Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 495–504

Stability of Gorenstein n-flat modules

C. Selvaraj and R. Udhayakumar

Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 16D10, 16D40, 16E30.

Keywords and phrases: Gorenstein n-flat module, Two-degree Gorenstein n-flat module, GF_n-closed ring.

The first author was partially supported by UGC Project grant No. 41-773/2012 (SR) and the second was supported by UGC Non SAP BSR grant No. 4-1/2008(BSR).

Abstract. In this paper we are concerned with the stability of the class of Gorenstein n-flat modules. We give an answer for the following natural question in the setting of a left GF_n-closed ring R: Given an exact sequence of Gorenstein n-flat R-modules $\mathbf{G} = \cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$ such that the complex $H \otimes_R \mathbf{G}$ is exact for each Gorenstein n-absolutely pure right R-module H, is the module $M := Im(G_0 \to G^0)$ a Gorenstein n-flat module?

1 Introduction

Sather-Wagstaff et al. proved in [10] that iterating the process used to define Gorenstein projective modules exactly leads to the Gorenstein projective modules. Also, they established in [11] a stability of the subcategory of Gorenstein flat modules under a procedure to build R-modules from complete resolutions. Further Samir Bouchiba et al. in [3] proved over a left GF-closed ring R, the stability of the Gorenstein flat modules under the very process used to define these entities. Recently Z. Wang and Z. Liu in [13] proved that the two-degree strongly Gorenstein flat modules are nothing more than the strongly Gorenstein flat modules. Motivated by these works, we are concerned with the stability of the class of Gorenstein n-flat modules introduced in [12].

On the other hand, in [1], Bennis defined and studied the notion of left GF-closed rings. These are rings for which $\mathcal{GF}(R)$ (class of Gorenstein flat R-modules) is closed under extensions, that is for any exact sequence $0 \to A \to B \to C \to 0$ of R-modules if $A, C \in \mathcal{GF}(R)$, then $B \in \mathcal{GF}(R)$. In this paper we introduced the concept of a left GF_n-closed ring (see Definition 3.2) where n is a non-negative integer and give a characterization of this ring and show that the two-degree Gorenstein n-flat modules and Gorenstein n-flat modules are coincide when R is a left GF_n-closed ring.

Throughout this paper, R denotes an associative ring with identity element. All modules, if not otherwise specified, are assumed to be left R-modules. Let $\mathcal{M}(R)$ denote the category of left R-modules. For an R-module M, we use M^+ to denote the character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M. Let M and N be R-modules. $Hom(M, N)$ (resp. $Ext^i(M, N)$) means $Hom_R(M, N)$ (resp. $Ext^i_R(M, N)$), and similarly $M \otimes N$ (resp. $Tor_i(M, N)$) denotes $M \otimes_R N$ (resp. $Tor^R_i(M, N)$) for an integer $i \geq 1$. A left R-module M is called n-flat [8] if $Tor_1(N, M) = 0$ holds for all finitely presented right R-modules N with projective dimension $\leq n$ and a right R-module M is called n-absolutely pure [8] if $Ext^1(N, M) = 0$ holds for all finitely presented right R-modules N with projective dimension $\leq n$. This paper is organized as follows. In Section 2, we recall some known definitions and introduce two-degree Gorenstein n-flat modules as well as we initiate the main theorem of this paper. In Section 3, we introduce the definition of a left GF_n-closed ring and give a characterization of this ring. In the last section over a left GF_n-closed ring, we show that the two-degree Gorenstein n-flat modules are nothing more than that the Gorenstein n-flat modules.

2 Gorenstein n-flat modules

In this section we recall some known definitions and introduce two-degree Gorenstein n-flat modules as well as we initiate the main theorem of this paper.

Definition 2.1. [12] A left R-module M is said to be Gorenstein n-flat, if there exists an exact sequence of n-flat left R-modules,

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

such that $M \cong Im(F_0 \to F^0)$ and such that $E \otimes_R -$ leaves the sequence exact whenever E is an n-absolutely pure right R-module.

Definition 2.2. [12] A right R-module M is said to be Gorenstein n-absolutely pure, if there exists an exact sequence of n-absolutely pure right R-modules

$$\cdots \to A_1 \to A_0 \to A^0 \to A^1 \to \cdots$$

such that $M \cong Im(A_0 \to A^0)$ and such that $Hom_R(E, -)$ leaves the sequence exact whenever E is an n-absolutely pure right R-module.

Next, we introduce the following definition

Definition 2.3. A left R-module M is called two-degree Gorenstein n-flat if there exists an exact sequence of Gorenstein n-flat left R-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ and it remains exact after applying $H \otimes_R -$ for any Gorenstein n-absolutely pure right R-module H.

Let $\mathcal{GF}_n(R)$, $\mathcal{GA}_n(R)$ and $\mathcal{G}^{(2)}\mathcal{F}_n(R)$ are denotes the class of all Gorenstein n-flat left, Gorenstein n-absolutely pure right and two-degree Gorenstein n-flat left modules over R respectively. Also denote $\mathcal{G}_i^{(2)}\mathcal{F}_n(R)$ the subcategory of $\mathcal{M}(R)$ for which there exists an exact sequence of Gorenstein n-flat R-modules

$$\cdots \to G_1 \to G_0 \to G^0 \to G^1 \to \cdots$$

such that $M \cong Im(G_0 \to G^0)$ and it remains exact after applying $E \otimes_R -$ for any n-absolutely pure right R-module E. It is routine to check that

$$\mathcal{GF}_n(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}_n(R).$$

Our main theorem proves that these inequalities turn out to be equalities when R is a left GF_n-closed ring as is stated next.
Main Theorem: Let R be a left GF_n-closed ring. Then $\mathcal{GF}_n(R) = \mathcal{G}^{(2)}\mathcal{F}_n(R) = \mathcal{G}_i^{(2)}\mathcal{F}_n(R)$.

3 GF_n-closed ring

In this section we introduce the definition of a left GF_n-closed ring and give a characterization of this ring. First recall the following definitions:

Definition 3.1. Let R be a ring and let $\mathfrak{X}$ be a class of left R-modules.

(1) $\mathfrak{X}$ is closed under extensions: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions A and C are in $\mathfrak{X}$ implies B is in $\mathfrak{X}$.

(2) $\mathfrak{X}$ is closed under kernels of epimorphisms: If for every short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, the conditions B and C are in $\mathfrak{X}$ implies A is in $\mathfrak{X}$.

(3) $\mathfrak{X}$ is projectively resolving: If it contains all projective modules and it is closed under both extensions and kernels of epimorphisms. i.e., for every short exact sequence of R-modules $0 \to A \to B \to C \to 0$ with $C \in \mathfrak{X}$, the conditions $A \in \mathfrak{X}$ and $B \in \mathfrak{X}$ are equivalent.

Definition 3.2. A ring R is said to be left GF_n-closed if $\mathcal{GF}_n(R)$ is closed under extensions.

Since Gorenstein n-flat modules need not be Gorenstein flat (see [12, Example 3.3]), we get left GF_n-closed ring need not be left GF-closed ring. Also recall that a ring R is called right n-coherent [8] (for integers $n > 0$ or $n = \infty$) if every finitely generated submodule of a free right R-module whose projective dimension is $\leq n-1$ is finitely presented.

Example 3.3. Every right n-coherent ring is left GF_n-closed.

Proof. It follows from [12, Proposition 3.4], [12, Lemma 3.5] and [7, Proposition 1.4]. □

We begin with the following result.

Lemma 3.4. *The following are equivalent for a left R-module M:*

(1) M is Gorenstein n-flat;

(2) M satisfies the two following conditions:

(i) $Tor_i(E, M) = 0$ for all $i > 0$ and all n-absolutely pure right R-modules E; and

(ii) There exists an exact sequence of left R-modules $0 \to M \to F^0 \to F^1 \to \cdots$, where each F^i is n-flat, such that $E \otimes_R -$ leaves the sequence exact whenever E is an n-absolutely pure right R-module;

(3) There exists a short exact sequence of left R-modules $0 \to M \to F \to G \to 0$, where F is n-flat and G is Gorenstein n-flat.

Proof. Using the definition of Gorenstein n-flat modules, the equivalence $(1) \Leftrightarrow (2)$ is obtained by standard argument. Also, by definition, we get immediately the implication $(1) \Rightarrow (3)$.

We prove the implication $(3) \Rightarrow (2)$. Suppose that there exists a short exact sequence of left R-modules:

$$(\alpha) : 0 \to M \to F \to G \to 0$$

where F is n-flat and G is Gorenstein n-flat. Let E be an n-absolutely pure right R-module. Since G is Gorenstein n-flat and by the equivalence $(1) \Leftrightarrow (2)$, $Tor_{i+1}(E, G) = 0$ for all $i \geq 0$. Then, use the long exact sequence,

$$Tor_{i+1}(E, G) \to Tor_i(E, M) \to Tor_i(E, F),$$

to get $Tor_i(E, M) = 0$ for all $i > 0$.

On the other hand, since G is Gorenstein n-flat, there is an exact sequence of left R-modules:

$$(\beta) : 0 \to G \to F^0 \to F^1 \to \cdots,$$

where each F^i is n-flat, such that $E \otimes_R -$ leaves the sequence exact whenever E is an n-absolutely pure right R-module. Assembling the sequences (α) and (β), we get the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & F \arrow[rr] \arrow[dr] & & F^0 \arrow[r] & F^1 \arrow[r] & \ldots \\
& & & G \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

such that $E \otimes_R -$ leaves the upper exact sequence exact whenever E is an n-absolutely pure right R-module, as desired. □

Lemma 3.5. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of left R-modules. If A is Gorenstein n-flat and C is n-flat, then B is Gorenstein n-flat*

Proof. Since A is Gorenstein n-flat, there exists a short exact sequence of left R-modules $0 \to A \to F \to G \to 0$, where F is n-flat and G is Gorenstein n-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, dashed] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F \arrow[r, dashed] \arrow[d] & F' \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & G \arrow[r, equal] \arrow[d] & G \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the sequence $0 \to F \to F' \to C \to 0$, both F and C are n-flat, hence so is F'. Then, by the middle vertical sequence and from Lemma 3.4, B is Gorenstein n-flat, as desired. □

Now we give the main theorem of this Section which is analog to Theorem 2.3 in [1]

Theorem 3.6. *The following conditions are equivalent for a ring R:*

(1) R is left GF_n-closed;

(2) The class $\mathcal{GF}_n(R)$ is projectively resolving;

(3) For every short exact sequence of left R-modules $0 \to G_1 \to G_0 \to M \to 0$, where G_0 and G_1 are Gorenstein n-flat. If $Tor_1(E, M) = 0$ for all n-absolutely pure right R-modules E, then M is Gorenstein n-flat.

Proof. $(1) \Rightarrow (2)$. To claim that the class $\mathcal{GF}_n(R)$ is projectively resolving, it suffices to prove that it is closed under kernels of epimorphisms (see Definition 3.1). Then, consider a short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, where B and C are Gorenstein n-flat. We prove that A is Gorenstein n-flat. Since B is Gorenstein n-flat, there exists a short exact sequence of left R-modules $0 \to B \to F \to G \to 0$, where F is n-flat and G is Gorenstein n-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & A \arrow[r] & F \arrow[r, dashed] \arrow[d] & D \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

By the right vertical sequence and since R is left GF_n-closed, the R-module D is Gorenstein n-flat. Therefore, by the middle horizontal sequence and Lemma 3.4, A is Gorenstein n-flat, as desired.

$(1) \Rightarrow (3)$. Since G_1 is Gorenstein n-flat, there exists a short exact sequence of left R-modules $0 \to G_1 \to F_1 \to H \to 0$, where F_1 is n-flat and H is Gorenstein n-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F_1 \arrow[r, dashed] \arrow[d] & D \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the short exact sequence $0 \to G_0 \to D \to H \to 0$ both G_0 and H are Gorenstein n-flat, then so is D since R is left GF_n-closed. Then, there exists a short exact sequence of left R-modules $0 \to D \to F \to G \to 0$, where F is n-flat and G is Gorenstein n-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & F_1 \arrow[r] & F \arrow[r, dashed] \arrow[d] & F' \arrow[r] \arrow[d] & 0 \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

We want to show that F' is n-flat. Consider the sequence $0 \to M \to F' \to G \to 0$. Let E be an n-absolutely pure right R-module. By the exact sequence,

$$0 = Tor_1(E, M) \to Tor_1(E, F') \to Tor_1(E, G) = 0,$$

we get

$$(*) \quad Tor_1(E, F') = 0.$$

On the other hand, consider the sequence $0 \to F_1 \to F \to F' \to 0$. By Rotman (1979, Lemma 3.51), we have the following short exact sequence of character modules:

$$(\beta) = 0 \to (F')^+ \to F^+ \to (F_1)^+ \to 0.$$

From [8, Lemma 5], F^+ and $(F_1)^+$ are n-absolutely pure right R-modules. Then, by $(*)$ and from Cartan and Eilenberg (1956, Proposition 5.1, p. 120),

$$Ext^1((F_1)^+, (F')^+) \cong (Tor_1((F_1)^+, F'))^+ = 0.$$

Then, the sequence (β) splits, and so $(F')^+$ is n-absolutely pure being a direct summand of the n-absolutely pure right R-module F^+. Therefore, F' is a n-flat left R-module by [8, Lemma 5]. Finally, by Lemma 3.4 and the short exact sequence $0 \to M \to F' \to G \to 0$, M is Gorenstein n-flat.

(3) $\Rightarrow$ (1). Consider a short exact sequence of left R-modules $0 \to A \to B \to C \to 0$, where A and C are Gorenstein n-flat. We prove that B is Gorenstein n-flat. Let E be an n-absolutely pure right R-module. Applying the functor $E \otimes_R -$ to the short exact sequence $0 \to A \to B \to C \to 0$, we get the long exact sequence,

$$Tor_i(E, A) \to Tor_i(E, B) \to Tor_i(E, C).$$

Then, $Tor_i(E, B) = 0$ for all $i > 0$ (since A and C are Gorenstein n-flat and by Lemma 3.4).

On the other hand, since C is Gorenstein n-flat, there exists a short exact sequence of left R-modules $0 \to G \to F \to C \to 0$, where F is n-flat and C is Gorenstein n-flat. Consider the following pullback diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G \arrow[r, equal] \arrow[d] & G \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & D \arrow[r, dashed] \arrow[d, dashed] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Also, since A is Gorenstein n-flat, there exists a short exact sequence of left R-modules $0 \to A \to F' \to G' \to 0$, where F' is n-flat and G' is Gorenstein n-flat. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, dashed] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & F' \arrow[r, dashed] \arrow[d] & D' \arrow[r] \arrow[d] & F \arrow[r] & 0 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

In the short exact sequence $0 \to F' \to D' \to F \to 0$ both F' and F are n-flat, then so is D'. Then, by the short exact sequence $0 \to D \to D' \to G' \to 0$ and from Lemma 3.4, D is Gorenstein n-flat. Finally, consider the short exact sequence $0 \to G \to D \to B \to 0$. We have G and D are Gorenstein n-flat, and $Tor_i(E, B) = 0$ for all $i > 0$ and all n-absolutely pure right R-modules E. Therefore, by (3), B is Gorenstein n-flat. This completes the proof. □

Corollary 3.7. *If R is a left GF_n-closed ring, then the class $\mathcal{GF}_n(R)$ is closed under direct summands.*

Proof. Use [7, Proposition 1.4], [12, Proposition 3.4] and Theorem 3.6. □

4 Stability of Gorenstein n-flat modules

To prove the main theorem of this paper, we need the following definitions and results. First, let us call Gorenstein G n-flat module, any element of $\mathcal{G}_i^{(2)}\mathcal{F}_n(R)$, which is defined in Section 2.

Definition 4.1. An R-module M is called a strongly Gorenstein n-flat module if there exists an exact sequence of R-modules,

$$0 \to M \to F \to M \to 0$$

such that F is a n-flat R-module and $E \otimes_R -$ leaves this sequence exact whenever E is an n-absolutely pure right module over R.

Next, we introduce strongly Gorenstein G n-flat module.

Definition 4.2. An R-module M is called a strongly Gorenstein G n-flat module if there exists an exact sequence of R-modules $0 \to M \to G \to M \to 0$ such that G is Gorenstein n-flat over R and $E \otimes_R -$ leaves this sequence exact for each n-absolutely pure right R-module E.

Proposition 4.3. *(1) Any strongly Gorenstein G n-flat module is Gorenstein G n-flat.*

(2) The family of Gorenstein G n-flat modules is stable under arbitrary direct sums.

Proof. (1) It is straightforward.
(2) It is straightforward, since any direct sum of Gorenstein n-flat modules is Gorenstein n-flat by [12, Proposition 3.4] and since, for each positive integer m, $Tor_m(B, \bigoplus_i A_i) \cong \bigoplus_i Tor_m(B, A_i)$ for any family of modules A_i and any right module B by [9, Theorem 8.10]. □

Proposition 4.4. *Let M be an R-module. Then the following statements hold.*

(1) Given an exact sequence of R-modules

$$0 \to K \to G_1 \to G_2 \to \cdots \to G_m \to M \to 0$$

such that $G_1, G_2, \cdots, G_m$ are Gorenstein n-flat modules, then

$$Tor_{m+i}(E, M) \cong Tor_i(E, K)$$

for each n-absolutely pure right R-module E and each integer $i \geq 1$.

(2) If M is a Gorenstein G n-flat R-module, then $Tor_i(E, M) = 0$ for each n-absolutely pure right R-module E and each integer $i \geq 1$.

Proof. (1) It suffices to handle the case $m = 1$. So, let $0 \to K \to G \to M \to 0$ be an exact sequence such that G is Gorenstein n-flat. Let E be a right n-absolutely pure R-module. Applying the functor $E \otimes_R -$ to this sequence yields the following exact sequence:

$$Tor_{i+1}(E, G) = 0 \to Tor_{i+1}(E, M) \to Tor_i(E, K) \to Tor_i(E, G) = 0$$

for each integer $i \geq 1$. This ensures that

$$Tor_{i+1}(E, M) \cong Tor_i(E, K)$$

for each integer $i \geq 1$, as desired.

(2) Let M be a Gorenstein G n-flat module. Then there exists a short exact sequence $0 \to K \to G \to M \to 0$ such that G is a Gorenstein n-flat module, K is a Gorenstein G n-flat module and

$$0 \to E \otimes K \to E \otimes G \to E \otimes M \to 0$$

is exact whenever E is an n-absolutely pure right R-module. Hence, $Tor_1(E, M) = 0$ for each n-absolutely pure right module E. Reiterating this process and using (1), we get $Tor_i(E, M) = 0$ for each n-absolutely pure right module E and each integer $i \geq 1$. □

The next result establishes an analog version of Proposition 3.6 in [2] for the Gorenstein G n-flat notion.

Proposition 4.5. *Let M be an R-module. Then the following statements are equivalent:*

(1) M is a strongly Gorenstein G n-flat module.

(2) There exists an exact sequence $0 \to M \to G \to M \to 0$ such that G is a Gorenstein n-flat module, and $Tor_1(E, M) = 0$ for any n-absolutely pure R-module E.

(3) There exists an exact sequence $0 \to M \to G \to M \to 0$ such that G is a Gorenstein n-flat module and, for any right n-absolutely pure R-module E, the following sequence is exact

$$0 \to E \otimes M \to E \otimes G \to E \otimes M \to 0.$$

Proof. (1) $\Rightarrow$ (2) holds by Proposition 4.4
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are straightforward, this completes the proof. □

The next result is analog to Theorem 3.5 in [2].

Proposition 4.6. *Let M be a Gorenstein G n-flat R-module. Then M is a direct summand of a strongly Gorenstein G n-flat module.*

Proof. Let M be a Gorenstein G n-flat module and $\mathbf{G} = \cdots \longrightarrow G_2 \xrightarrow{d_2} G_1 \xrightarrow{d_1} G_0 \xrightarrow{d_0} G_{-1} \xrightarrow{d_{-1}} G_{-2} \xrightarrow{d_{-2}} \cdots$ be a complete Gorenstein n-flat resolution such that $M = Im(d_0)$. Let $M_i := Im(d_i)$ for each integer i. As $\mathcal{GF}_n(R)$ is stable under direct sums, it is easily seen that the following sequence is a complete Gorenstein n-flat resolution:

$$\mathbf{G}' = \cdots \to \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \bigoplus_{i\in\mathbb{Z}} G_i \xrightarrow{\oplus_i d_i} \cdots$$

such that $Im(\bigoplus_i d_i) = \bigoplus_i M_i$. Then $\bigoplus_i M_i$ is a strongly Gorenstein G n-flat module so that M is a direct summand of a strongly Gorenstein G n-flat module, as contended. □

For easiness, we adopt the following definition.

Definition 4.7. Let M be a strongly Gorenstein G n-flat module. An R-module N is called an M_n-type module if there exists an exact sequence $0 \to M \to N \to H \to 0$ such that H is a Gorenstein n-flat module.

Proposition 4.8. *Let M be a strongly Gorenstein G n-flat module and N an M_n-type module. Then,*

(1) $Tor_i(E, N) = 0$ for each n-absolutely pure right R-module E and for each integer $i \geq 1$.

(2) If R is a left GF_n-closed ring, then there exists an exact sequence $0 \to N \to F \to L \to 0$ such that F is an n-flat module and L is an M_n-type module.

Proof. (1) If $0 \to M \to N \to H \to 0$ is an exact sequence such that H is a Gorenstein n-flat R-module, then, by considering the corresponding long exact sequence and by Proposition 4.4, we have $Tor_i(E, N) \cong Tor_i(E, M) = 0$ for each n-absolutely pure right module E and each integer $i \geq 1$.

(2) Assume that R is a left GF_n-closed ring. Let $0 \to M \to G \to M \to 0$ and $0 \to M \to N \to H \to 0$ be exact sequences such that G and H are Gorenstein n-flat R-modules. Consider the following pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r, dashed] \arrow[d] & T \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
 & H \arrow[r, equal] \arrow[d] & H \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Since G and H are Gorenstein n-flat modules, we get, as R is left GF_n-closed, T is Gorenstein n-flat. Then there exists a short exact sequence $0 \to T \to F \to K \to 0$ such that F is a n-flat R-module and K is a Gorenstein n-flat R-module. Hence, we get the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & N \arrow[r] & F \arrow[r, dashed] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

as desired. □

Corollary 4.9. *Let R be a left GF_n-closed ring. Let M be a strongly Gorenstein G n-flat module and N an M_n-type module. Then N is a Gorenstein n-flat R-module.*

Proof. First, observe that by Proposition 4.8 there exist a n-flat module F_0 and an M_n-type module L such that the following sequence $0 \to N \to F_0 \to L \to 0$ is exact and stays exact after applying the functor $E \otimes_R -$ for each n-absolutely pure right module E. Then, it suffices to iterate Proposition 4.8(2) to get a resolution $0 \to N \to F_0 \to F_1 \to F_2 \to \cdots$ of n-flat modules, which remains exact after applying the functor $E \otimes_R -$ for each n-absolutely pure right R-module E. Now, Proposition 4.8(1) completes the proof. □

Proof of the main theorem. In view of the inclusions $\mathcal{GF}_n(R) \subseteq \mathcal{G}^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{G}_i^{(2)}\mathcal{F}_n(R)$, it suffices to prove that $\mathcal{G}_i^{(2)}\mathcal{F}_n(R) \subseteq \mathcal{GF}_n(R)$. Since R is left GF_n-closed, by Corollary 3.7, $\mathcal{GF}_n(R)$ is stable under direct summands. Thus, it suffices, by Proposition 4.6, to prove that any strongly Gorenstein G n-flat module is Gorenstein n-flat. Then, let M be a strongly Gorenstein G n-flat module. There exists an exact sequence $0 \to M \to G \to M \to 0$ such that G is a Gorenstein n-flat module and $Tor_i(E, M) = 0$ for each n-absolutely pure right module E and each integer $i \geq 1$ by Proposition 4.5. As G is Gorenstein n-flat, there exists an exact sequence of R-modules $0 \to G \to F \to G_1 \to 0$ such that F is a n-flat module and G_1 is a Gorenstein n-flat module. Then we get the following pushout diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & G \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & M \arrow[r] & F \arrow[r, dashed] \arrow[d] & M_1 \arrow[r] \arrow[d] & 0 \\
 & & G_1 \arrow[r, equal] \arrow[d] & G_1 \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

Hence, M_1 is an M_n-type R-module. It follows from Corollary 4.9 that M_1 is a Gorenstein n-flat module. As R is left GF_n-closed and G_1 is Gorenstein n-flat, we get M is a Gorenstein n-flat R-module, as desired. □

References

[1] D. Bennis, Rings over which the class of Gorenstein flat modules is closed under extentions, *Comm. Algebra* 37(3) (2009), 855-868.

[2] D. Bennis and N. Mahdou, Strongly Gorenstein projective, injective, and flat modules, *J. Pure Appl. Algebra* 210 (2007), 437-445.

[3] S. Bouchiba and M. Khaloui, Stability of Gorenstein flat modules, *Glasgow. Math. J* 54 (2012), 169-175.

[4] H. Carten and S. Eilenberg, Homological Algebra, Princeton University Press, Princeton, 1956.

[5] E. Enochs and O. M. G. Jenda, Relative Homological Algebra, *De Gruyter Expositions in Mathematics* No 30, New York, 2000.

[6] E. Enochs, O. M. G. Jenda and B. Torrecillas, Gorenstein flat modules, Nanjing Daxue Shuxue Bannian Kan 10 (1993), 1-9.

[7] H. Holm, Gorenstein Homological Dimensions, *Journal of Pure and Applied Algerba* 189 (2004), 167-193.

[8] S. B. Lee, n-coherent rings, *Comm. Algebra* 30 (3) (2002), 1119-1126.

[9] J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[10] S. Sather-Wagstaff, T. Sharif and D. White, Stability of Gorenstein categories, *J. Lond. Math. Soc.* 77(2) (2008), 481-502.

[11] S. Sather-Wagstaff, T. Sharif and D. White, AB-contexts and stability for Gorenstein flat modules with respect to semidualizing modules, *Algebra Represent. Theory* 14(3) (2011), 403-428.

[12] C. Selvaraj, R.Udhayakumar and A.Umamaheswaran, Gorenstein n-flat modules and their covers, *Asian-European J. Math.* Vol. 7, No. 3 (2014) 1450051 (13 pages).

[13] Z. Wang and Z. Liu, Stability of Strongly Gorenstein Flat Modules, *Vietnam J. Math.* 42 (2014), 171-178.

Author information

C. Selvaraj and R. Udhayakumar, Department of Mathematics, Periyar University, Salem - 636 011, TN, India.
E-mail: selvavlr@yahoo.com; udhayaram_v@yahoo.co.in

Received: February 14, 2014.

Accepted: April 3, 2014.