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On Generalized Semiradical Formula

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Abstract. Some properties of semiprime submodules are given. Semiradical of a module and the generalized semiradical formula are defined. Then we showed that Noetherian modules satisfy the generalized semiradical formula.

1 Introduction

Throughout all rings are commutative and all modules are unitary. Let R be a ring and M be an R-module. A proper submodule N of M is prime if whenever $rm \in N$, for some $r \in R, m \in M$ then $m \in N$ or $rM \subseteq N$. A proper submodule N of an R-module M is semiprime, if whenever $r^k m \in N$ for some $r \in R, m \in M$ and $k \in \mathbb{Z}^+$, then $rm \in N$. The envelope of N in M is defined as the set

$$E_M(N) = \{rm : r \in R, m \in M \quad and \quad r^k m \in N \quad for \quad some \quad k \in \mathbb{Z}^+\}.$$

Semiprime submodules can be defined in terms of their envelopes, that is, a proper submodule N is semiprime if and only if $\langle E_M(N)\rangle = N$. This characterization can be used to show that semiprime submodules need not be prime despite the fact that every prime submodule is semiprime. For example, if $R = \mathbb{Q}[x, y, z]$, $M = R^3$ and $N = \langle z\mathbf{e}_1, y\mathbf{e}_1, xy\mathbf{e}_2, xy\mathbf{e}_3, xz\mathbf{e}_2 + x^2z\mathbf{e}_3\rangle$, then by [6] Theorem 2.5, $\langle E_M(N)\rangle = N$. Hence N is a semiprime submodule of M with $N : M = \langle xy\rangle$. On the other hand N is not a prime submodule; if we take $r = z$ and $m = (0, x, x^2)$, then $rm = z(0, x, x^2) = (0, xz, x^2z) \in N$ but $r = z \notin N : M$ and $m = (0, x, x^2) \notin N$.

If N is a proper submodule of an R-module M, then the prime radical of N, $rad_M(N)$, is the intersection of all prime submodules containing N. The semiradical of N, denoted by $srad_M(N)$, is defined as the intersection of all semiprime submodules of M containing N. If there is no semiprime submodule containing N, then $srad_M(N) = M$. We shall denote the semiradical of M by $srad_M(0)$. Since $rad_M(N)$ is semiprime, we have

$$N \subseteq \langle E_M(N)\rangle \subseteq srad_M(N) \subseteq rad_M(N).$$

In section 2, we study some properties of semiprime submodules. In section 3, semiprime radical is defined and it is shown that for domains, the study of semiprime radical of any modules reduces to torsion modules. Also the equality $srad_M(N) = \langle E_M(N)\rangle$ is investigated for some special cases. In section 4, we define generalized semiradical formula and showed that Noetherian modules satisfy the generalized semiradical formula.

2 Semiprime Submodules

If N is a prime submodule of an R-module M, then it is well known that $N : M$ is a prime ideal. If N is semiprime, we have the following.

Lemma 2.1. If N is a semiprime submodule of an R-module M, then $N : M$ is a semiprime ideal.

Proof. Let $x \in \sqrt{N : M}$. Then $x^k M \subseteq N$ for some $k \in \mathbb{Z}^+$. Since N is semiprime, $xM \subseteq N$. Hence $\sqrt{N : M} = N : M$. This implies that $N : M$ is a radical ideal which means that $N : M$ is a semiprime ideal. □

Lemma 2.2. Let N be a primary submodule. Then N is semiprime submodule iff $N : M$ is a semiprime ideal.

Proof. Suppose $N : M$ is semiprime ideal. Let $r^k m \in N$ where $r \in R, m \in M - N$ and $k \in \mathbb{Z}^+$. Since N is primary and $N : M$ is semiprime, $r \in \sqrt{N : M} = N : M$. Hence $rm \in N$. Otherside is clear by the above lemma. □

The following lemma shows that a semiprime submodule is prime if it is primary.

Lemma 2.3. Let M be an R-module and N be a proper submodule of M. Then N is prime submodule of M if and only if N is primary and semiprime.

Proof. Assume that N is primary and semiprime. Let $am \in N$ for $a \in R, m \in M$. Since N is primary, either $m \in N$ or $a \in \sqrt{N : M}$. By Lemma 2.1, $m \in N$ or $a \in N : M$. Hence N is prime submodule. The converse is clear. □

We also have the followings.

Lemma 2.4. Let M be an R-module. Assume that N and K are submodules of M such that $K \subseteq N$ with $N \neq M$. Then, if K and N/K are semiprime submodules then N is also semiprime.

Proof. Let $r^t m \in N$ for $r \in R, m \in M$ and $t \in \mathbb{Z}^+$. Then $r^t(m + K) = r^t m + K \in N/K$. If $r^t m \in K$, then $rm \in K \subseteq N$ since K is semiprime. Now, we may assume that $r^t m \notin K$. Then $r^t(m + K) \in N/K$ and N/K is semiprime implies that $r(m + K) = rm + K \in N/K$. Hence $rm \in N$. □

Lemma 2.5. Let $M = K \oplus L$ be the direct sum of submodules K, L and N be semiprime submodule of K. Then $N \oplus L$ is a semiprime submodule of M.

Proof. Let $r \in R, m \in M$ and $r^t m \in N \oplus L$ for some $t \in \mathbb{Z}^+$. Then there exist elements $n \in N, l \in L$ such that $r^t m = n + l$. Since $M = K \oplus L$, there exists an element $k \in K$ such that $r^t k \in N$. Since N is semiprime, $rk \in N$. Hence $rm \in N \oplus L$. □

If we replace the term prime with semiprime in [4], then we get the following which is straightforward.

Lemma 2.6. Let M, M' be R-modules with $\phi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $Ker\phi \subseteq N$. Then

(i) If P is a semiprime submodule of M containing N, then $\phi(P)$ is a semiprime submodule of M' containing $\phi(N)$.

(ii) If P' is a semiprime submodule of M' containing $\phi(N)$, then $\phi^{-1}(P')$ is a semiprime submodule of M containing N.

Let K and N be any submodules of an R-module M where $N \subseteq K$. If we consider the canonical epimorphism $\phi : M \to M/N$, then by Lemma 2.6 it is clear that K is a semiprime submodule of M if and only if K/N is semiprime submodule of M/N.

3 Semiprime Radical

Let N be a submodule of an R-module. The semiradical of N, $srad_M(N)$, is the intersection of all semiprime submodules of M containing N. Since intersection of semiprime submodules is semiprime, $srad_M(N)$ is semiprime. Hence $srad_M(N)$ is the smallest semiprime submodule of M containing N. In this section we will give generalization of [1] and [3] to semiprime radical. The following two lemmas are generalization of [3] Lemma 4 and Lemma 6.

Lemma 3.1. Let R be a ring, M be an R-module and N, K be submodules of M with $K \subseteq N$. Then $srad_N(K) \subseteq srad_M(K)$.

Proof. Let P be any semiprime submodule of M with $K \subseteq P$. If $N \subseteq P$, then $srad_N(K) \subseteq P$. If $N \nsubseteq P$, then $N \cap P$ is a semiprime submodule of N. Hence $srad_N(K) \subseteq N \cap P \subseteq P$. Thus in any case $srad_N(K) \subseteq srad_M(K)$. □

Lemma 3.2. Let M be the direct sum of the R-modules M_i, $i \in I$. Let $N = \oplus N_i$ be a submodule of M such that N_i is a submodule of M_i for all $i \in I$. Then $srad_M(N) = \oplus srad_{M_i}(N_i)$.

Proof. By Lemma 3.1, $srad_{M_i}(N_i) \subseteq srad_M(N)$ for all $i \in I$. Let $m \in srad_M(N)$ and $m \notin \bigoplus_i srad_{M_i}(N_i)$. Then there exists $j \in I$ such that $\pi_j(m) \notin srad_{M_j}(N_j)$ where $\pi_j : M \to M_j$ denotes the canonical projection. There exists a semiprime submodule P_j of M_j such that $\pi_j(m) \notin P_j$. By Lemma 2.5, $K = P_j \bigoplus(\bigoplus_{i \neq j} M_i)$ is semiprime submodule of M containing N. Since $\pi_j(m) \notin P_j$, $m \notin K$. Then $m \notin srad_M(N)$. Therefore $srad_M(N) = \bigoplus_i srad_{M_i}(N_i)$. □

Since every prime submodule is semiprime, $T(M)$ and PM are semiprime submodules of a module M over a domain where P is maximal ideal of R, [3].

The general form of [1], Proposition 1.3 is

Proposition 3.3. Let R be a domain and M be an R-module with torsion submodule $T(M)$. If N is a submodule of $T(M)$, then N is semiprime submodule of $T(M)$ if and only if N is semiprime submodule of M.

Proof. Suppose N is semiprime submodule of $T(M)$. Let $0 \neq r \in R, m \in M$ with $r^k m \in N$ for some $k \in \mathbb{Z}^+$. Since $T(M)$ is semiprime submodule of M, $rm \in T(M)$. Then there exists nonzero $s \in R$ such that $s(rm) = 0$. Since $sr \neq 0$, we have $m \in T(M)$ which implies that $rm \in N$. Thus, N is a semiprime submodule of M. The converse is clear. □

Now we can show that for domains, the study of semiprime radicals of any modules reduces to torsion modules.

Corollary 3.4. Let R be a domain and M be an R-module with torsion submodule $T(M)$. Then $srad_M(0) = srad_{T(M)}(0)$.

Proof. Since $T(M)$ is a submodule of M, by Lemma 3.1 $srad_{T(M)}(0) \subseteq srad_M(0)$. Now, suppose $srad_{T(M)}(0) = \bigcap N$ where N is a semiprime submodule of $T(M)$. By Proposition 3.3, N is also semiprime submodule of M. Hence $srad_M(0) \subseteq srad_{T(M)}(0)$. □

We also have the following corollary which is the generalization of [1], Lemma 1.7.

Corollary 3.5. Let R be a domain and M be a left R-module with torsion submodule $T(M)$. Then

$$srad_M(0) \subseteq \bigcap\{PT(M) : P \;\; is \;\; a \;\; maximal \;\; ideal \;\; of \;\; R\}.$$

Proof. By Corollary 3.4. □

Note that any submodule N of a module M satisfies the radical formula (s.t.r.f) if $rad_M(N) = \langle E_M(N) \rangle$. It is said that M satisfies the radical formula if for every submodule N of M, $rad_M(N) = \langle E_M(N) \rangle$. A ring R satisfies the radical formula, if every R-module s.t.r.f.. Modules which satisfy the radical formula was studied in [2], [3] and [4]. In the same manner, we say that M satisfies the semiradical formula (s.t.s.r.f.) if for any submodule N of M, $srad_M(N) = \langle E_M(N) \rangle$.

It is well-known that for an ideal I of R, $\sqrt{\sqrt{I}} = \sqrt{I}$; but the envelope of a submodule does not satisfy an equation similiar to this one as the following example shows.

Example 3.6. Let $R = \mathbb{Q}[x, y, z]$ and let M be an R-submodule $R \oplus R$. Consider the submodule $N = \langle z^2\mathbf{e}_1, z^2\mathbf{e}_2, yz\mathbf{e}_2, y^2\mathbf{e}_1 + z\mathbf{e}_2, y^2\mathbf{e}_2, y\mathbf{e}_1 + x^3\mathbf{e}_2 \rangle$. N is $p = \langle z, y \rangle$-primary, so by [6] Theorem 2.5,

$$\langle E_M(N) \rangle = N + \langle z, y \rangle M = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_2 \rangle.$$

Primary decompostion is $\langle E_M(N) \rangle = Q_1 \cap Q_2$ where

$$Q_1 = \langle \mathbf{e}_2, z\mathbf{e}_1, y\mathbf{e}_1 \rangle \text{ is } \langle z, y \rangle - \text{primary},$$

$$Q_2 = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x^3\mathbf{e}_1, x^3\mathbf{e}_2 \rangle \text{ is } \langle x, y, z \rangle - \text{primary}.$$

Hence,

$$\langle E_M(\langle E_M(N) \rangle) \rangle = \langle z\mathbf{e}_1, z\mathbf{e}_2, y\mathbf{e}_1, y\mathbf{e}_2, x\mathbf{e}_2 \rangle \neq \langle E_M(N) \rangle.$$

In [2], Azizi and Nikseresht defined the kth envelope of N recursively by $E_0(N) = N, E_1(N) = E_M(N), E_2(N) = E_M(\langle E_M(N)\rangle)$ and $E_k(N) = E_M(\langle E_{k-1}(N)\rangle)$ for every submodule N of M. It is easy to show that

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle \subseteq srad_M(N) \subseteq rad_M(N)$$

where $\langle E_\infty(N)\rangle = \bigcup\limits_{k=0}^{\infty} \langle E_k(N)\rangle$.

It is clear that, $\langle E_\infty(N)\rangle$ is semiprime and thus $\langle E_\infty(N)\rangle = srad_M(N)$. Therefore we have the following equivalent conditions.

Theorem 3.7. The followings are equivalent.

(i) A module M satisfies the semiradical formula;

(ii) $\langle E_i(N)\rangle = \langle E_j(N)\rangle$ for all i, j;

(iii) $\langle E_M(N)\rangle = \langle E_2(N)\rangle$ for all submodules N of M.

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are clear.

$(iii) \Rightarrow (i)$ $\langle E_M(N)\rangle = \langle E_2(N)\rangle$ implies that $\langle E_M(N)\rangle$ is semiprime submodule and hence M s.t.s.r.f.

□

By Theorem 3.7, we can conclude that a module M s.t.s.r.f. if and only if $\langle E_M(N)\rangle$ is either M or a semiprime submodule of M for every submodule N of M; R s.t.s.r.f. if and only if either $\langle E_M(0)\rangle = M$ or $\langle E_M(0)\rangle$ is semiprime submodule of M for every non-zero R-module M. Now, we will investigate the equality $srad_M(N) = \langle E_M(N)\rangle$.

Lemma 3.8. Let M, M' be R-modules with $\phi : M \to M'$ an R-module epimorphism and N be a submodule of M such that $Ker\phi \subseteq N$. Then $\phi(srad_M(N)) = srad_{M'}(\phi(N))$.

Lemma 3.9. Let N be a submodule of a module M. Then $srad_{M/N}(0) = srad_M(N)/N$.

Proof. Consider the canonical epimorphism $\pi : M \to M/N$. Since $Ker\pi = N$, we can apply Lemma 3.8. Then $\pi(srad_M(N)) = srad_{M/N}(\pi(N)) = srad_{M/N}(0)$. Let $srad_M(N) = \bigcap\limits_i P_i$. Then we have;

$$srad_{M/N}(0) = \phi(\bigcap_i P_i) = \bigcap_i (P_i/N) = (\bigcap_i P_i)/N = srad_M(N)/N.$$

□

Corollary 3.10. Let N be a submodule of a module M and N' be a submodule of a module M' such that $M/N \cong M'/N'$. Then $srad_M(N) = \langle E_M(N)\rangle$ if and only if $srad_{M'}(N') = \langle E_{M'}(N')\rangle$.

Proof. It is clear by the definition of envelope that $\langle E_{M/N}(0)\rangle = \langle E_M(N)\rangle/N$, also by Lemma 3.9, we have

$$\begin{aligned} srad_M(N) = \langle E_M(N)\rangle &\Leftrightarrow srad_M(N)/N = \langle E_M(N)\rangle/N \\ &\Leftrightarrow srad_{M/N}(0) = \langle E_{M/N}(0)\rangle \\ &\Leftrightarrow srad_{M'/N'}(0) = \langle E_{M'/N'}(0)\rangle \\ &\Leftrightarrow srad_{M'}(N')/N' = \langle E_{M'}(N')\rangle/N' \\ &\Leftrightarrow srad_{M'}(N') = \langle E_{M'}(N'). \end{aligned}$$

□

Corollary 3.11. Let N, L be submodules of M such that $M = N + L$ and $srad_L(N \cap L) = \langle E_L(N \cap L)\rangle$. Then $srad_M(N) = \langle E_M(N)\rangle$.

Proof. Note that $M/N = (N + L)/N \cong L/N \cap L$. Apply Corollary 3.10. □

Lemma 3.12. Let M be the direct sum of the R-modules M_i, $i \in I$. Let $N = \oplus N_i$ be a submodule of M such that N_i is a submodule of M_i for all $i \in I$. Then $srad_M(N) = \langle E_M(N)\rangle$ if and only if $srad_{M_i}(N_i) = \langle E_{M_i}(N_i)\rangle$ for each i.

Proof. Assume $srad_{M_i}(N_i) = \langle E_{M_i}(N_i)\rangle$ for each i. By Lemma 3.2,

$$srad_M(N) = \bigoplus_{i\in I} srad_{M_i}(N_i) = \bigoplus_{i\in I}\langle E_{M_i}(N_i)\rangle.$$

The result follows by [8], Lemma 2.3.

Conversely, since N_i is a submodule of M_i and $N = \oplus N_i$, by Lemma 3.2

$$srad_{M_i}(N_i) \subseteq srad_M(N_i) \subseteq srad_M(N) = \langle E_M(N)\rangle.$$

If $m \in srad_{M_i}(N_i)$, then by the definition of envelope it is easy to show that $m \in \langle E_{M_i}(N_i)\rangle$. □

By Lemma 3.2 and the above lemma, we have the following result.

Corollary 3.13. Let R be any ring and M be any projective R-module. Then $srad_M(0) = \langle E_M(0)\rangle$.

Proof. Since M is projective, there exists a free R-module F such that M is the direct summand of F. Then there exist an index set Λ and cyclic submodules F_λ of F such that $F = \bigoplus_\Lambda F_\lambda$ where $\lambda \in \Lambda$ by [7]. By Lemma 3.2, $srad_F(0) = \bigoplus_\Lambda srad_{F_\lambda}(0)$ and since every cyclic module s.t.r.f., $srad_{F_\lambda}(0) = \langle E_{F_\lambda}(0)\rangle$ for all $\lambda \in \Lambda$. Hence $srad_F(0) = \langle E_F(0)\rangle$ by Lemma 3.12 and thus $srad_M(0) = \langle E_M(0)\rangle$. □

Since every prime submodule is semiprime, this result can also be obtained from [3], Corollary 8.

4 Generalized Semiradical Formula

Suppose that N is a submodule of an R-module M. We say that N satisfies the generalized semiradical formula (s.t.g.s.r.f.) in M if

(i) $N = srad_M(N)$ or

(ii) there exists a submodule L of M such that $N \subseteq L \subsetneq srad_M(N)$ and $srad_M(N) = \langle E_M(L)\rangle$.

A module M s.t.g.s.r.f. if every submodule of M s.t.g.s.r.f. in M. A ring R s.t.g.s.r.f. as a ring if every R-module s.t.g.s.r.f. Clearly a module s.t.s.r.f. implies that s.t.g.s.r.f.

Proposition 4.1. If M s.t.g.s.r.f., then every homomorphic image of M s.t.g.s.r.f.

Proof. Assume M s.t.g.s.r.f. Let N be any submodule of M and K be any submodule of M containing N. If K/N is semiprime submodule of M/N, then $srad_M(K) = K$. So,

$$srad_{M/N}(K/N) = srad_M(K)/N = K/N.$$

Now, assume K/N is not semiprime submodule of M. So, K is not semiprime submodule of M. Since M s.t.g.s.r.f., there exists a submodule L of M with $K \subseteq L \subsetneq srad_M(K)$ and $srad_M(K) = \langle E_M(L)\rangle$. Then we have

$$\begin{aligned} srad_{M/N}(K/N) &= srad_M(K)/N \\ &= \langle E_M(L)\rangle/N \\ &= \langle E_{M/N}(L/N)\rangle. \end{aligned}$$

where $K/N \subseteq L/N \subsetneq srad_{M/N}(K/N)$. Therefore M/N s.t.g.s.r.f. □

Theorem 4.2. If M is Noetherian, then M s.t.g.s.r.f.

Proof. Let N be any submodule of M. Then we have the chain

$$N = \langle E_0(N)\rangle \subseteq \langle E_1(N)\rangle \subseteq \langle E_2(N)\rangle \subseteq \cdots\cdots \subseteq \langle E_\infty(N)\rangle = srad_M(N) \subseteq rad_M(N).$$

Since M is Noetherian, there exists a natural number k such that the chain terminates, that is $\langle E_k(N)\rangle = \langle E_{k+1}(N)\rangle$. Hence $\langle E_k(N)\rangle$ is a semiprime submodule and $srad_M(N) = \langle E_k(N)\rangle$.

We may assume that k is the smallest such number. If $k = 0$, then $srad_M(N) = N$. If $k \geq 1$, then $\langle E_k(N)\rangle = \langle E_M(\langle E_{k-1}(N)\rangle)\rangle$. By the minimality of k, we have $N \subseteq \langle E_{k-1}(N)\rangle \subsetneq srad_M(N)$ and $srad_M(N) = \langle E_M(\langle E_{k-1}(N)\rangle)\rangle$. Therefore N s.t.g.s.r.f. in M. □

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