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Structure Space of Prime Ideals of Γ-Semirings

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Abstract The space of prime ideals of a Γ-semiring endowed with the hull kernel toplogy. Various properties of the space of prime ideals of a Γ-semiring endowed with the hull kernel toplogy are studied.

1 Introduction

As a generalization of a Γ-ring and a semiring the notion of a Γ-semiring was introduced by Rao [11]. Various characterizations of a semiring were done in [2, 8, 9]. Also some work on a Γ-semiring was given in [5, 6, 7, 11]. The Structure spaces of a semiring was studied by Adhikari and Das in [1] while the structure spaces of a Γ-semigroup by Chattppadhay and Kar in [4]. In this paper efforts are taken for the study of structure spaces of prime ideals of a Γ-semiring.

The set $\wp$ of all prime ideals in a Γ-semiring S endowed with the hull kernel toplogy τ. Various topological properties of the space $(\wp,\ \tau)$ are studied. Necessary and sufficient conditions for the space $(\wp,\ \tau)$ to be T_1, T_2, T_3 are furnished. It is observed that space $(\wp,\ \tau)$ is a compact space if and only if for any collection $\{a_i\}_{i\in\Lambda} \subset S$ there exists a finite subcollection $\{a_1, a_2,\ a_3, \cdots\cdots, a_n\}$ in S such that $I \in \wp$ there exist a_i such that $a_i \notin I$.

2 Preliminaries

First we recall some definitions of the basic concepts of a Γ-semiring that we need in sequel. For this we follow Dutta and Sardar [5]. Also for the basic concepts of topology we follow Kelly [10].

Definition 2.1. Let S and Γ be two additive commutative semigroups. S is called a Γ-semiring if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ denoted by $a\alpha b$; for all $a, b \in S$ *and* $\alpha \in \Gamma$ satisfying the following conditions:
(i) $a\alpha\,(b + c) = (a\ \alpha b) + (a\ \alpha c)$
(ii) $(b + c)\,\alpha a = (b\ \alpha a) + (c\ \alpha a)$
(iii) $a(\alpha + \beta)c = (a\ \alpha c) + (a\ \beta c)$
(iv) $a\alpha\,(b\beta c) = (a\alpha b)\,\beta c$; for all $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$.

Obviously, every semiring S is a Γ-semiring.

Definition 2.2. An element $0 \in S$ is said to be an absorbing zero if $0\alpha a = 0 = a\alpha 0,\ a + 0 = 0 + a = a$; for all $a \in S$ *and* $\alpha \in \Gamma$.

Now onwards S denotes a Γ-semiring with absorbing zero unless otherwise stated.

Definition 2.3. A nonempty subset T of S is called a left (respectively right) ideal of S if T is a subsemigroup of $(S, +)$ and $x\alpha a \in T$ (respectively $a\alpha x \in T$) for all $a \in T,\ x \in\ S$ and $\alpha \in \Gamma$.

Definition 2.4. If T is both left and right ideal of S, then T is known as an ideal of S.

Definition 2.5. An ideal P of S is called a prime ideal if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of S.

Definition 2.6. A prime ideal P of S is said to be a minimal prime ideal if there does not exist any other prime ideal of S containing P properly.

A proper ideal M of S is said to be a maximal ideal if there does not exist any other proper ideal of S containing M properly.

(a) denotes an ideal generated by $a \in S$ and is defined as $(a) = N_0 a + S\Gamma a$, where N_0 denotes the set of non negative integers.

3 Prime Ideal Space

Let $\wp$ denote the collection of all prime ideals of S. For any subset A of $\wp$ we define $\overline{A} = \{I \in \wp \mid \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$.

Then Further we have

Theorem 3.1. *The function $A \to \overline{A}$ is a closure operator on $\wp$.*

Proof :- obviously $\overline{\phi} = \phi$. (i) By the definition of $\overline{A}$ for each α, $I_\alpha \in A$. Therefore $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha$ implies $I_\alpha \in \overline{A}$. Hence $A \subseteq \overline{A}$.
(ii) Let $I_\beta \in \overline{\overline{A}}$. Then $\bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. But $\bigcap_{I_\gamma \in A} I_\gamma \subseteq I_\alpha$. As this is true for all $\alpha \in \Lambda$, where Λ denotes the indexing set. We get $\bigcap_{I_\gamma \in A} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta$. This gives $I_\beta \in \overline{A}$. Thus $\overline{\overline{A}} \subseteq \overline{A}$. As by (i) $\overline{A} \subseteq \overline{\overline{A}}$, the result follows.
(iii) Assume that $A \subseteq B$. Then as $\bigcap_{I_\alpha \in B} I_\alpha \subseteq \bigcap_{I_\alpha \in A} I_\alpha \subseteq I$ we get $\overline{A} \subseteq \overline{B}$.
(iv) By (iii) $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Now let $I \in \overline{A \cup B}$. Then $\bigcap_{I_\alpha \in A \cup B} I_\alpha \subseteq I$. Obviously $\bigcap_{I_\alpha \in A \cup B} I_\alpha = (\bigcap_{I_\alpha \in A} I_\alpha) \cap (\bigcap_{I_\alpha \in B} I_\alpha)$. Now $\bigcap_{I_\alpha \in A} I_\alpha$ and $\bigcap_{I_\alpha \in B} I_\alpha$ are ideals of S and $(\bigcap_{I_\alpha \in A} I_\alpha) \Gamma (\bigcap_{I_\alpha \in B} I_\alpha) \subseteq (\bigcap_{I_\alpha \in A} I_\alpha) \cap (\bigcap_{I_\alpha \in B} I_\alpha) \subseteq I$. As I is a prime ideal of S, we get $\bigcap_{I_\alpha \in A} I_\alpha \subseteq I$ or $\bigcap_{I_\alpha \in B} I_\alpha \subseteq I$. Hence $I \in \overline{A}$ or $I \in \overline{B}$. Thus $I \in \overline{A} \cup \overline{B}$. This shows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.Combining both the inclusions we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

The closure operator $A \to \overline{A}$ induces a topology τ on $\wp$. This topology is the hull kernel topology and the space $(\wp, \tau)$ is called the structure space of a Γ-semiring S.

For any ideal I of S, define $V(I) = \{J \in \wp \mid I \subseteq J\}$. As a special property of $V(I)$ we have

Theorem 3.2. *Any closed set in $\wp$ is of the form $V(I)$, for some ideal I of S.*

Proof :- Let A be any closed set in $\wp$. Then $\overline{A} = A$. Therefore $A = \{I \in \wp \mid \bigcap_{I_\alpha \in A} I_\alpha \subseteq I\}$.Define $I = \bigcap_{I_\alpha \in A} I_\alpha$. Then I is an ideal of S and $A = V(I)$. Now for any $J \in \overline{V(I)}$ implies $I \subseteq \bigcap_{I_\alpha \in A} I_\alpha \subseteq J$. Hence $J \in \overline{V(I)}$ gives $\bigcap_{I_\alpha \in V(I)} I_\alpha \subseteq J$. This implies $J \in V(I)$. $\overline{V(I)} \subseteq V(I)$. Thus $V(I) = \overline{V(I)}$. □

Remark 3.3. We define $U(I) = \wp \setminus V(I) = \{J \in \wp \mid I \nsubseteq J\}$. Similar to the Theorem3.2, we have $U(I)$ is an open set, where $U(I)$denotes the complement of $V(I)$ in $\wp$ and I is an ideal of S.

If I is an ideal of S generated by $a \in S$ that is $I = \langle a \rangle$. Then $V(I) = V(\langle a \rangle)$. Hence we define for any $a \in S$, $V(a) = \{J \in \wp \mid a \in J\}$ and $\wp \setminus V(a) = U(a) = \{J \in \wp \mid a \notin J\}$. Then we have the following results.

Theorem 3.4. *$\{U(a) \mid a \in S\}$ forms a base for open sets for the hull kernel topology τ on $\wp$ and the space is a T_0 space.*

Proof :- Let G be any open set in τ. Then by Remark3.3, we have $G = U(I)$, for some ideal I of S. For any $J \in G$ we have $I \nsubseteq J$. Select $a \in I$ such that $a \notin J$. Hence $J \in U(a)$. Let $K \in U(a)$. Then we have $a \notin K$. This gives that $I \nsubseteq K$. Therefore $K \in G$. Hence $U(a) \subseteq G$. Thus we get $J \in U(a) \subseteq G$. Then $G = \bigcup_{a \in G} U(a)$. Therefore $\{U(a) \mid a \in S\}$ forms an open base for the hull kernel topology τ on $\wp$. Let I and J be two distinct elements of $\wp$. Assume that $a \in I \setminus J$. But then $J \in U(a)$ and $I \notin U(a)$.
Therefore $(\wp, \tau)$ is a T_0 space. □

Theorem 3.5. *If S is a Γ-semiring with unity 1, then $(\wp, \tau)$ is a T_1 space if and only if every prime ideal of S is maximal.*

Proof :- Suppose that $(\wp, \tau)$ is a T_1space. Let $P \in \wp$ such that P is not maximal. Then there exists a maximal ideal M of S such that $P \subset M$. As $(\wp, \tau)$ is a T_1 space and $P \neq M$, there exist basic open sets $U(a)$ and $U(b)$ such that $P \in U(a)$, $M \notin U(a)$ and $P \notin U(b)$, $M \in U(b)$. As $b \in P$ we get $b \in M$ and hence $M \notin U(b)$; a contradiction. Hence every prime ideal of S is maximal. Conversely, suppose that every prime ideal of S is maximal. To show that structure $(\wp, \tau)$ is T_1. Let I and J be two distinct elements of $\wp$. Then by assumption either $I \nsubseteq J$ and $J \nsubseteq I$. This shows that there exist $a, b \in S$ such that $a \in I$, $b \in J$ but $a \notin J$,$b \notin I$. Then we have $I \in U(b)$, $J \in U(a)$ but $I \notin U(a)$, $J \notin U(b)$. Thus $(\wp, \tau)$ is a T_1space. □

Theorem 3.6. *$(\wp, \tau)$ is a Hausdorff space if and only if for any two distinct pair of elements I and J of $\wp$ there exists $a, b \in S$ such that $a \notin I$, $b \notin J$ and there does not exist any element K of $\wp$ such that $a \notin K$ and $b \notin K$.*

Proof :- Suppose that the structure space $(\wp,\ \tau)$ is a Hausdorff space. Then for any two distinct elements I and J of $\wp$ there exists two open sets $U\,(a)$ and $U\,(b)$ such that $I \in U\,(a)$, $J \in U\,(b)$ and $U\,(a) \cap U\,(b) = \emptyset$. But then $a \notin I$ and $b \notin J$. Let if possible there exist K in $\wp$ such that $a \notin K$ and $b \notin K$. Then $K \in U\,(a)$ and $K \in U\,(b)$ gives $K \in U\,(a) \cap U\,(b) = \emptyset$, which is a contradiction . Thus there does not exist any element K of $\wp$ such that $a \notin K$ and $b \notin K$. Conversely, suppose that given condition holds. To show the space $(\wp,\ \tau)$ is a Hausdorff space. Let I and J be two distinct elements of $\wp$. Then by assumption there exists $a, b \in S$ such that $a \notin I$, $b \notin J$. This gives $I \in U\,(a)$, $J \in U\,(b)$. Again by assumption there does not exist any element K of $\wp$ such that $a \notin K$ and $b \notin K$. Therefore there does not exist any element K of $\wp$ such that $K \in U\,(a) \cap U\,(b)$. Hence $U\,(a) \cap U\,(b) = \emptyset$. Therefore $(\wp,\ \tau)$ is a Hausdorff space. □

Every Hausdorff space being a T_1 space we get,

Corollary 3.7. *If $(\wp,\ \tau)$ is a Hausdorff space, then no prime ideal contains any other prime ideal.(OR If $(\wp,\ \tau)$ is a Hausdorff space, then prime ideal of S is a minimal prime ideal). In other words If $(\wp,\ \tau)$ is a Hausdorff space, then the set of all minimal prime ideals and maximal ideals coincide.*

Theorem 3.8. *If $(\wp,\ \tau)$ is a Hausdorff space containing more than one element, then there exist $a, b \in S$ such that $\wp = U\,(a) \cup U\,(b) \cup V\,(I)$, where I is an ideal generated by $a,\ b$ in S.*

Proof :- Suppose that $(\wp,\ \tau)$ is a Hausdorff space containing more than one element. Let J and K be any two elements of $\wp$ such that $J \neq K$. $J \neq K$ and $(\wp,\ \tau)$ is a Hausdorff space imply there exist two open sets say $U\,(a)$ and $U\,(b)$such that $J \in U\,(a)$, $K \in U\,(b)$ and $U\,(a) \cap U\,(b) = \emptyset$. Let I be the ideal generated by $a, b \in S$. Now for any $K \in \wp$, $a \notin K$, $b \notin K$. In this case $K \in U\,(a)$ and $K \in U\,(b)$that is $K \in U\,(a) \cap U\,(b)$, which is not possible as $U\,(a) \cap U\,(b) = \emptyset$. Hence either $a \in K$, $b \in K$ then $K \in U\,(a) \cup U\,(b) \cup V(I)$. Thus $K \in \wp$ implies $K \in U\,(a) \cup U\,(b) \cup V(I)$. Therefore $\wp \subseteq U\,(a) \cup U\,(b) \cup V(I)$. But $U\,(a) \cup U\,(b) \cup V(I) \subseteq \wp$. Hence $\wp = U\,(a) \cup U\,(b) \cup V(I)$. □

Theorem 3.9. *$(\wp,\ \tau)$ is a regular space if and only if for any $I \in \wp$ and $a \notin I$, for $a \in S$ there exist an ideal J of S and $b \in S$ such that $I \in U\,(b) \subseteq V(J) \subseteq U(a)$.*

Proof :- Suppose that structure space $(\wp,\ \tau)$ is a regular space. Let $I \in \wp$ and $a \notin I$, for $a \in S$. As $a \notin I$, we have $I \in U\,(a)$. $U\,(a)$ is an open set of $\wp$ implies $V\,(a) = \wp \backslash U\,(a)$ is a closed set of $\wp$ not containing I. As $(\wp,\ \tau)$ is a regular space, there exist two open sets say G and H such that $I \in G$, $\wp \backslash U\,(a) \subseteq H$ and $G \cap H = \emptyset$. $\wp \backslash U\,(a) \subseteq H$ gives $\wp \backslash H \subseteq U\,(a)$. H is an open set of $\wp$ implies $\wp \backslash H$ is a closed set. Therefore $\wp \backslash H = V(J)$ for some ideal J of S. $\wp \backslash G = V(K)$ for some ideal K in S (see Theorem3.2). Then we have $H \subseteq V(K)$. Since $I \in G$ that is $I \notin \wp \backslash G = V(K)$ implies $K \nsubseteq I$. $K \nsubseteq I$gives there exist $b \in K$. but $b \notin I$. As $b \notin I$ then $I \in U\,(b)$. Now to show that $H \subseteq V(b)$. Let $T \in H = V(K)$. Then $K \subseteq T$. But $b \in K$ gives $b \in T$, it follows that $T \in V\,(b)$. Therefore $H \subseteq V(b)$. Hence $\wp \backslash V\,(b) \subseteq \wp \backslash H = V(J)$. That is $U\,(b) \subseteq V(J)$. Thus we get for any $I \in \wp$ there exist an ideal J of S and $b \in S$ such that $I \in U\,(b) \subseteq V(J) \subseteq U(a)$.

Conversely, suppose that for any $I \in \wp$ and $a \notin I$, for $a \in S$ there exists an ideal J of S and, $b \in S$ such that $I \in U\,(b) \subseteq V(J) \subseteq U(a)$. To show the space $(\wp,\ \tau)$ is a regular space. Let $I \in \wp$ and $V(K)$ be any closed set of $\wp$ not containing I. $I \notin V(K)$ implies $K \nsubseteq I$. Therefore there exists $a \in K$ but $a \notin I$. This gives $I \in U\,(a)$. By the assumption there exist an ideal J of S and $b \in S$ such that $I \in U\,(b) \subseteq V(J) \subseteq U(a)$. $a \in K$ gives $K \in V\,(a)$. Thus we have $U\,(a) \cap V\,(K) = \emptyset$ then $V(K) \subseteq \wp \backslash U(a) \subseteq \wp \backslash V(J)$. As $V(J)$ is a closed set , we have $\wp \backslash V(J)$ is an open set of $\wp$ containing closed set $V(K)$. Hence $U(b) \subseteq V(J)$ implies $U\,(b) \cap (\wp \backslash V\,(J)) = \emptyset$.Thus there exist two disjoint open sets $U\,(b)$ and $(\wp \backslash V\,(J))$ such that $V(K) \subseteq \wp \backslash V(J)$ and $I \in U\,(b)$. Therefore the space $(\wp,\ \tau)$ is a regular space. □

The space $(\wp,\ \tau)$ is a T_0space (see Theorem3.4) and every regular T_0 space is a T_3 space. Hence we get

Corollary 3.10. *$(\wp,\ \tau)$ is a T_3 space if and only if for any $I \in \wp$ and $a \notin I$, for $a \in S$ there exist an ideal J of S and $b \in S$ such that $I \in U\,(b) \subseteq V(J) \subseteq U(a)$.*

We know that if S contains an unit element, then the structure space $(\wp,\ \tau)$ is a compact space. Otherwise we have

Theorem 3.11. *$(\wp,\ \tau)$ is a compact space if and only if for any collection $\{a_i\}_{i \in \Lambda} \subset S$ there exists a finite subcollection $\{a_1, a_2,\ a_3, \cdots\cdots, a_n\}$ in S such that $I \in \wp$ there exist a_i such that $a_i \notin I$.*

Proof :- Suppose that structure space $(\wp,\ \tau)$ is a compact space. Let $\{U(a_i)|a_i \in S\}$ be forms an open cover of $(\wp,\ \tau)$. Then this open cover has a finite subcover $\{U(a_i) \mid i = 1, 2, \cdots, n\}$. Let I be any element of $\wp$. Then $I \in \{U(a_i) \mid i = 1, 2, \cdots, n\}$. Therefore $I \in U(a_i)$ for some $a_i \in S$. Hence $a_i \notin I$. Thus $\{a_1, a_2,\ a_3, \cdots\cdots, a_n\}$ is the required finite subcollection of elements of S such that $a_i \notin I$. Conversely, suppose that given condition hold. To show the space $(\wp,\ \tau)$ is a compact space. Let $\{U(a_i)|a_i \in S\}$ be forms an open cover of $(\wp,\ \tau)$. Assume that no finite subcollection of $\{U(a_i)|a_i \in S\}$ be forms a cover of $\wp$. This shows that for any finite set $\{a_1, a_2,\ a_3, \cdots\cdots, a_n\}$ of elements of S, $U(a_1) \cup U(a_2) \cup \cdots\cdots \cup U(a_n) \neq \wp$. Therefore $\wp \backslash [U(a_1) \cup U(a_2) \cup \cdots\cdots \cup U(a_n)] \neq \emptyset$. Then $V(a_1) \cap V(a_2) \cap \cdots\cdots \cap V(a_n) \neq \emptyset$. This implies there exist $I \in \wp$ such that $I \in V(a_1) \cap V(a_2) \cap \cdots\cdots \cap V(a_n)$, gives that $a_1, a_2,\ a_3, \cdots\cdots, a_n \in I$. Which is a contradiction to the hypothesis. Hence our assumption $\{U(a_i)|a_i \in S\}$ has no finite subcover which covers $\wp$ is wrong. $\{U(a_i)|a_i \in S\}$ has finite subcover which covers $\wp$. Therefore the space $(\wp,\ \tau)$is a compact space. □

By the Theorem3.11 immediately we get

Corollary 3.12. *If S is finitely generated , then the space $(\wp,\ \tau)$ is compact.*

Arbitrary intersection of prime ideals is a semiprime ideal in S but need not be a prime ideal. In the following theorem we give a sufficient condition for intersection of prime ideals of S to be a prime ideal.

Theorem 3.13. *Let $\{P_i|i \in \Lambda\}$ be the collection of prime ideals of S such that $\{P_i|i \in \Lambda\}$ forms a chain of ideals. Then $\bigcap_{i\in\Lambda} P_i$ is a prime ideal of S.*

Proof :- Clearly $\bigcap_{i\in\Lambda} P_i$ is an ideal of S. Let A and B be any two ideals of S such that $A\Gamma B \subseteq \bigcap_{i\in\Lambda} P_i$. Assume that $A \nsubseteq \bigcap_{i\in\Lambda} P_i$ and $B \nsubseteq \bigcap_{i\in\Lambda} P_i$. Then there exist i and j such that $A \nsubseteq P_i$ and $B \nsubseteq P_j$. As $\{P_i|i \in \Lambda\}$ forms a chain of ideals, we have either $P_i \subseteq P_j$ or $P_j \subseteq P_i$. Assume $P_j \subseteq P_i$. Then $A \nsubseteq P_j$. $A\Gamma B \subseteq \bigcap_{i\in\Lambda} P_i \subseteq P_j$ and P_j is a prime ideal of S imply $A \subseteq P_j$ or $B \subseteq P_j$, which is a contradiction. Therefore either $A \subseteq \bigcap_{i\in\Lambda} P_i$ or $B \subseteq \bigcap_{i\in\Lambda} P_i$. Hence $\bigcap_{i\in\Lambda} P_i$ is a prime ideal of S. □

As in [4] for Γ-semigroup we define

Definition 3.14. The space $(\wp,\ \tau)$ is called irreducible if for any decomposition $\wp = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are closed subsets of $\wp$, then either $\wp = \mathcal{A}$ or $\wp = \mathcal{B}$.

Theorem 3.15. *Let $\mathcal{A}$ be a closed subset of $\wp$. Then $\mathcal{A}$ is irreducible if and only if $\bigcap_{P_i\in\mathcal{A}} P_i$ is a prime ideal of S.*

Proof :- Assume that $\mathcal{A}$ is irreducible. To Prove that $\bigcap_{P_i\in\mathcal{A}} P_i$ is a prime ideal of S. Let B and C be any two ideals of S such that $B\Gamma C \subseteq \bigcap_{P_i\in\mathcal{A}} P_i$. Then $B\Gamma C \subseteq P_i$, for each i. As P_i is a prime ideal of S, we have $B \subseteq P_i$or $C \subseteq P_i$ for each i. Then $P_i \in \mathcal{A} \cap \overline{\mathcal{B}}$ or $P_i \in \mathcal{A} \cap \overline{\mathcal{C}}$ give $P_i \in (\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\mathcal{A} \cap \overline{\mathcal{C}})$. Therefore $\mathcal{A} = (\mathcal{A} \cap \overline{\mathcal{B}}) \cup (\mathcal{A} \cap \overline{\mathcal{C}})$. $(\mathcal{A} \cap \overline{\mathcal{B}})$ and $(\mathcal{A} \cap \overline{\mathcal{C}})$ are closed subsets of $\mathcal{A}$ and $\mathcal{A}$ is irreducible imply $\mathcal{A} = (\mathcal{A} \cap \overline{\mathcal{B}})$ or $\mathcal{A} = (\mathcal{A} \cap \overline{\mathcal{C}})$. Hence $\mathcal{A} \subseteq \overline{\mathcal{B}}$ or $\mathcal{A} \subseteq \overline{\mathcal{C}}$. This shows that $B \subseteq \bigcap_{P_i\in\mathcal{A}} P_i$ or $C \subseteq \bigcap_{P_i\in\mathcal{A}} P_i$. Therefore $\bigcap_{P_i\in\mathcal{A}} P_i$ is a prime ideal of S. Conversely, suppose that $\bigcap_{P_i\in\mathcal{A}} P_i$ is a prime ideal of S. To show that $\mathcal{A}$ is irreducible. Let $\mathcal{B}$ and $\mathcal{C}$ are closed subsets of $\mathcal{A}$ such that $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$. Clearly $\bigcap_{P_i\in\mathcal{A}} P_i \subseteq \bigcap_{P_i\in\mathcal{B}} P_i$ and $\bigcap_{P_i\in\mathcal{A}} P_i \subseteq \bigcap_{P_i\in\mathcal{C}} P_i$. Also $\bigcap_{P_i\in\mathcal{A}} P_i = \bigcap_{P_i\in\mathcal{B}} P_i = (\bigcap_{P_i\in\mathcal{B}\cup\mathcal{C}} P_i) \cap (\bigcap_{P_i\in\mathcal{C}} P_i)$. As $\bigcap_{P_i\in\mathcal{B}} P_i$ and $\bigcap_{P_i\in\mathcal{C}} P_i$ are ideals of S, we have
$(\bigcap_{P_i\in\mathcal{B}} P_i)\Gamma(\bigcap_{P_i\in\mathcal{C}} P_i) \subseteq \bigcap_{P_i\in\mathcal{B}} P_i$ and $(\bigcap_{P_i\in\mathcal{B}} P_i)\Gamma(\bigcap_{P_i\in\mathcal{C}} P_i) \subseteq \bigcap_{P_i\in\mathcal{C}} P_i$.
Therefore $(\bigcap_{P_i\in\mathcal{B}} P_i)\ \Gamma\, (\bigcap_{P_i\in\mathcal{C}} P_i) \subseteq (\bigcap_{P_i\in\mathcal{B}} P_i) \cap (\bigcap_{P_i\in\mathcal{C}} P_i) = \bigcap_{P_i\in\mathcal{A}} P_i$. But $\bigcap_{P_i\in\mathcal{A}} P_i$ is a prime ideal of S. Then we have $\bigcap_{P_i\in\mathcal{B}} P_i \subseteq \bigcap_{P_i\in\mathcal{A}} P_i$ or $\bigcap_{P_i\in\mathcal{C}} P_i \subseteq \bigcap_{P_i\in\mathcal{A}} P_i$. Therefore $\bigcap_{P_i\in\mathcal{B}} P_i = \bigcap_{P_i\in\mathcal{A}} P_i$ or $\bigcap_{P_i\in\mathcal{C}} P_i = \bigcap_{P_i\in\mathcal{A}} P_i$. Now for any $P_k \in \mathcal{A}$, $\bigcap_{P_i\in\mathcal{B}} P_i = \bigcap_{P_i\in\mathcal{A}} P_i \subseteq P_k$ or $\bigcap_{P_i\in\mathcal{C}} P_i = \bigcap_{P_i\in\mathcal{A}} P_i \subseteq P_k$. As $\mathcal{B}$ and $\mathcal{C}$ are closed subsets of $\mathcal{A}$, we have $P_i \subseteq P_k$, for all $P_i \in \mathcal{B}$ or $P_i \subseteq P_k$, for all $P_i \in \mathcal{C}$. Therefore $\mathcal{A} \subseteq \mathcal{B}$ or$\mathcal{A} \subseteq \mathcal{C}$. Thus $\mathcal{A} = \mathcal{B}$ or$\mathcal{A} = \mathcal{C}$. Hence $\mathcal{A}$ is irreducible. □

For any subset $\mathcal{A}$ of $\wp$ we define $r\,(\mathcal{A}) = \bigcap_{I_k\in\wp} I_k$. Obviously $r\,(\wp) = \bigcap_{I_k\in\wp} I_k$ is the $\wp$-radical of S. Always $r\,(\wp) \subseteq r(\ A)$. We know that $\mathcal{A} \subseteq \wp$ is dense in $\wp$ if $\overline{\mathcal{A}} = \wp$. We characterise dense sets in $\wp$ as follows

Theorem 3.16. *The subset $\mathcal{A}$ of $\wp$ is dense in $\wp$ if and only if $r\,(\mathcal{A}) = r(\ \wp)$.*

Proof :- Assume that the subset $\mathcal{A}$ of $\wp$ is dense in $\wp$. As $\mathcal{A} \subseteq \wp$, we have $r\,(\wp)\ \subseteq r(\mathcal{A})$. Only to show that $r(\mathcal{A}) \subseteq r(\wp)$. $\overline{\mathcal{A}} = \wp$ gives $\overline{\mathcal{A}} = \{I \in \wp \mid \bigcap_{I_\alpha\in A} I_\alpha \subseteq I\} = \wp$. $P \in \wp$ implies $P \in \overline{\mathcal{A}}$.

Then $r(\mathcal{A}) \subseteq P$. As this true for each $P \in \wp$ we get $r(\mathcal{A}) = \bigcap_{I_\alpha \in A} I_\alpha \subseteq \bigcap_{I_\alpha \in \wp} I_\alpha = r(\wp)$. Hence $r(\mathcal{A}) = r(\wp)$. Conversely assume that $r(\mathcal{A}) = r(\wp)$. To show $\overline{\mathcal{A}} = \wp$. Suppose that $\wp \backslash \overline{\mathcal{A}} \neq \emptyset$. Then there is a prime ideal say P of S such that $P \in \wp \backslash \overline{\mathcal{A}}$ that is $P \in \wp$ and $P \in \widetilde{\overline{\mathcal{A}}}$ i.e. $P \notin \overline{\mathcal{A}}$. $P \notin \overline{\mathcal{A}}$ implies there exists any open set say $U(I)$ containing P such that $U(I) \cap \overline{\mathcal{A}} \backslash \{P\} = \emptyset$. That is open set of $\wp$ containing P does not contains any other element of $\mathcal{A}$ other than P. Therefore $r(\wp) = \bigcap_{I_\alpha \in \wp} I_\alpha \subset r(\mathcal{A}) = \bigcap_{I_\alpha \in A} I_\alpha$. Then $r(\mathcal{A}) \neq r(\wp)$. Hence by contrapositive method result holds. $\square$

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