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On left centralizers of prime rings with involution

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

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Abstract. The objective of the present paper is to prove that if a prime ring with involution of characteristic different from two admits a nonzero left centralizer T such that $[T(x), x^*] = 0$ for all $x \in R$, then R is normal. Further, we characterize normal rings and two sided centralizers among all prime rings with involution satisfying certain identities involving left centralizers.

1. Introduction

This research is inspired by the work of Divinsky [8] and Vukman [24] . Throughout this article, R will represent an associative ring with centre $Z(R)$. We denote by $Q_l(R)$, $Q_m(R)$, $Q_s(R)$ and C, the maximal left ring of quotients, maximal right ring of quotients, the symmetric ring of Quotients and the extended centroid of a prime ring R. For the explanation of $Q_l(R)$, $Q_m(R)$, $Q_s(R)$ and C we refer the reader to [5]. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and make extensive use of basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = [x, y]z + y[x, z]$ for all $x, y, z \in R$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds for all $x, y \in R$. A ring equipped with an involution is known as ring with involution or $*$-ring. An element x in a ring with involution $'*'$ is said to hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. An element $x \in R$ is said to be normal if $xx^* = x^*x$ for all $x \in R$. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [11], where further references can be looked.

Let R be any ring. An additive mapping $T : R \to R$ is said to be a left centralizer (resp. right centralizer) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer in case T is a left and a right centralizer of R. Following [1], an additive mapping $T : R \to R$ is said to be a left $*$-centralizer (resp. reverse left $*$-centralizer) if $T(xy) = T(x)y^*$ (resp. $T(xy) = T(y)x^*$) holds for all $x, y \in R$, where R is ring with involution. This concept appears naturally in C^*-algebras. In ring theory it is more common to work with module homomorphisms. Ring theorists would write that $T : R_R \to R_R$ is a homomorphism of a ring module R into itself. For a semiprime ring R all such homomorphisms are of the form $T(x) = qx$ for all $x \in R$, where q is an element of Martindale left ring of quotients Q_r (see Chapter 2 in [5]). If R has the identity element then $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$ and some fixed element $a \in R$. Staring with the paper by Zalar [27], during the last some years the study of centralizers becomes an active area of research in semi(prime) rings, C^*-algebras and H^*-algebras (see for instance, [1], [3], [4], [9], [10], [14], [16], [21], [22], [23], [24], [25] and [26] for details).

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A derivation d_a is called inner if there exists $a \in R$ such that $d_a(x) = [a, x]$ for all $x \in R$. A mapping f of R into itself is called centralizing if $[f(x), x] \in Z(R)$ holds for all $x \in R$; in the special case when $[f(x), x] = 0$ holds for all $x \in R$, the mapping f is said to be commuting. The history of commuting and centralizing mappings goes back to 1995 when Divinsky [8] proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. Two years later, Posner [20] has proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative (Posner's second theorem). In [13], Mayne obtained an analogous result for automorphisms of prime rings. Very recently, Oukhtite [18] has established Posner's theorem for Jordan ideals in rings with involution. Over the last some decades, several authors have proved commutativity theorems for prime and semiprime rings with or without involution admitting automorphisms, left centralizers or derivations which are centralizing or commuting on an appropriate subset of the ring (viz., [4], [6], [11], [14], [15], [16], [17], [18], [19]).

Let R be a ring with involution $'*'$ and S be a nonempty subset of R. Following [2], a mapping f of R into itself is called $*$-centralizing on S if $[f(x), x^*] \in Z(R)$ for all $x \in R$; and is called $*$-commuting on S if $[f(x), x^*] = 0$ for all $x \in R$. Notice that for any central element a, the map $x \mapsto ax^*$ is $*$-commuting and $*$-centralizing but neither commuting nor centralizing on R. Thus, it is reasonable to study the behaviour of such mappings in the setting of prime rings with involution. The main purpose of this paper is to study $*$-commuting mapping in prime rings with involution. Further, we prove that if a prime ring R with involution admitting a nonzero left centralizer T, then either R is normal or T is both sided centralizer if any one of the following conditions hold: (i) $T(xx^*) \pm xx^* = 0,$ (ii) $T(x^*x) \pm x^*x = 0,$ (iii) $xT(x^*) \pm T(x)x^* = 0,$ (iv) $T(x)T(x^*) \pm xx^* = 0,$ (v) $T(x^*)T(x) \pm x^*x = 0$ for all $x \in R$.

2. Some preliminaries

We begin with the following lemma:

Lemma 2.1. Let R be a prime ring with involution $'*'$. If T is a nonzero left centralizer of R such that $T(xx^*) = 0$ for all $x \in R$ or $T(x^*x) = 0$ for all $x \in R$, then R is normal.

Proof. In view of our hypothesis we have $T(xx^*) = 0$ for all $x \in R$. This can be further written as $T(x)x^* = 0$ for all $x \in R$. Linearizing the last relation, we obtain $T(x)y^* + T(y)x^* = 0$ for all $x, y \in R$. Substituting yx for y in the above expression and using the given hypothesis we find that $T(y)xx^* = 0$ for all $x, y \in R$. Replacing x by x^* in the last relation, we get $T(y)x^*x = 0$ for all $x, y \in R$. Last two relations yields that $T(y)[x, x^*] = 0$ for all $x, y \in R$. Replace y by yr to get $T(y)r[x, x^*] = 0$ for all $x, y, r \in R$ i.e., $T(y)R[x, x^*] = (0)$ for all $x,\ y \in R$. Thus by the primeness of R, we conclude that either $T(y) = 0$ for all $y \in R$ or $[x, x^*] = 0$ for all $x \in R$. Since $T(y) = 0$ for all $y \in R$, gives a contradiction. Thus the only possibility is $[x, x^*] = 0$ for all $x \in R$. Which proves that R is normal.

By the similar arguments, we obtain the same conclusion in the case $T(x^*x) = 0$ for all $x \in R$. This proves the lemma.

Lemma 2.2. Let R be a prime ring with involution $'*'$ of characteristic different from two. Suppose there exists $a \notin Z(R)$ such that $[h, a] = 0$ for all $h \in H(R)$, then R is normal.

Proof. We have

$$[h, a] = 0 \; for \; all \; h \in H(R). \tag{0.1}$$

If $h \in H(R),\ k \in S(R)$, then $hk - kh \in H(R)$ and therefore from (0.1), we have $[hk, a] - [kh, a] = 0$ for all $h \in H(R)$ and $k \in S(R)$. This implies that

$$h[k, a] + [h, a]k - k[h, a] - [k, a]h = 0 \; for \; all \; h \in H(R) \; and \; k \in S(R).$$

Application of (0.1) yields that

$$h[k, a] - [k, a]h = 0 \; for \; all \; h \in H(R) \; and \; k \in S(R). \tag{0.2}$$

Also, we have

$$h[h_1, a] - [h_1, a]h = 0 \; for \; all \; h, h_1 \in H(R). \tag{0.3}$$

Since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h_1 + k$ where $h_1 \in H(R), k \in S(R)$ and therefore making use of (0.2) and (0.3), we obtain

$$2h[x, a] - 2[x, a]h = 0 \; for \; all \; x \in R \; and \; h \in H(R).$$

Since $char(R) \neq 2$, the last relation forces that

$$h[x, a] - [x, a]h = 0 \; for \; all \; x \in R \; and \; h \in H(R). \tag{0.4}$$

Now, since the mapping $x \mapsto [x, a]$ is a derivation and so in view of Herstein's result [[12], Theorem] we conclude that $h \in Z(R)$ for all $h \in H(R)$. Thereby proving R is normal.

3. The Main results

We shall start our investigations with our first theorem which is inspired by the work of Divinsky [8].

Theorem 3.1. Let R be a prime ring with involution $'*'$ such that $char(R) \neq 2$. Let T be a nonzero left centralizer of R such that $[T(x), x^*] = 0$ for all $x \in R$. Then R is normal.

Proof. We have $[T(x), x^*] = 0$ for all $x \in R$. Replacing x by x^*, we obtain $[T(x^*), x] = 0$ for all $x \in R$. Let $f : R \to R$ be defined by $f(x) = T(x^*)$ for all $x \in R$. Then, it is easy to verify that f is a reverse left $*$-centralizer and hence $[f(x), x] = 0$ for all $x \in R$. In view of [[7], Theorem 3.2], we conclude that $f(x) = \mu x + \nu(x)$ for all $x \in R$, where $\mu \in C$, the extended centroid of R and $\nu : R \to C$ is an additive mapping. Define a new map $g : R \to R$ such that $g(x) = f(x)^*$ for all $x \in R$. Then clearly g is a left R-module homomorphism (i.e., right centralizer). Hence there exists $p \in Q_m(R)$ such that $g(x) = xp$ for all $x \in R$ (see [[5], Chapter 2] for details). Therefore, we obtain $f(x) = \lambda x^*$ for all $x \in R$, where $\lambda = p^*$. Hence, we get

$$\lambda x^* - \mu x \in C$$

for all $x \in R$. Since the identity involves involution, so it is a functional identity or the so-called g-identity (see [[5], Chapter 6]). In view of [[5], Theorem 6.4.6], we conclude that $\lambda x^* - \mu x \in C$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $\lambda - \mu \in C$. This implies that $[\lambda, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T(x) = f(x^*) = \lambda x$$

for all $x \in R$, where $\lambda \in C$. Since $T \neq 0$, it follows that $\lambda \neq 0$. Thus we conclude that $0 = [T(x), x^*] = [\lambda x, x^*] = \lambda[x, x^*]$ for all $x \in R$. Hence, by the primeness of R, R is normal. This proves the theorem completely.

The above theorem has following interesting consequences:

Corollary 3.2. Let R be a prime ring with involution $'*'$ of characteristic different from two. Let T be a nonzero left centralizer of R such that $[T(x), x^*] = 0$ for all $x \in R$. Then there exists $\lambda \in C$, the extended centroid of R such that $T(x) = \lambda x$ for all $x \in R$.

Proof. The proof follows from the above theorem.

Theorem 3.3. Let R be a noncommutative prime ring with involution $'*'$ of characteristic different from two. Let T_1 and T_2 be two nonzero left centralizers of R such that $T_1(x)x^* - x^*T_2(x) = 0$ for all $x \in R$. Then R is normal.

Proof. By the given hypothesis, we have

$$T_1(x)x^* - x^*T_2(x) = 0 \tag{0.5}$$

for all $x \in R$. On linearizing (0.5), we get

$$T_1(x)y^* + T_1(y)x^* - x^*T_2(y) - y^*T_2(x) = 0 \tag{0.6}$$

for all $x, y \in R$. Replacing y by xy in (0.6), we arrive at

$$T_1(x)y^*x^* + T_1(x)yx^* - x^*T_2(x)y - y^*x^*T_2(x) = 0 \tag{0.7}$$

for all $x, y \in R$. Using (0.5) in (0.7), we obtain $T_1(x)y^*x^* + T_1(x)yx^* - T_1(x)x^*y - y^*T_1(x)x^* = 0$. This can be further written as $[T_1(x), y^*]x^* + T_1(x)[y, x^*] = 0$ for all $x, y \in R$. Since $T_1(x) = \lambda_1 x$ (where $\lambda_1 \in Q_l(R)$) for all $x \in R$. Thus $[\lambda_1 x, y^*]x^* + \lambda_1 x[y, x^*] = 0$ for all $x \in R$. Since the above identity is a g-identity (see [[5], Chapter 6]). In view of [[5], Theorem 6.4.6], we conclude that $[\lambda_1 x, y^*]x^* + \lambda_1 x[y, x^*] = 0$ for all $x \in Q_s(R)$, the symmetric ring of quotients. Note that $Q_s(R)$ has the identity element 1. Replacing x by 1 in the above expression, we see that $[\lambda_1, y] = 0$ for all $y \in Q_s(R)$. Thus,

$$T_1(x) = \lambda_1 x$$

for all $x \in R$, where $\lambda_1 \in C$. Since $T_1 \neq 0$, it follows that $\lambda_1 \neq 0$. Also $T_2(x) = \lambda_2 x$, where $\lambda_2 \in Q_l(R)$. Hence from (0.5), $\lambda_1 xx^* - x^*\lambda_2 x = 0$ for all $x \in R$. Since the above identity is a g-identity. Thus by [[5], Theorem 6.4.6], we obtain $\lambda_1 xx^* - x^*\lambda_2 x = 0$ for all

$x \in Q_s(R)$, the symmetric ring of quotients. Replacing x by 1 in the above expression, we see that $\lambda_1 = \lambda_2 = \lambda(say)$. Thus $T_2(x) = T_1(x) = \lambda x$ for all $x \in R$, where $0 \neq \lambda \in C$. Hence we conclude that $0 = T_1(x)x^* - x^*T_2(x) = \lambda xx^* - x^*\lambda x = \lambda(xx^* - x^*x)$ for all $x \in R$. Thus by the primeness of R, R is normal. This proves the theorem completely.

As an immediate consequence of Theorem 3.3, we have the following corollary:

Corollary 3.4. Let R be a prime ring with involution $'*'$ of characteristic different from two. Let T_1 and T_2 be two nonzero left centralizers of R such that $T_1(x)x^* - x^*T_2(x) = 0$ for all $x \in R$. Then there exists $\lambda \in C$, the extended centroid of R such that $T_1(x) = T_2(x) = \lambda x$ for all $x \in R$.

Proof. The proof follows from the above theorem.

It would be interesting to know whether Theorem 3.1 and Theorem 3.3 hold in the case of arbitrary rings. Following example justifies this fact:

Example 3.1. Let F be a field and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in F \right\}$. Define mappings $T : R \longrightarrow R$, and $* : R \longrightarrow R$ such that
$T \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & c & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}.$
Then, it is easy to verify that T satisfies all the requirements of Theorem 3.1. However, R is not normal.

Example 3.2. Consider the ring as in Example 1, define mappings $T_1, T_2 : R \longrightarrow R$ such that $T_1 \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_2 \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & o \\ 0 & 0 & 0 \end{pmatrix}.$
Then, it is straightforward to check that T_1 and T_2 satisfy all the requirements of Theorem 3.3. However, R is not normal.

The aim of the rest in this paper is to characterize both sided centralizers and normal rings among all prime rings with involution involving certain identities. We begin with the following result:

Theorem 3.4. Let R be a prime ring with involution $'*'$. Let T be a left centralizer of R such that $T(xx^*) \pm xx^* = 0$ for all $x \in R$. Then either T is a centralizer or R is normal.

Proof. First we consider the case $T(xx^*) - xx^* = 0$ for all $x \in R$. This can be further written as

$$T(x)x^* - xx^* = 0 \; for \; all \; x \in R. \tag{0.8}$$

Linearizing the above relation, we get

$$T(x)y^* + T(y)x^* - xy^* - yx^* = 0 \; for \; all \; x, y \in R. \tag{0.9}$$

Replacing y by yx in (0.9), we obtain

$$T(x)x^*y^* + T(y)xx^* - xx^*y^* - yxx^* = 0 \; for \; all \; x, y \in R.$$

Application of (0.8) yields that

$$T(y)xx^* - yxx^* = 0 \; for \; all \; x, y \in R. \tag{0.10}$$

Substituting zy for y in (0.10), we have

$$T(z)yxx^* - zyxx^* = 0 \; for \; all \; x, y, z \in R. \tag{0.11}$$

Left multiplication to (0.10) by z yields that

$$zT(y)xx^* - zyxx^* = 0 \; for \; all \; x, y, z \in R. \tag{0.12}$$

Subtracting (0.11) from (0.12), we obtain

$$zT(y)xx^* - T(z)yxx^* = 0 \; for \; all \; x, y, z \in R. \tag{0.13}$$

Substituting yr for y in (0.13) to get $zT(y)rxx^* - T(z)yrxx^* = 0 \; for \; all \; x, y, z, r \in R$. Which can be further written as

$$(zT(y) - T(z)y)rxx^* = 0 \; for \; all \; x, y, z \; and \; r \in R. \tag{0.14}$$

Replacing x by x^* in (0.14), we find that

$$(zT(y) - T(z)y)rx^*x = 0 \; for \; all \; x, y, z, r \in R. \tag{0.15}$$

Subtracting (0.15) from (0.14), we obtain

$$(zT(y) - T(z)y)r[x, x^*] = 0 \; for \; all \; x, y, z, r \in R.$$

This implies that $(zT(y) - T(z)y)R[x, x^*] = (0) \; for \; all \; x, y, z \in R$. Thus by the primeness of R we have either $zT(y) - T(z)y = 0$ for all $y, z \in R$ or $[x, x^*] = 0$ for all $x \in R$. Now if $zT(y) - T(z)y = 0$ for all $y, z \in R$ i.e., $zT(y) = T(z)y$ for all $y, z \in R$. Then T is also a right centralizer of R and hence a centralizer of R. On the other hand if $[x, x^*] = 0$ for all $x \in R$, then R is normal.

By the same arguments, we obtain the same conclusion in case $T(xx^*) + xx^* = 0$ for all $x \in R$. This proves the theorem.

By similar arguments as above with necessary variation, we can prove the following theorem:

Theorem 3.5. Let R be a prime ring with involution $'*'$. Let T be a left centralizer of R such that $T(x^*x) \pm x^*x = 0$ for all $x \in R$. Then either T is a centralizer or R is normal.

Theorem 3.6. Let R be a prime ring with involution $'*'$. Let T be a left centralizer of R such that $xT(x^*) \pm T(x)x^* = 0$ for all $x \in R$. Then either T is a centralizer or R is normal.

Proof. First we consider the case

$$xT(x^*) - T(x)x^* = 0 \; for \; all \; x \in R. \tag{0.16}$$

Linearizing the above relation, we get

$$xT(y^*) - T(x)y^* + yT(x^*) - T(y)x^* = 0 \; for \; all \; x, y \in R. \tag{0.17}$$

Replacing y by yx in (0.17) and using (0.16), we obtain

$$T(y)xx^* - yxT(x^*) = 0 \; for \; all \; x, y \in R. \tag{0.18}$$

Substituting zy for y in (0.18), we have

$$T(z)yxx^* - zyxT(x^*) = 0 \; for \; all \; x, y, z \in R. \tag{0.19}$$

Left multiplying (0.18) by z yields that

$$zT(y)xx^* - zyxT(x^*) = 0 \; for \; all \; x, y, z \in R. \tag{0.20}$$

Subtracting (0.19) from (0.20), we obtain

$$T(z)yxx^* - zT(y)xx^* = 0 \; for \; all \; x, y, z \in R. \tag{0.21}$$

Substituting yr for y in (0.21), to get

$$(zT(y) - T(z)y)rxx^* = 0 \; for \; all \; x, y, z \; and \; r \in R. \tag{0.22}$$

The above equation is same as (0.14) and henceforward using the same approach as we have used in the last paragraph of the proof of Theorem 3.4, we get the required result. This proves the theorem.

Theorem 3.7. Let R be a prime ring with involution $'*'$. Let T be a left centralizer of R such that $T(x)T(x^*) \pm xx^* = 0$ for all $x \in R$. Then either T is a centralizer or R is normal.

Proof. First we consider situation

$$T(x)T(x^*) - xx^* = 0 \; for \; all \; x \in R. \tag{0.23}$$

Replacing x by $x+y$, we get

$$T(x)T(y^*) - xy^* + T(y)T(x^*) - yx^* = 0 \ for \ all \ x, y \in R. \tag{0.24}$$

Substituting yx for y in (0.24), we obtain

$$T(x)T(x^*)y^* - xx^*y^* + T(y)xT(x^*) - yxx^* = 0 \ for \ all \ x, y \in R.$$

In view of (0.23), the above expression reduces to

$$T(y)xT(x^*) - yxx^* = 0 \ for \ all \ x, y \in R. \tag{0.25}$$

Replace y by zy in (0.25), to get

$$T(z)yxT(x^*) - zyxx^* = 0 \ for \ all \ x, y, z \in R. \tag{0.26}$$

Left multiplying (0.25) by z, we get

$$zT(y)xT(x^*) - zyxx^* = 0 \ for \ all \ x, y, z \in R. \tag{0.27}$$

Subtracting (0.27) from (0.26), we obtain

$$zT(y)xT(x^*) - T(z)yxT(x^*) = 0 \ for \ all \ x, y, z \in R. \tag{0.28}$$

Substituting yr for y in (0.28), we find that

$$zT(y)rxT(x^*) - T(z)yrxT(x^*) = 0 \ for \ all \ x, y, z, r \in R.$$

This implies that

$$(zT(y) - T(z)y)rxT(x^*) = 0 \ for \ all \ x, y, z \ and \ r \in R. \tag{0.29}$$

That is, $(zT(y) - T(z)y)RxT(x^*) = (0) \ for \ all \ x, y, z \in R$. Thus by the primeness of R we find that either $zT(y) - T(z)y = 0$ for all $y, z \in R$ or $xT(x^*) = 0$ for all $x \in R$. If $zT(y) - T(z)y = 0$ i.e., $T(z)y = zT(y)$ for all $y, z \in R$, then T is also a right centralizer and hence a centralizer on R. On the other hand, suppose $xT(x^*) = 0$ for all $x \in R$. This gives $T(y)xT(x^*) = 0$ for all $x, y \in R$. Hence (0.25) reduces to $yxx^* = 0 \ for \ all \ x, y \in R$. Replacing y by yr in the above relation, we obtain

$$yrxx^* = 0 \ for \ all \ x, y \in R. \tag{0.30}$$

Replacing x by x^* in (0.30), we have

$$yrx^*x = 0 \ for \ all \ x, y \in R. \tag{0.31}$$

Subtracting (0.31) from (0.30), we obtain

$$yr[x, x^*] = 0 \ for \ all \ x, y, r \in R.$$

This implies that $[x, x^*]R[x, x^*] = (0) \ for \ all \ x \in R$. Since R is prime, the last expression forces that $[x, x^*] = 0$ for all $x \in R$.

Similar conclusion holds for the case $T(x)T(x^*) + xx^* = 0$ for all $x \in R$. This finishes the second case, and so the theorem is proved.

Using similar approach with necessary variations we can establish the following:

Theorem 3.8. Let R be a prime ring with involution $'*'$. Let T be a left centralizer of R such that $T(x^*)T(x) \pm x^*x = 0$ for all $x \in R$. Then either T is a centralizer or R is normal.

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