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I- CONVERGENCE OF DIFFERENCE SEQUENCES OF FUZZY REAL NUMBERS DEFINED BY ORLICZ FUNCTION

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Abstract In this article we introduce some *I*-convergent difference sequence spaces of fuzzy real numbers defined by Orlicz function and study their different properties like completeness, solidity, symmetricity etc.

1 Introduction

The notion of I-convergence of real valued sequence was studied at the initial stage by Kostyrko, Salát and Wilczyñski [5] which generalizes and unifies different notions of convergence of sequences. The notion was further studied by Salát, Tripathy and Ziman [10], Debnath and Debnath [1] and many others.

The notion of fuzzy sets was introduced by Zadeh [15]. After that many authors have studied and generalized this notion in many ways, due to the potential of the introduced notion. Also it has wide range of applications in almost all the branches of studied in particular science, where mathematics is used. It attracted many workers to introduce different types of fuzzy sequence spaces.

Bounded and convergent sequences of fuzzy numbers were studied by Matloka [9]. Later on sequences of fuzzy numbers have been studied by Kaleva and Seikkala [3], Tripathy and Sarma ([13], [14]) and many others.

An *Orlicz function* is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of *M* is replaced by

$$M(x + y) \leq M(x) + M(y),$$

then this function is called the *modulus function*.

Kizmaz [5] defined the difference sequence spaces $\ell(\Delta), c(\Delta), c_0(\Delta)$ for crisp sets as follows: $Z(\Delta) = \{X = (X_k) : \Delta X_k \in Z\}$ Where $Z = \ell, c$ and c_0 and $\Delta X_k = X_k - X_{k+1}$.

2 Definitions and Background

Let X be a non-empty set, then a non-void class $I \subseteq 2^X$ (power set of X) is called an *ideal* if *I* is additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$) and hereditary (i.e. $A \in I$ and $B \subseteq A \Rightarrow B \in I$).

An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \neq 2^X$. A non-trivial ideal I is said to be *admissible* if *I* contains every finite subset of *N*. A non-trivial ideal *I* is said to be *maximal* if there does not exist any non-trivial ideal $J \neq I$ containing *I* as a subset.

Let X be a non-empty set, then a non-void class $F \subseteq 2^X$ is said to be a filter in X if $\phi \notin F; A, B \in F \Rightarrow A \cap B \in F$ and $A \in F, A \subseteq B \Rightarrow B \in F$. For any ideal I, there is a filter $\Psi(I)$ corresponding to I, given by $\Psi(I) = \{K \subseteq N : N \backslash K \in I\}$.

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line *R*. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|)$$

It is known that (D, d) is a complete metric space.

A fuzzy real number *X* is a fuzzy set on R i.e. a mapping $X : R \to L(= [0, 1])$ associating each real number *t* with its grade of membership $X(t)$.

The α- level set $[X]^{\alpha}$ of a fuzzy real number *X* for $0 < \alpha \leq 1$, defined as

$$X^{\alpha} = \{t \in R : X(t) \geq \alpha\}.$$

A fuzzy real number *X* is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number *X* is called *normal*.

A fuzzy real number *X* is said to be *upper semi-continuous* if for each $\varepsilon > 0, X^{-1}([0, a+\varepsilon))$, is open for all $a \in L$ in the usual topology of *R*.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X(t)|$ of $X \in L(R)$ is defined as

$$|X(t)| = \begin{cases} \max\{X(t), X(-t)\}, \text{if } t \geq 0 \\ 0, \text{if } t < 0 \end{cases}$$

Let $\bar{d}$: *L*(*R*) $\times$ *L*(*R*) $\to$ *R* be defined by

$$\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d(X^{\alpha}, Y^{\alpha})$$

Then $\bar{d}$ defines a metric on *L*(*R*).

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0, if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\bar{d}(X_k, X_0) < \varepsilon$ for all $k \geq k_0$.

A fuzzy real valued sequence space E^F is said to be *solid* if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \leq |X_k|$, for all $k \in N$.

A sequence space E^F is said to be *symmetric* if $(X_{\pi}(k)) \in E^F$, whenever $(X_k) \in E^F$, π is a permutation on *N*.

A sequence $X = (X_k)$ of fuzzy numbers is said to be *I*- convergent if there exists a fuzzy number X_0 such that for all $\varepsilon > 0$, the set $\{n \in N : \bar{d}(X_k, X_0) \geq \varepsilon\} \in I$.

We write $I - lim X_k = X_0$.

A sequence (X_k) of fuzzy numbers is said to be *I*- bounded if there exists a real number μ such that the set $\{k \in N : \bar{d}(X_k, \bar{0}) > \mu\} \in I$.

Throughout $c^{I(F)}$, $c_0^{I(F)}$ and $\ell_{\infty}^{I(F)}$ denote the spaces of fuzzy real-valued *I*-convergent, *I*-null and *I*- bounded sequences respectively.

We define the following classes of sequences:

$$(c^I)^F(M,\Delta) = \left\{(X_k) : \left\{k : M\left(\frac{\overline{d}(\Delta X_k, L)}{r}\right) \geq \varepsilon \text{ for some } r > 0 \text{ and } L \in R(I)\right\} \in I\right\}$$

$$(c_0^I)^F(M,\Delta) = \left\{(X_k) : \left\{k : M\left(\frac{\overline{d}(\Delta X_k, \overline{0})}{r}\right) \geq \varepsilon \text{ for some } r > 0\right\} \in I\right\}$$

3 Main Results

Theorem 3.1. *The spaces* $(m^I)^F(M,\Delta)$ *and* $(m_0^I)^F(M,\Delta)$ *are complete metric spaces with respect to the metric given by*

$$f(X,Y) = \overline{d}(X_1, Y_1) + \inf\left\{r > 0 : \sup_k M\left(\frac{\overline{d}(\Delta X_k, \Delta Y_k)}{r}\right) \leq 1\right\}$$

Proof. Let (X^n) be a Cauchy sequence in $(m^I)^F(M,\Delta)$, where $(X^n) = (X_k^n)$.

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M\left(\frac{rx_0}{3}\right) \geq 1$ and $m_0 \in N$ such that

$$f(X^n, X^m) < \frac{\varepsilon}{rx_0}, \text{ for all } n, m \geq m_0.$$

By definition of *f* we have,

$$\overline{d}\,(X_1^m, Y_1^n) < \varepsilon$$

(X_1^m) is a Cauchy sequence of fuzzy real numbers and so $\lim_m X_1^m$ exist.

Again $M\left(\frac{\overline{d}(\Delta X_k^m, \Delta X_k^n)}{f(X^m, X^n)}\right) \leq 1 \leq M\left(\frac{rx_0}{3}\right)$

$\Rightarrow \overline{d}(\Delta X_k^m, \Delta X_k^n) < \frac{\varepsilon}{3}$, for all *n*, $m \geq m_0$.

Thus (ΔX_k^m) is a Cauchy sequence of fuzzy real numbers and so $\lim_m \Delta X_k^m = \Delta X_k$ exist.

Moreover using the existence of $\lim_m X_1^m$ we can conclude that so $\lim_m X_k^m$ exist.
Using continuity of M, $M\left(\frac{\overline{d}(\Delta X_k^m, \Delta X_k)}{r}\right) \leq 1$
Taking infimum of such r's we get

$$f(X^n, X) < \frac{\varepsilon}{rx_0} < \varepsilon \text{ for all } n \geq m_0.$$

Thus (X^n) converges to X.
Since $X^m, X^n \in (m^I)^F(M, \Delta)$ so there exist fuzzy numbers Y_m, Y_k such that

$$A = \left\{k \in N : M\left(\frac{\overline{d}(\Delta X_k^m, Y_k)}{r}\right) < M\left(\frac{\varepsilon}{3r}\right)\right\} \in \psi(I)$$

$$= \left\{k \in N : \overline{d}(\Delta X_k^m, Y_k) < \frac{\varepsilon}{3}\right\} \in \psi(I).$$

$$B = \left\{k \in N : \overline{d}(\Delta X_k^n, Y_m) < \frac{\varepsilon}{3}\right\} \in \psi(I).$$

Now $A \cap B \in \psi(I)$ and let $k \in A \cap B$.
Then $\overline{d}(Y_k, Y_m) \leq \overline{d}(Y_k, \Delta X_k^n) + \overline{d}(\Delta X_k^n, \Delta X_k^m) + \overline{d}(\Delta X_k^m, Y_m)$
$< \varepsilon$ for all $n, m \geq m_0$.

Thus (Y_k) is a Cauchy sequence of fuzzy real numbers. So there exists a fuzzy real number Y such that $\lim Y_k = Y$. To show that I-$\lim \Delta X_k = Y$.
This follows from above inequalities as

$$\overline{d}(\Delta X_k, Y) \leq \overline{d}(\Delta X_k, \Delta X_k^m) + \overline{d}(\Delta X_k^m, Y_k) + \overline{d}(Y_k, Y)$$

$$< \eta.$$

Thus I-$\lim X_k = Y$. Hence $(X_k) \in (m^I)^F(M, \Delta)$. □

Property 3.2. *The sequence spaces* $(c^I)^F(M, \Delta)$, $(c_0^I)^F(M, \Delta)$,$(m^I)^F(M, \Delta)$ *and* $(m_0^I)^F(M, \Delta)$ *are not symmetric.*

Proof. For this result consider the following example.

Example 3.3. Let $I = I_\delta$ and $M(x) = x$. Consider the sequence $(X_k) \in (c_0^I)^F(M, \Delta) \subset \left(c^I\right)^F(M, \Delta)$ as follows:

$$X_1(t) = \begin{cases} 1, \text{if } -1 \leq t \leq 0 \\ 0, \text{otherwise} \end{cases}$$

$$X_k(t) = \begin{cases} 1, \text{if } -\left(\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right) + \frac{1}{k}\right) \leq t \leq -\sum_{r=1}^{k-1}\left(\frac{1}{2r}\right) \\ 0, \text{otherwise} \end{cases}$$

For each $\alpha \in (0, 1]$ we have $[X_1]^\alpha = [-1, 0]$ and for $k \geq 2$,

$$[X_k]^\alpha = \left[-\left(\sum_{r=1}^{k-1}\frac{1}{2r} + \frac{1}{k}\right), -\sum_{r=1}^{k-1}\frac{1}{2r},\right]$$

Then for all $\alpha \in (0, 1]$ and $k \in N$ we have,

$$[\Delta X_k]^\alpha = \left[-(2k)^{-1}, \left\{(2k)^{-1} + (k+1)^{-1}\right\}\right]$$

Hence $\Delta X_k \to \overline{0}$ as $k \to \infty$. Thus $(X_k) \in (c_0^I)^F(M, \Delta) \subset \left(c^I\right)^F(M, \Delta)$
Let the sequence (Y_k) be a rearrangement of (X_k), such that

$$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, X_{25}, X_7 ....)$$

That is,

$(Y_k) = X_{\left(\frac{k+1}{2}\right)^2}$, for all k odd.
$= X_{\left(\frac{2n+k}{2}\right)}$, for all k even and $n(n-1) < \frac{k}{2} \leq n(n+1)$, $n \in N$.

Then for $k = 1$, we have

$$[\Delta Y_1]^\alpha = [X_1]^\alpha - [X_2]^\alpha = [-0.5, 1] \text{ for each } \alpha \in (0, 1]$$

. For $k > 1$ odd and $n \in N$ satisfying $n\,(n-1) < \frac{k}{2} \le n(n+1)$ we have,

$$[\Delta Y_k]^\alpha = \left[X_{\left(\frac{k+1}{2}\right)^2}\right]^\alpha - \left[X_{\left(n+\frac{k+1}{2}\right)}\right]^\alpha$$

= $\left[-\left(\sum_{r=n+\frac{k+1}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{2r} + \frac{1}{\left(\frac{k+1}{2}\right)^2}\right), -\left(\sum_{r=n+\frac{k+1}{2}}^{\left(\frac{k+1}{2}\right)^2-1} \frac{1}{2r} +\right) + \frac{1}{\left(n+\frac{k+1}{2}\right)}\right]$ for all $\alpha \in (0, 1]$.

For k even and $n \in N$ satisfying $n\,(n-1) < \frac{k}{2} \le n(n+1)$ we have ,

$$[\Delta Y_k]^\alpha = \left[X_{\left(n+\frac{k}{2}\right)}\right]^\alpha - \left[X_{\left(\frac{k+2}{2}\right)^2}\right]^\alpha$$

= $\left[\left(\sum_{r=n+\frac{k}{2}}^{\left(\frac{k+2}{2}\right)^2-1} \left(\frac{1}{2r}\right) - \frac{1}{\left(n+\frac{k}{2}\right)}\right), \left(\sum_{r=n+\frac{k}{2}}^{\left(\frac{k+2}{2}\right)^2-1} \left(\frac{1}{2r}\right) + \frac{1}{\left(\frac{k+2}{2}\right)^2}\right)\right]$ for all $\alpha \in (0, 1]$.

Here it is observed that the values of (ΔY_k) increases with

$$\Delta Y_4(t) = \begin{cases} 1, \text{if } 0.2759 \le t \le 0.7200 \\ 0, \text{otherwise} \end{cases}$$

for $k > 3$ and k is even and decreases for $k > 3$ and k odd. Therefore the sequence can not converge to a point.

Hence $(Y_k) \notin \left(c^I\right)^F(M,\ \Delta)\left(\supset (c_0^I)^F(M, \Delta)\right)$. This completes the proof.

□

Theorem 3.4. *The sequence spaces $(c_0^I)^F(M, \Delta)$, $(c^I)^F(M,\ \Delta)$, $(m^I)^F(M\ ,\Delta)$ and $(m_0^I)^F(M,\ \Delta)$ are not solid.*

Proof. We prove the result for $(c^I)^F(M, \Delta)$. For the other spaces the result can be proved similarly.

Consider the sequence of fuzzy real numbers given by

$$(X_k)(t) = \begin{cases} t+1, & 0 \le t \le 1 \\ -t+1, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Then $[(X_k)]^\alpha = [\alpha - 1, 1 - \alpha]$. From this it is clear that (ΔX_k) is convergent.

Consider the sequence (Y_k) defined by

$$Y_k = \begin{cases} X_k, & \text{for all even } k \\ \overline{0}, & \text{otherwise} \end{cases}$$

Here (ΔY_k) is not convergent. □

Property 3.5. *The sequence spaces $(c^I)^F(M,\ \Delta)$, $(c_0^I)^F(M, \Delta)$,$(m^I)^F(M\ ,\Delta)$ and $(m_0^I)^F(M,\ \Delta)$ are not convergence free.*

Proof. For this result consider the following example.

Example 3.6. Let $I = I_\delta$ and $M(x) = x$. Consider the sequence $(X_k) \in (c_0^I)^F(M, \Delta) \subset \left(c^I\right)^F(M,\ \Delta)$ as follows:

For $k \ne i^2, i \in N$

$$X_k(t) = \begin{cases} 1, \text{ if } 0 \le t \le k^{-1} \\ 0, \text{otherwise} \end{cases}$$

and for $k = i^2, i \in N, X_k(t) = \overline{0}$.

Then for $\alpha \in (0, 1]$ we have,

$$[X_k]^\alpha = \begin{cases} [0,0], \text{if } k = i^2 \\ [0, k^{-1}], \text{if } k \neq i^2 \end{cases}$$

and

$$[\Delta X_k]^\alpha = \begin{cases} \left[-(k+1)^{-1}, 0\right], \text{ for } k = i^2 \\ \left[0, k^{-1}\right], \text{for } k = i^2 - 1 \text{ with } i \neq 1 \\ \left[-(k+1)^{-1}, k^{-1}\right] \text{ otherwise} \end{cases}$$

Hence $\Delta X_k \to \overline{0}$ as $k \to \infty$. Thus $(X_k) \in (c_0^I)^F(M, \Delta) \subset \left(c^I\right)^F(M,\ \Delta)$
Let (Y_k) be another sequence such that '
For $k \neq i^2, i \in N$,

$$Y_k(t) = \begin{cases} 1, \text{if } 0 \leq t \leq k \\ 0, \text{otherwise} \end{cases}$$

And for $k = i^2, i \in N$, $Y_k(t) = \overline{0}$.
Now for all $\alpha \in (0,1]$ we have,

$$[Y_k]^\alpha = \begin{cases} [0,0], \text{if } k = i^2 \\ [0, k], \text{if } k \neq i^2 \end{cases}$$

and

$$[\Delta Y_k]^\alpha = \begin{cases} [-(k+1), 0], \text{for } k = i^2 \\ [0, k], \text{ for } k = i^2 - 1 \text{ with } i \neq 1 \\ [-(k+1), k] \text{ otherwise} \end{cases}$$

This implies, $(Y_k) \notin (c_0^I)^F(M, \Delta) \subset \left(c^I\right)^F(M, \Delta)$ Hence $\left(c^I\right)^F(M, \Delta)$, $(c_0^I)^F(M, \Delta)$ are not convergence free. Similarly the other spaces are also not convergence free.

□

Theorem 3.7. *Z(M_1, Δ) ⊆ Z(M_2oM_1, Δ) for Z =*$c^{I(F)}$, $c_0^{I(F)}$*and* $\ell_\infty^{I(F)}$

Proof. Let Z = $(c^I)^F$ and $(X_k) \in (c^I)^F$(*M*, Δ). Then

$$\left\{k : M\left(\frac{\overline{d}(\Delta X_k, L)}{r}\right) \geq \varepsilon, \text{ for some } r > 0\right\} \in I.$$

Since M_2 is continuous, so for $\varepsilon > 0$ there exist $\eta > 0$ such that $M_2(\varepsilon) = \eta$. The result follows from

$$M_2\left(M_1\left(\frac{\overline{d}(\Delta X_k, L)}{r}\right)\right) \geq M_2(\varepsilon){=}\eta.$$

□

Theorem 3.8. *Z(M, Δ)* ⊆ $(\ell_\infty^I)^F(M, \Delta)$ *for Z =* $(c^I)^F$, $(c_0^I)^F$.

Proof. The proofs are obvious. □

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