

Stability of Gorenstein n -flat modules

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. In this paper we are concerned with the stability of the class of Gorenstein n -flat modules. We give an answer for the following natural question in the setting of a left GF_n -closed ring R : Given an exact sequence of Gorenstein n -flat R -modules $\mathbf{G} = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$ such that the complex $H \otimes_R \mathbf{G}$ is exact for each Gorenstein n -absolutely pure right R -module H , is the module $M := \text{Im}(G_0 \rightarrow G^0)$ a Gorenstein n -flat module?

1 Introduction

Sather-Wagstaff et al. proved in [10] that iterating the process used to define Gorenstein projective modules exactly leads to the Gorenstein projective modules. Also, they established in [11] a stability of the subcategory of Gorenstein flat modules under a procedure to build R -modules from complete resolutions. Further Samir Bouchiba et al. in [3] proved over a left GF -closed ring R , the stability of the Gorenstein flat modules under the very process used to define these entities. Recently Z. Wang and Z. Liu in [13] proved that the two-degree strongly Gorenstein flat modules are nothing more than the strongly Gorenstein flat modules. Motivated by these works, we are concerned with the stability of the class of Gorenstein n -flat modules introduced in [12].

On the other hand, in [1], Bennis defined and studied the notion of left GF -closed rings. These are rings for which $\mathcal{GF}(R)$ (class of Gorenstein flat R -modules) is closed under extensions, that is for any exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of R -modules if $A, C \in \mathcal{GF}(R)$, then $B \in \mathcal{GF}(R)$. In this paper we introduced the concept of a left GF_n -closed ring (see Definition 3.2) where n is a non-negative integer and give a characterization of this ring and show that the two-degree Gorenstein n -flat modules and Gorenstein n -flat modules are coincide when R is a left GF_n -closed ring.

Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R -modules. Let $\mathcal{M}(R)$ denote the category of left R -modules. For an R -module M , we use M^+ to denote the character module $\text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M . Let M and N be R -modules. $\text{Hom}(M, N)$ (resp. $\text{Ext}^i(M, N)$) means $\text{Hom}_R(M, N)$ (resp. $\text{Ext}_R^i(M, N)$), and similarly $M \otimes N$ (resp. $\text{Tor}_i(M, N)$) denotes $M \otimes_R N$ (resp. $\text{Tor}_i^R(M, N)$) for an integer $i \geq 1$. A left R -module M is called n -flat [8] if $\text{Tor}_1(N, M) = 0$ holds for all finitely presented right R -modules N with projective dimension $\leq n$ and a right R -module M is called n -absolutely pure [8] if $\text{Ext}^1(N, M) = 0$ holds for all finitely presented right R -modules N with projective dimension $\leq n$. This paper is organized as follows. In Section 2, we recall some known definitions and introduce two-degree Gorenstein n -flat modules as well as we initiate the main theorem of this paper. In Section 3, we introduce the definition of a left GF_n -closed ring and give a characterization of this ring. In the last section over a left GF_n -closed ring, we show that the two-degree Gorenstein n -flat modules are nothing more than that the Gorenstein n -flat modules.

2 Gorenstein n -flat modules

In this section we recall some known definitions and introduce two-degree Gorenstein n -flat modules as well as we initiate the main theorem of this paper.

Definition 2.1. [12] A left R -module M is said to be Gorenstein n -flat, if there exists an exact sequence of n -flat left R -modules,

$$\cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

such that $M \cong \text{Im}(F_0 \rightarrow F^0)$ and such that $E \otimes_R -$ leaves the sequence exact whenever E is an n -absolutely pure right R -module.

Definition 2.2. [12] A right R -module M is said to be Gorenstein n -absolutely pure, if there exists an exact sequence of n -absolutely pure right R -modules

$$\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$$

such that $M \cong \text{Im}(A_0 \rightarrow A^0)$ and such that $\text{Hom}_R(E, -)$ leaves the sequence exact whenever E is an n -absolutely pure right R -module.

Next, we introduce the following definition

Definition 2.3. A left R -module M is called two-degree Gorenstein n -flat if there exists an exact sequence of Gorenstein n -flat left R -modules

$$\cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$$

such that $M \cong \text{Im}(G_0 \rightarrow G^0)$ and it remains exact after applying $H \otimes_R -$ for any Gorenstein n -absolutely pure right R -module H .

Let $\mathcal{GF}_n(R)$, $\mathcal{GA}_n(R)$ and $\mathcal{G}^{(2)}\mathcal{F}_n(R)$ are denotes the class of all Gorenstein n -flat left, Gorenstein n -absolutely pure right and two-degree Gorenstein n -flat left modules over R respectively. Also denote $\mathcal{G}_i^{(2)}\mathcal{F}_n(R)$ the subcategory of $\mathcal{M}(R)$ for which there exists an exact sequence of Gorenstein n -flat R -modules

$$\cdots \rightarrow G_1 \rightarrow G_0 \rightarrow G^0 \rightarrow G^1 \rightarrow \cdots$$

such that $M \cong \text{Im}(G_0 \rightarrow G^0)$ and it remains exact after applying $E \otimes_R -$ for any n -absolutely pure right R -module E . It is routine to check that

$$\mathcal{GF}_n(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}_n(R).$$

Our main theorem proves that these inequalities turn out to be equalities when R is a left GF_n -closed ring as is stated next.

Main Theorem: Let R be a left GF_n -closed ring. Then $\mathcal{GF}_n(R) = \mathcal{G}^{(2)}\mathcal{F}_n(R) = \mathcal{G}_i^{(2)}\mathcal{F}_n(R)$.

3 GF_n -closed ring

In this section we introduce the definition of a left GF_n -closed ring and give a characterization of this ring. First recall the following definitions:

Definition 3.1. Let R be a ring and let \mathfrak{X} be a class of left R -modules.

- (1) \mathfrak{X} is closed under extensions: If for every short exact sequence of left R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the conditions A and C are in \mathfrak{X} implies B is in \mathfrak{X} .
- (2) \mathfrak{X} is closed under kernels of epimorphisms: If for every short exact sequence of left R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, the conditions B and C are in \mathfrak{X} implies A is in \mathfrak{X} .
- (3) \mathfrak{X} is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms. i.e., for every short exact sequence of R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.

Definition 3.2. A ring R is said to be left GF_n -closed if $\mathcal{GF}_n(R)$ is closed under extensions.

Since Gorenstein n -flat modules need not be Gorenstein flat (see [12, Example 3.3]), we get left GF_n -closed ring need not be left GF -closed ring. Also recall that a ring R is called right n -coherent [8] (for integers $n > 0$ or $n = \infty$) if every finitely generated submodule of a free right R -module whose projective dimension is $\leq n - 1$ is finitely presented.

Example 3.3. Every right n -coherent ring is left GF_n -closed.

Proof. It follows from [12, Proposition 3.4], [12, Lemma 3.5] and [7, Proposition 1.4]. □

We begin with the following result.

Lemma 3.4. *The following are equivalent for a left R -module M :*

- (1) M is Gorenstein n -flat;
- (2) M satisfies the two following conditions:
 - (i) $Tor_i(E, M) = 0$ for all $i > 0$ and all n -absolutely pure right R -modules E ; and
 - (ii) There exists an exact sequence of left R -modules $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \dots$, where each F^i is n -flat, such that $E \otimes_R -$ leaves the sequence exact whenever E is an n -absolutely pure right R -module;
- (3) There exists a short exact sequence of left R -modules $0 \rightarrow M \rightarrow F \rightarrow G \rightarrow 0$, where F is n -flat and G is Gorenstein n -flat.

Proof. Using the definition of Gorenstein n -flat modules, the equivalence (1) \Leftrightarrow (2) is obtained by standard argument. Also, by definition, we get immediately the implication (1) \Rightarrow (3).

We prove the implication (3) \Rightarrow (2). Suppose that there exists a short exact sequence of left R -modules:

$$(\alpha) : 0 \rightarrow M \rightarrow F \rightarrow G \rightarrow 0$$

where F is n -flat and G is Gorenstein n -flat. Let E be an n -absolutely pure right R -module. Since G is Gorenstein n -flat and by the equivalence (1) \Leftrightarrow (2), $Tor_{i+1}(E, G) = 0$ for all $i \geq 0$. Then, use the long exact sequence,

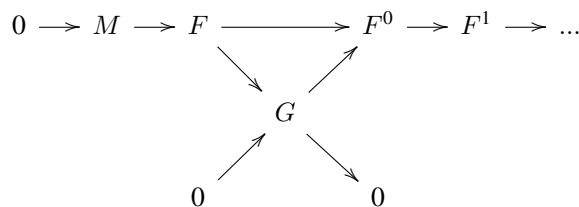
$$Tor_{i+1}(E, G) \rightarrow Tor_i(E, M) \rightarrow Tor_i(E, F),$$

to get $Tor_i(E, M) = 0$ for all $i > 0$.

On the other hand, since G is Gorenstein n -flat, there is an exact sequence of left R -modules:

$$(\beta) : 0 \rightarrow G \rightarrow F^0 \rightarrow F^1 \rightarrow \dots,$$

where each F^i is n -flat, such that $E \otimes_R -$ leaves the sequence exact whenever E is an n -absolutely pure right R -module. Assembling the sequences (α) and (β) , we get the following commutative diagram:



such that $E \otimes_R -$ leaves the upper exact sequence exact whenever E is an n -absolutely pure right R -module, as desired. □

Lemma 3.5. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of left R -modules. If A is Gorenstein n -flat and C is n -flat, then B is Gorenstein n -flat*

Proof. Since A is Gorenstein n -flat, there exists a short exact sequence of left R -modules $0 \rightarrow A \rightarrow F \rightarrow G \rightarrow 0$, where F is n -flat and G is Gorenstein n -flat. Consider the following pushout

diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \vdots & & \parallel \\
 0 & \longrightarrow & F & \dashrightarrow & F' & \longrightarrow & C \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & G & \equiv & G & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

In the sequence $0 \rightarrow F \rightarrow F' \rightarrow C \rightarrow 0$, both F and C are n -flat, hence so is F' . Then, by the middle vertical sequence and from Lemma 3.4, B is Gorenstein n -flat, as desired. \square

Now we give the main theorem of this Section which is analog to Theorem 2.3 in [1]

Theorem 3.6. *The following conditions are equivalent for a ring R :*

- (1) R is left GF_n -closed;
- (2) The class $\mathcal{GF}_n(R)$ is projectively resolving;
- (3) For every short exact sequence of left R -modules $0 \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$, where G_0 and G_1 are Gorenstein n -flat. If $\text{Tor}_1(E, M) = 0$ for all n -absolutely pure right R -modules E , then M is Gorenstein n -flat.

Proof. (1) \Rightarrow (2). To claim that the class $\mathcal{GF}_n(R)$ is projectively resolving, it suffices to prove that it is closed under kernels of epimorphisms (see Definition 3.1). Then, consider a short exact sequence of left R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where B and C are Gorenstein n -flat. We prove that A is Gorenstein n -flat. Since B is Gorenstein n -flat, there exists a short exact sequence of left R -modules $0 \rightarrow B \rightarrow F \rightarrow G \rightarrow 0$, where F is n -flat and G is Gorenstein n -flat. Consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \vdots \\
 0 & \longrightarrow & A & \longrightarrow & F & \dashrightarrow & D \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G & \equiv & G \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

By the right vertical sequence and since R is left GF_n -closed, the R -module D is Gorenstein n -flat. Therefore, by the middle horizontal sequence and Lemma 3.4, A is Gorenstein n -flat, as desired.

(1) \Rightarrow (3). Since G_1 is Gorenstein n -flat, there exists a short exact sequence of left R -modules $0 \rightarrow G_1 \rightarrow F_1 \rightarrow H \rightarrow 0$, where F_1 is n -flat and H is Gorenstein n -flat. Consider the

following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \longrightarrow & F_1 & \dashrightarrow & D & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & H & \xlongequal{\quad} & H & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

In the short exact sequence $0 \rightarrow G_0 \rightarrow D \rightarrow H \rightarrow 0$ both G_0 and H are Gorenstein n -flat, then so is D since R is left GF_n -closed. Then, there exists a short exact sequence of left R -modules $0 \rightarrow D \rightarrow F \rightarrow G \rightarrow 0$, where F is n -flat and G is Gorenstein n -flat. Consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & F_1 & \longrightarrow & D & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & F_1 & \longrightarrow & F & \dashrightarrow & F' \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G & \xlongequal{\quad} & G \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

We want to show that F' is n -flat. Consider the sequence $0 \rightarrow M \rightarrow F' \rightarrow G \rightarrow 0$. Let E be an n -absolutely pure right R -module. By the exact sequence,

$$0 = \text{Tor}_1(E, M) \rightarrow \text{Tor}_1(E, F') \rightarrow \text{Tor}_1(E, G) = 0,$$

we get

$$(*) \quad \text{Tor}_1(E, F') = 0.$$

On the other hand, consider the sequence $0 \rightarrow F_1 \rightarrow F \rightarrow F' \rightarrow 0$. By Rotman (1979, Lemma 3.51), we have the following short exact sequence of character modules:

$$(\beta) = 0 \rightarrow (F')^+ \rightarrow F^+ \rightarrow (F_1)^+ \rightarrow 0.$$

From [8, Lemma 5], F^+ and $(F_1)^+$ are n -absolutely pure right R -modules. Then, by $(*)$ and from Cartan and Eilenberg (1956, Proposition 5.1, p. 120),

$$\text{Ext}^1((F_1)^+, (F')^+) \cong (\text{Tor}_1((F_1)^+, F'))^+ = 0.$$

Then, the sequence (β) splits, and so $(F')^+$ is n -absolutely pure being a direct summand of the n -absolutely pure right R -module F^+ . Therefore, F' is a n -flat left R -module by [8, Lemma 5]. Finally, by Lemma 3.4 and the short exact sequence $0 \rightarrow M \rightarrow F' \rightarrow G \rightarrow 0$, M is Gorenstein n -flat.

(3) \Rightarrow (1). Consider a short exact sequence of left R -modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, where A and C are Gorenstein n -flat. We prove that B is Gorenstein n -flat. Let E be an n -absolutely pure right R -module. Applying the functor $E \otimes_R -$ to the short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$, we get the long exact sequence,

$$\text{Tor}_i(E, A) \rightarrow \text{Tor}_i(E, B) \rightarrow \text{Tor}_i(E, C).$$

Then, $Tor_i(E, B) = 0$ for all $i > 0$ (since A and C are Gorenstein n -flat and by Lemma 3.4).

On the other hand, since C is Gorenstein n -flat, there exists a short exact sequence of left R -modules $0 \rightarrow G \rightarrow F \rightarrow C \rightarrow 0$, where F is n -flat and C is Gorenstein n -flat. Consider the following pullback diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & G & \xlongequal{\quad} & G & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \longrightarrow & D & \dashrightarrow & F \longrightarrow 0 \\
 & & \parallel & & \vdots & & \downarrow \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Also, since A is Gorenstein n -flat, there exists a short exact sequence of left R -modules $0 \rightarrow A \rightarrow F' \rightarrow G' \rightarrow 0$, where F' is n -flat and G' is Gorenstein n -flat. Consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A & \longrightarrow & D & \longrightarrow & F \longrightarrow 0 \\
 & & \downarrow & & \vdots & & \parallel \\
 0 & \longrightarrow & F' & \dashrightarrow & D' & \longrightarrow & F \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & G' & \xlongequal{\quad} & G' & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0. & &
 \end{array}$$

In the short exact sequence $0 \rightarrow F' \rightarrow D' \rightarrow F \rightarrow 0$ both F' and F are n -flat, then so is D' . Then, by the short exact sequence $0 \rightarrow D \rightarrow D' \rightarrow G' \rightarrow 0$ and from Lemma 3.4, D is Gorenstein n -flat. Finally, consider the short exact sequence $0 \rightarrow G \rightarrow D \rightarrow B \rightarrow 0$. We have G and D are Gorenstein n -flat, and $Tor_i(E, B) = 0$ for all $i > 0$ and all n -absolutely pure right R -modules E . Therefore, by (3), B is Gorenstein n -flat. This completes the proof. \square

Corollary 3.7. *If R is a left GF_n -closed ring, then the class $\mathcal{GF}_n(R)$ is closed under direct summands.*

Proof. Use [7, Proposition 1.4], [12, Proposition 3.4] and Theorem 3.6. \square

4 Stability of Gorenstein n -flat modules

To prove the main theorem of this paper, we need the following definitions and results. First, let us call Gorenstein G n -flat module, any element of $\mathcal{G}_i^{(2)}\mathcal{F}_n(R)$, which is defined in Section 2.

Definition 4.1. An R -module M is called a strongly Gorenstein n -flat module if there exists an exact sequence of R -modules,

$$0 \rightarrow M \rightarrow F \rightarrow M \rightarrow 0$$

such that F is a n -flat R -module and $E \otimes_R -$ leaves this sequence exact whenever E is an n -absolutely pure right module over R .

Next, we introduce strongly Gorenstein G n -flat module.

Definition 4.2. An R -module M is called a strongly Gorenstein G n -flat module if there exists an exact sequence of R -modules $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ such that G is Gorenstein n -flat over R and $E \otimes_R -$ leaves this sequence exact for each n -absolutely pure right R -module E .

Proposition 4.3. (1) Any strongly Gorenstein G n -flat module is Gorenstein G n -flat.
 (2) The family of Gorenstein G n -flat modules is stable under arbitrary direct sums.

Proof. (1) It is straightforward.
 (2) It is straightforward, since any direct sum of Gorenstein n -flat modules is Gorenstein n -flat by [12, Proposition 3.4] and since, for each positive integer m , $Tor_m(B, \bigoplus_i A_i) \cong \bigoplus_i Tor_m(B, A_i)$ for any family of modules A_i and any right module B by [9, Theorem 8.10]. \square

Proposition 4.4. Let M be an R -module. Then the following statements hold.

(1) Given an exact sequence of R -modules

$$0 \rightarrow K \rightarrow G_1 \rightarrow G_2 \rightarrow \dots \rightarrow G_m \rightarrow M \rightarrow 0$$

such that G_1, G_2, \dots, G_m are Gorenstein n -flat modules, then

$$Tor_{m+i}(E, M) \cong Tor_i(E, K)$$

for each n -absolutely pure right R -module E and each integer $i \geq 1$.

(2) If M is a Gorenstein G n -flat R -module, then $Tor_i(E, M) = 0$ for each n -absolutely pure right R -module E and each integer $i \geq 1$.

Proof. (1) It suffices to handle the case $m = 1$. So, let $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ be an exact sequence such that G is Gorenstein n -flat. Let E be a right n -absolutely pure R -module. Applying the functor $E \otimes_R -$ to this sequence yields the following exact sequence:

$$Tor_{i+1}(E, G) = 0 \rightarrow Tor_{i+1}(E, M) \rightarrow Tor_i(E, K) \rightarrow Tor_i(E, G) = 0$$

for each integer $i \geq 1$. This ensures that

$$Tor_{i+1}(E, M) \cong Tor_i(E, K)$$

for each integer $i \geq 1$, as desired.

(2) Let M be a Gorenstein G n -flat module. Then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$ such that G is a Gorenstein n -flat module, K is a Gorenstein G n -flat module and

$$0 \rightarrow E \otimes K \rightarrow E \otimes G \rightarrow E \otimes M \rightarrow 0$$

is exact whenever E is an n -absolutely pure right R -module. Hence, $Tor_1(E, M) = 0$ for each n -absolutely pure right module E . Reiterating this process and using (1), we get $Tor_i(E, M) = 0$ for each n -absolutely pure right module E and each integer $i \geq 1$. \square

The next result establishes an analog version of Proposition 3.6 in [2] for the Gorenstein G n -flat notion.

Proposition 4.5. Let M be an R -module. Then the following statements are equivalent:

- (1) M is a strongly Gorenstein G n -flat module.
- (2) There exists an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ such that G is a Gorenstein n -flat module, and $Tor_1(E, M) = 0$ for any n -absolutely pure R -module E .
- (3) There exists an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ such that G is a Gorenstein n -flat module and, for any right n -absolutely pure R -module E , the following sequence is exact

$$0 \rightarrow E \otimes M \rightarrow E \otimes G \rightarrow E \otimes M \rightarrow 0.$$

Proof. (1) \Rightarrow (2) holds by Proposition 4.4
 (2) \Rightarrow (3) and (3) \Rightarrow (1) are straightforward, this completes the proof. \square

The next result is analog to Theorem 3.5 in [2].

Proposition 4.6. *Let M be a Gorenstein G n -flat R -module. Then M is a direct summand of a strongly Gorenstein G n -flat module.*

Proof. Let M be a Gorenstein G n -flat module and $\mathbf{G} = \cdots \rightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \xrightarrow{d_{-2}} \cdots$ be a complete Gorenstein n -flat resolution such that $M = \text{Im}(d_0)$. Let $M_i := \text{Im}(d_i)$ for each integer i . As $\mathcal{GF}_n(R)$ is stable under direct sums, it is easily seen that the following sequence is a complete Gorenstein n -flat resolution:

$$\mathbf{G}' = \cdots \rightarrow \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i \in \mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \cdots$$

such that $\text{Im}(\bigoplus_i d_i) = \bigoplus_i M_i$. Then $\bigoplus_i M_i$ is a strongly Gorenstein G n -flat module so that M is a direct summand of a strongly Gorenstein G n -flat module, as contended. \square

For easiness, we adopt the following definition.

Definition 4.7. Let M be a strongly Gorenstein G n -flat module. An R -module N is called an M_n -type module if there exists an exact sequence $0 \rightarrow M \rightarrow N \rightarrow H \rightarrow 0$ such that H is a Gorenstein n -flat module.

Proposition 4.8. *Let M be a strongly Gorenstein G n -flat module and N an M_n -type module. Then,*

- (1) $\text{Tor}_i(E, N) = 0$ for each n -absolutely pure right R -module E and for each integer $i \geq 1$.
- (2) If R is a left GF_n -closed ring, then there exists an exact sequence $0 \rightarrow N \rightarrow F \rightarrow L \rightarrow 0$ such that F is an n -flat module and L is an M_n -type module.

Proof. (1) If $0 \rightarrow M \rightarrow N \rightarrow H \rightarrow 0$ is an exact sequence such that H is a Gorenstein n -flat R -module, then, by considering the corresponding long exact sequence and by Proposition 4.4, we have $\text{Tor}_i(E, N) \cong \text{Tor}_i(E, M) = 0$ for each n -absolutely pure right module E and each integer $i \geq 1$.

(2) Assume that R is a left GF_n -closed ring. Let $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ and $0 \rightarrow M \rightarrow N \rightarrow H \rightarrow 0$ be exact sequences such that G and H are Gorenstein n -flat R -modules. Consider the following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \vdots & & \parallel \\
 0 & \longrightarrow & N & \dashrightarrow & T & \longrightarrow & M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & H & \xlongequal{\quad} & H & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Since G and H are Gorenstein n -flat modules, we get, as R is left GF_n -closed, T is Gorenstein n -flat. Then there exists a short exact sequence $0 \rightarrow T \rightarrow F \rightarrow K \rightarrow 0$ such that F is a n -flat R -module and K is a Gorenstein n -flat R -module. Hence, we get the following pushout

diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & N & \longrightarrow & T & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \vdots \\
 0 & \longrightarrow & N & \longrightarrow & F & \dashrightarrow & L \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & K & \equiv & K \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

as desired. □

Corollary 4.9. *Let R be a left GF_n -closed ring. Let M be a strongly Gorenstein G n -flat module and N an M_n -type module. Then N is a Gorenstein n -flat R -module.*

Proof. First, observe that by Proposition 4.8 there exist a n -flat module F_0 and an M_n -type module L such that the following sequence $0 \rightarrow N \rightarrow F_0 \rightarrow L \rightarrow 0$ is exact and stays exact after applying the functor $E \otimes_R -$ for each n -absolutely pure right module E . Then, it suffices to iterate Proposition 4.8(2) to get a resolution $0 \rightarrow N \rightarrow F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \dots$ of n -flat modules, which remains exact after applying the functor $E \otimes_R -$ for each n -absolutely pure right R -module E . Now, Proposition 4.8(1) completes the proof. □

Proof of the main theorem. In view of the inclusions $\mathcal{GF}_n(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}_n(R)$, it suffices to prove that $\mathcal{G}_i^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{GF}_n(R)$. Since R is left GF_n -closed, by Corollary 3.7, $\mathcal{GF}_n(R)$ is stable under direct summands. Thus, it suffices, by Proposition 4.6, to prove that any strongly Gorenstein G n -flat module is Gorenstein n -flat. Then, let M be a strongly Gorenstein G n -flat module. There exists an exact sequence $0 \rightarrow M \rightarrow G \rightarrow M \rightarrow 0$ such that G is a Gorenstein n -flat module and $Tor_i(E, M) = 0$ for each n -absolutely pure right module E and each integer $i \geq 1$ by Proposition 4.5. As G is Gorenstein n -flat, there exists an exact sequence of R -modules $0 \rightarrow G \rightarrow F \rightarrow G_1 \rightarrow 0$ such that F is a n -flat module and G_1 is a Gorenstein n -flat module. Then we get the following pushout diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & M \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \vdots \\
 0 & \longrightarrow & M & \longrightarrow & F & \dashrightarrow & M_1 \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & G_1 & \equiv & G_1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0.
 \end{array}$$

Hence, M_1 is an M_n -type R -module. It follows from Corollary 4.9 that M_1 is a Gorenstein n -flat module. As R is left GF_n -closed and G_1 is Gorenstein n -flat, we get M is a Gorenstein n -flat R -module, as desired. □

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