gamma(A)$ is g-invariant, this yields $g^n(I) \subseteq \gamma(A)$ for all $n = 0, 1, 2, \ldots$. Let $J = \sum_{n=0}^{\infty} g^n(I)$. It is clear that J is an additive subgroup of R such that $g(J) \subseteq J$. We now prove by induction on n that $g^n(I)R \subseteq \sum_{k=0}^{n} g^k(I)$ for all $n \geq 0$ to show that J is a right ideal of R. For $n = 0$, this has already been established since $IR \subseteq I$. Let $n > 0$. Then, $g(g^n(I)R) = g^{n+1}(I)R + g^n(I)d(R)$. And it follows that

$$\begin{aligned} g^{n+1}(I)R &\subseteq g(g^n(I)R) + g^n(I)d(R) \\ &\subseteq g\left(\sum_{k=0}^{n} g^k(I)\right) + g^n(I)R \\ &\subseteq \sum_{k=1}^{n+1} g^k(I) + \sum_{k=0}^{n} g^k(I) \\ &= \sum_{k=0}^{n+1} g^k(I). \end{aligned}$$

Hence, by induction, $JR \subseteq J$, that is, J is a right ideal of R. Similarly, one can show that J is also a left ideal of R and so J is an ideal of R.

So $J = \sum_{n=0}^{\infty} g^n(I)$ is a nonzero ideal of R which is contained in $\gamma(A)$. Hence, A contains the ideals Jg and $Jd(x)g$ for all $x \in R$. This completes the proof. □

2.3 The structure of the Lie ring G

Let R be a ring with identity element 1. We recall that $gDer(R)$ is the set of all generalized derivations on R in the sense of Nakajima.

Since R is an unital ring, we know that the sets $gDer(R)$ and $BDer(R)$ coincide. So we can assume that the set $gDer(R)$ is a Lie ring under the product

$$[f, f'] = ff' - f'f$$

for $f, f' \in gDer(R)$ and when R is commutative $gDer(R)$ also has an R-module structure given by $(rf)(x) = rf(x)$ for any $r, x \in R$ and $f \in gDer(R)$. And, because of the same reason the other definitions we used for Brešar generalized derivations (g, d) such as g-invariant subset, g-ideal and g-prime ring can be used for all $(f, d') \in gDer(R)$ where $d' = f - f(1)_l$.

So we can recall some definitions to use in this section. For any subset G of $gDer(R)$, an ideal of R will be called a G-ideal if $f(G) \subseteq G$ and $d(G) \subseteq G$ for all $(f, d) \in G$ where $d = f - f(1)_l$. Note that R is G -prime if the product of any two nonzero G-ideals of R is nonzero.

Firstly, we wanted to extend Theorem 2.7 to the case where G is a Lie subring and an R-submodule of $gDer(R)$. And, we asked this question: Is G a prime Lie ring if R is G-prime? The answer is not known in general, but to obtain an analogous result for generalized derivations in the sense of Nakajima, we certainly need to suppose some additional conditions.

From now on, we fix the notations that R is a commutative ring with identity element 1, G is a nonzero Lie subring and also an R-submodule of $gDer\,(R)$.

Let $f_1, f_2 \in G$ and $r, s \in R$. The composition $f_1\,(rf_2)$ is given by

$$f_1\,(rf_2) = f_1\,(r)\,f_2 + rd_1f_2$$

where $d_1 = f_1 - f_1\,(1)_l$, and $[rf_1, f_2] = r\,[f_1, f_2] - f_2\,(r)\,f_1 + rf_2\,(1)\,f_1$, $[f_1, sf_2] = s\,[f_1, f_2] + f_1\,(s)\,f_2 - sf_1\,(1)\,f_2$. And also we have

$$[rf_1, sf_2] = rs\,[f_1, f_2] + rf_1\,(s)\,f_2 - sf_2\,(r)\,f_1 - rs\,(f_1\,(1)\,f_2 - f_2\,(1)\,f_1) \qquad (2.6)$$

We will investigate the following: if R is G-prime and $\mathrm{char}R = 2$ is it true that G is prime?

Definition 2.9. *Let L be an ideal of G. Set $\tilde{L} = \{g \in G \mid Rg \subseteq L\}$ and $g(L) = \sum_{\alpha \in L} (\alpha(R) - R\alpha(1))R$. Also define the set $\mathrm{Ann}_R(L) = \{r \in R \mid rL = 0\}$.*

Lemma 2.10. *Let L be an ideal of G. Then $g(L) = \sum_{\alpha \in L} (\alpha(R) - R\alpha(1))R$ and also $\mathrm{Ann}_R(L)$ are G-ideals of R.*

Proof. If $f \in G, \alpha \in L$ and $x \in R$, then

$$\begin{aligned}
f(\alpha(x) - x\alpha(1)) &= [f, \alpha](x) + \alpha f(x) - f(x)\alpha(1) - x[f, \alpha](1) \\
&\quad -x\alpha f(1) + xf(1)\alpha(1) \\
&= ([f, \alpha](x) - x[f, \alpha](1)) + (\alpha f(x) - f(x)\alpha(1)) \\
&\quad -x(\alpha f(1) - f(1)\alpha(1)) \\
&\in ([f, \alpha](R) - R[f, \alpha](1)) + (\alpha(R) - R\alpha(1))R.
\end{aligned}$$

So $f((\alpha(R) - R\alpha(1))R) = f(\alpha(R) - R\alpha(1))R + (\alpha(R) - R\alpha(1))d(R) \subseteq g(L)$ where $d = f - f(1)_l$. And since d is a derivation of R, then we have

$$\begin{aligned}
d((\alpha(R) - R\alpha(1))R) &= d(\alpha(R) - R\alpha(1))R + (\alpha(R) - R\alpha(1))d(R) \\
&= (f - f(1)_l)(\alpha(R) - R\alpha(1))R + (\alpha(R) - R\alpha(1))d(R) \\
&= f(\alpha(R) - R\alpha(1))R - f(1)(\alpha(R) - R\alpha(1))R \\
&\quad +(\alpha(R) - R\alpha(1))d(R) \\
&\subseteq g(L).
\end{aligned}$$

So it follows that $g(L)$ is a G-ideal of R. If $x \in \mathrm{Ann}_R(L)$ and $\alpha \in L$, we get $x\alpha = 0$ and therefore for any $f \in G$ we have $0 = f(x\alpha) = f(x)\alpha + xd\alpha$ where $d = f - f(1)_l$. Then

$$\begin{aligned} f(x)\alpha &= -xd\alpha = -x(f - f(1)_l)\alpha = -xf\alpha + xf(1)\alpha \\ &= -x[f,\alpha] - x\alpha f = 0 \end{aligned}$$

since R is commutative and L is an ideal of G. Thus $f(x)\alpha = 0$ for all $\alpha \in L$ and $f(x) \in \mathrm{Ann}_R(L)$. And since d is a derivation of R, then we get $0 = d(x\alpha) = d(x)\alpha + xd\alpha$, so by the similar way, $d(x)\alpha = -xd\alpha = 0$ for all $\alpha \in L$ and $d(x) \in \mathrm{Ann}_R(L)$. Therefore $\mathrm{Ann}_R(L)$ is a G-ideal of R. □

Lemma 2.11. *Let L be an ideal of G. Then $\tilde{L}$ is both an ideal and R-submodule of G.*

Proof. Let $g \in \tilde{L}, f \in G$ and $r \in R$. Since $Rrg \subseteq Rg \subseteq L$ then $\tilde{L} \subseteq L$ is an R-submodule of G. From (2.1), $[rg, 1.f] = r[g,f] - f(r)g + rf(1)g$ and we get

$$r[g,f] = [rg,f] - rf(1)g + f(r)g \in [L,G] + L = L.$$

So $\tilde{L}$ is an ideal of G. □

Lemma 2.12. *Let L be any nonzero ideal of G. If R is G-prime and r is an element in R such that $rL = 0$, then $r = 0$.*

Proof. Note that $\mathrm{Ann}_R(L).g(L) = 0$. Since $L \neq 0$ then we have $g(L) \neq 0$, so $r \in \mathrm{Ann}_R(L) = 0$ by G-primeness of R. □

The following result is a modification of a theorem of Jordan [4]:

Theorem 2.13. *Let L be an ideal of G. If L is an R-submodule of G, then $g(L)G \subseteq L$.*

Proof. Since L is an ideal of G, then $g(L) = \sum\limits_{\alpha\in L}(\alpha(R) - R\alpha(1))R$ is a G-ideal of R from Lemma 2.10. For any $f \in G, \alpha \in L$ and $x \in R$, then we get

$$(\alpha(x) - x\alpha(1))f = [\alpha, xf] - x[\alpha, f] \in L + RL \subseteq L$$

and this shows that $g(L)G \subseteq L$ since $(\alpha(R) - R\alpha(1))RG = (\alpha(R) - R\alpha(1))G$ for any $\alpha \in L$. □

Lemma 2.14. *Suppose that R is G-prime and let L, M are ideals of G such that $[L,M] = 0$, then we have $\tilde{L} = 0$ or $\tilde{M} = 0$.*

Proof. By Theorem 2.13, there exist G-ideals $A = g(\tilde{L})$ and $B = g(\tilde{M})$ of R such that $AG \subseteq \tilde{L}$ and $BG \subseteq \tilde{M}$. For any $f_1, f_2 \in G, a \in A, b \in B$, we have $[af_1, bf_2] \in [L,M] = 0$ and hence

$$ab[f_1,f_2] + af_1(b)f_2 - bf_2(a)f_1 - abf_1(1)f_2 + abf_2(1)f_1 = 0.$$

Since A is an ideal of R, replace a by xa and we get

$$\begin{aligned} 0 &= xab[f_1,f_2] + xaf_1(b)f_2 - bf_2(xa)f_1 - xabf_1(1)f_2 + xabf_2(1)f_1 \\ &= x(ab[f_1,f_2] + af_1(b)f_2 - bf_2(a)f_1 - abf_1(1)f_2 + abf_2(1)f_1) \\ &\quad -bf_2(x)af_1 + bxf_2(1)af_1 \\ &= -bf_2(x)af_1 + bf_2(1)axf_1 \\ &= -ba(f_2(x) - xf_2(1))f_1 \end{aligned}$$

for all $f_1, f_2 \in G$ and $a \in A, b \in B, x \in R$. Then $BA(g(G))^2 = 0$. Since $B = g(\tilde{M}) \subseteq g(G)$ and $A = g(\tilde{L}) \subseteq g(G)$, we have $g(\tilde{M})^2 g(\tilde{L})^2 = 0$. This implies that $g(\tilde{M}) = 0$ or $g(\tilde{L}) = 0$ since R is G-prime. Therefore, either $\tilde{M} = 0$ or $\tilde{L} = 0$. □

Now we study the special case when $\mathrm{char}R = 2$.

Theorem 2.15. *Let R be G-prime and $char(R) = 2$. If $f(1)^2(f^2(x) - xf^2(1)) \neq 0$ for all $x \in R, f \in G$, then the Lie ring G is prime.*

Proof. Suppose that G is not prime. Then there exist nonzero ideals L, M of G such that $[L, M] = 0$. For any $\beta \in M, \alpha \in L$, $f \in G$ and $x \in R$, we have $\alpha(x)\beta - x\alpha(1)\beta = [\alpha, x\beta] \in L$ since $[L, M] = 0$.
It follows that $[xf, \alpha(x)\beta - x\alpha(1)\beta] \in L$ and $[x\beta(x)f - x^2\beta(1)f, \alpha] \in L$. So

$$\begin{aligned} & x(f\alpha(x)\beta - \alpha(x)f(1)\beta - f(x)\alpha(1)\beta - xf\alpha(1)\beta) \\ = \ & [xf, \alpha(x)\beta - x\alpha(1)\beta] - [x\beta(x)f - x^2\beta(1)f, \alpha] \in L \end{aligned}$$

since $[\alpha, [\beta, xf]] \in [L, M] = 0$ for any $\beta \in M, \alpha \in L$, $f \in G$ and $x \in R$. Therefore, Lemma 2.14 implies that

$$f\alpha(x)\beta - \alpha(x)f(1)\beta - f(x)\alpha(1)\beta - xf\alpha(1)\beta \in \tilde{L} = 0$$

for any $\beta \in M, \alpha \in L$, $f \in G$ and $x \in R$. Hence

$$(f\alpha(x) - \alpha(x)f(1) - f(x)\alpha(1) - xf\alpha(1))\beta = 0. \tag{2.7}$$

By Lemma 2.12, we see that

$$(f\alpha(x) - \alpha(x)f(1) - f(x)\alpha(1) - xf\alpha(1)) = 0 \tag{2.8}$$

for any $\alpha \in L$, $f \in G$ and $x \in R$. Let $S_L = \{x \in R \mid xf \in L, \forall f \in G\}$ and $S_M = \{x \in R \mid xf \in M, \forall f \in G\}$. Note that if $s \in S_M$, then

$$f(s)f - sf(1)f = [f, sf] \in M$$

for any $f \in G$ and hence $f(s) - sf(1) \in S_M$. Similarly, if $s' \in S_L$, then $f(s') - s'f(1) \in S_L$. Moreover, since $0 \neq L \subseteq G$ and $0 \neq M \subseteq G$, we see that $S_L, S_M \neq 0$.
For any $s_1 \in S_L$, $\beta \in M$, $f \in G$, we have

$$\beta(s_1)f(1)\alpha + s_1\beta(f(1))\alpha = [\beta, s_1 f(1)\alpha] \in [M, L] = 0$$

since G is a nonzero R-submodule of $gDer(R)$. So it follows from Lemma 2.12 that

$$\beta(s_1)f(1) + s_1\beta(f(1)) = 0.$$

Now we replace β by $s_2 f(1)f$ in the last equation where $s_2 \in S_M$ and we see that

$$s_2 f(1)f(s_1)f(1) + s_1 s_2 f(1)f^2(1) = 0$$

for any $s_1 \in S_L$, $s_2 \in S_M$ and $f \in G$. Then, $S_M f(1)(f(s_1)f(1) + s_1 f^2(1)) = 0$ and hence

$$(S_M G)f(1)(f(s_1)f(1) + s_1 f^2(1)) = 0.$$

Since $S_M G$ is a nonzero ideal of G, we get

$$f(1)(f(s_1)f(1) + s_1 f^2(1)) = 0 \tag{2.9}$$

by Lemma 2.12. If we replace α by $s_1 f(1)f$ in (2.8), then we have

$$\begin{aligned} 0 &= f(s_1 f(1)f(x)) - s_1 f(1)f(x)f(1) - f(x)s_1 f(1)f(1) - xf(s_1 f(1)f(1)) \\ &= (f(s_1)f(1) + s_1 f^2(1))f(x) + s_1 f(1)(f^2(x) - xf^2(1)) - xf(1)(f(s_1)f(1) \\ &\quad + s_1 f^2(1)) \end{aligned}$$

for any $s_1 \in S_L$, $x \in R$ and $f \in G$. So it follows from (2.9), we see that

$$s_1 f(1)^2(f^2(x) - xf^2(1)) = 0$$

if we multiply the last equation by $f(1)$. Therefore we have $(S_L G)f(1)^2(f^2(x) - xf^2(1)) = 0$ for any $x \in R$ and $f \in G$. Since $S_L G$ is a nonzero ideal of G, we get $f(1)^2(f^2(x) - xf^2(1)) = 0$ by Lemma 2.12. □

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