A note on difference cordial graphs

R. Ponraj, S. Sathish Narayanan and R. Kala

Communicated by Ayman Badawi

MSC 2010 Classifications: 05C78.

Keywords and phrases: Lotus inside a circle, Pyramid, Permutation graphs, t-fold wheel, corona.

Abstract Let $G$ be a $(p, q)$ graph. Let $f : V (G) \rightarrow \{1, 2, \ldots, p\}$ be a function. For each edge $uv$, assign the label $|f(u) - f(v)|$. $f$ is called a difference cordial labeling if $f$ is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph which admits a difference cordial labeling is called a difference cordial graph. In this paper we investigate the difference cordial labeling behaviour of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web and some more standard graphs have been investigated. In this paper, R. Ponraj, S. Sathish Narayanan and R. Kala introduced difference cordial labeling in [6]. In [6, 7, 8, 9] difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web and some more standard graphs have been investigated. In this paper, we investigate the difference cordial labeling behavior of $K_2 + mK_1$, $K_n^c + 2K_2$, Sunflower graph, Lotus inside a circle, Pyramid, Permutation graphs, book with $n$ pentagonal pages, t-fold wheel, double fan. Let $\alpha$ be any real number. Then the symbol $|x|$ stands for the largest integer less than or equal to $x$ and $\lfloor x \rfloor$ stands for the smallest integer greater than or equal to $x$. Terms and definitions not defined here are used in the sense of Harary [3].

1 Introduction

Let $G = (V, E)$ be $(p, q)$ graph. In this paper we have considered only simple and undirected graphs. The number of vertices of $G$ is called the order of $G$ and the number of edges of $G$ is called the size $G$. Labeled graphs are used in several areas of science and technology such as astronomy, radar, circuit design and database management[2]. The origin of graph labeling is Graceful labeling which was introduced by Rosa [11] in the year 1967. In 1980, Cahit [1] introduced the cordial labeling of graphs. Cordiality behavior of numerous graphs were studied by several authors [4, 12, 14, 20, 5, 15, 16, 17, 18, 19, 13]. In this approach, R. Ponraj, S. Sathish Narayanan and R. Kala introduced difference cordial labeling in [6]. In [6, 7, 8, 9] difference cordial labeling behavior of several graphs like path, cycle, complete graph, complete bipartite graph, bistar, wheel, web and some more standard graphs have been investigated. In this paper, we investigate the difference cordial labeling behavior of $K_2 + mK_1$, $K_n^c + 2K_2$, Sunflower graph, Lotus inside a circle, Pyramid, Permutation graphs, book with $n$ pentagonal pages, t-fold wheel, double fan. Let $\alpha$ be any real number. Then the symbol $|x|$ stands for the largest integer less than or equal to $x$ and $\lfloor x \rfloor$ stands for the smallest integer greater than or equal to $x$. Terms and definitions not defined here are used in the sense of Harary [3].

2 Difference cordial labeling

Definition 2.1. Let $G$ be a $(p, q)$ graph. Let $f$ be a map from $V (G)$ to $\{1, 2, \ldots, p\}$. For each edge $uv$, assign the label $|f(u) - f(v)|$. $f$ is called difference cordial labeling if $f$ is a one to one map and $|e_f(0) - e_f(1)| \leq 1$ where $e_f(1)$ and $e_f(0)$ denote the number of edges labeled with 1 and not labeled with 1 respectively. A graph with a difference cordial labeling is called a difference cordial graph.

The following results (Theorem 2.1 to 2.9) are used in the subsequent section.

Theorem 2.2. [6] Any Path is a difference cordial graph.

Theorem 2.3. [6] Any Cycle is a difference cordial graph.

Theorem 2.4. [6] If $G$ is a $(p, q)$ difference cordial graph, then $q \leq 2p - 1$.

Theorem 2.5. [6] $K_n$ is difference cordial iff $n \leq 4$.

Theorem 2.6. [6] $K_{2,n}$ is difference cordial iff $n \leq 4$.

Theorem 2.7. [6] $K_{3,n}$ is difference cordial iff $n \leq 4$.

Theorem 2.8. [6] The wheel $W_n$ is difference cordial.

Theorem 2.9. [10] The ladder $L_n$ is difference cordial.

Theorem 2.10. [10] The prism $C_n \times P_2$ is difference cordial.
The join of two graphs $G_1$ and $G_2$ is denoted by $G_1 + G_2$ and whose vertex set is $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$

**Theorem 2.11.** Let $G$ be a $(p, q)$ graph. If $q > p + 1$, then $G + K_1$ is not difference cordial.

**Proof.** Obviously, $G + K_1$ is a $(p + 1, p + q)$ graph. Suppose $G + K_1$ is difference cordial, then by theorem 2.4, $p + q \leq 2(p + 1) - 1$. This implies $q \leq p + 1$, a contradiction. $$

**Theorem 2.12.** Let $G_1$ be a $(p_1, q_1)$ connected graph and $G_2$ be a $(p_2, q_2)$ connected graph with $G_1 \neq K_1$ and $p_2 > 3$ then $G_1 + G_2$ is not difference cordial.

**Proof.** The order and size of $G_1 + G_2$ are $p_1 + p_2$ and $q_1 + q_2 + p_1p_2$ respectively. Suppose $G_1 + G_2$ is difference cordial with $G_1 \neq K_1$ and $p_2 > 3$, then by theorem 2.4, $q_1 + q_2 + p_1p_2 \leq 2(p_1 + p_2) - 1 \Rightarrow p_1p_2 - p_1 - 1 \leq 2$, $p_1 > 2(p_1 - 1) - p_1 + 1 > 3(p_1 - 1) - p_1 + 1 > 2$, a contradiction. $$

**Theorem 2.13.** Let $G$ be a $(p, q)$ graph. Then $G + G$ is difference cordial iff $p \leq 3$ and $q \leq 1$.

**Proof.** The number of vertices and edges in $G + G$ are $2p$ and $2q + p^2$ respectively. Suppose $f$ is a difference cordial labeling of $G + G$, then by theorem 2.4, $2q + p^2 \leq 2(2p) - 1$. This implies $p \leq 3$. It follows that $q \leq 1$. When $p = 3$ and $q = 1$, the difference cordial labeling of $G + G$ is shown in figure 1.

![Figure 1](image1)

When $p = 3$ and $q = 0$, $G + G \cong K_{3, 3}$ which is difference cordial by theorem 2.7. When $p = 2$ and $q = 1$, $G + G \cong K_4$ which is difference cordial by theorem 2.5. When $p = 2$ and $q = 0$, $G + G \cong K_{2, 2}$ which is difference cordial by theorem 2.6. When $p = 1$, $q$ must be 0. Here, $G + G \cong P_2$ this is difference cordial by theorem 2.2.

**Theorem 2.14.** Let $G$ be a $(p, q)$ difference cordial graph with $k$ ($k > 1$) vertices of degree $p - 1$. Then $p \leq 7$.

**Proof.** Obviously, $e_f(1) \leq p - 1$. Let $u_i$ ($1 \leq i \leq k$) be the vertex of $G$ such that $deg(u_i) = p - 1$ ($1 \leq i \leq k$). Then $e_f(0) \geq (p - 3) + (p - 3) - 1 + \cdots + (p - 3) - (k - 1) = p - 3 - \frac{k(k - 1)}{2}$. This implies $e_f(0) - e_f(1) \geq k(p - 3) - \frac{k(k - 1)}{2} - p + 1$. It follows that $p \leq 7$. $$

**Theorem 2.15.** $K_2 + mK_1$ is difference cordial iff $m \leq 4$.

**Proof.** Suppose $K_2 + mK_1$ is difference cordial then by theorem 2.14, $m \leq 5$. Let $V(K_2 + mK_1) = \{u, v, w_i : 1 \leq i \leq m\}$ and $E(K_2 + mK_1) = \{uv, uw_i, vw_i : 1 \leq i \leq m\}$. When $m = 5$, the maximum value of $e_f(1)$ occur when $f(u) = 2, f(v) = 4, f(w_1) = 1, f(w_2) = 3, f(w_3) = 5, f(w_4) = 6$ and $f(w_5) = 7$. It follows that, $e_f(1) \leq 4$. $e_f(0) \geq 4 - 7 > 7$. This implies $e_f(0) - e_f(1) \geq 3$. Hence $K_2 + 5K_1$ is not difference cordial. Since, $K_2 + K_1 \cong C_3$, by theorem 2.3, $K_2 + K_1$ is difference cordial. The difference cordial labeling of $K_2 + 2K_1$, $K_2 + 3K_1$ and $K_2 + 4K_1$ are shown in figure 2.

![Figure 2](image2)

**Theorem 2.16.** $K_n^2 + 2K_2$ is difference cordial iff $n \leq 2$. 

Proof. Let \( V(K_n^c + 2K_2) = \{u, v, w, z, u_i : 1 \leq i \leq n\} \) and \( E(K_n^c + 2K_2) = \{uw, wz\} \cup \{uw_i, vwu_i, wu_i, zu_i : 1 \leq i \leq n\} \). Clearly, \( e_f(1) \leq n + 3 \). Since the degree of the vertices \( u, v, w \) and \( z \) is \( n + 1 \), \( e_f(0) \geq (n - 1) + (n - 1) - 1 + (n - 1) + (n - 1) - 1 \geq 4n - 6 \). Hence, \( e_f(0) - e_f(1) \geq 3n - 9 \). This implies \( n \leq 3 \). Suppose \( n = 3 \). Here, \( e_f(1) \leq 6 \). Also \( e_f(0) \geq q - e_f(1) \geq 8 \). Therefore, \( e_f(0) - e_f(1) \geq 2 \). Hence, \( K_n^c + 2K_2 \) is not difference cordial. For \( n \leq 2 \), the difference cordial labeling is given in figure 3.

![Figure 3](image)

The sunflower graph \( S_n \) is obtained by taking a wheel with central vertex \( u_0 \) and the cycle \( C_n : v_1 v_2 \ldots v_n v_1 \) and new vertices \( w_1 w_2 \ldots w_n \) where \( w_i \) is joined by vertices \( v_i, v_{i+1(\mod n)} \).

Theorem 2.17. The sunflower graph \( S_n \) is difference cordial, for all \( n \).

Proof. Define \( f : V(S_n) \to \{1, 2, \ldots, 2n + 1\} \) by \( f(v_0) = 1, f(v_i) = 2i, 1 \leq i \leq n, f(w_i) = 2i + 1, 1 \leq i \leq n \). Now \( e_f(0) = 2n \) and \( e_f(1) = 2n \). Therefore \( f \) is a difference cordial labeling.

The Lotus inside a circle \( LC_n \) is a graph obtained from the cycle \( C_n : u_1 u_2 \ldots u_n u_1 \) and a star \( K_{1,n} \) with central vertex \( u_0 \) and the end vertices \( v_1 v_2 \ldots v_n \) by joining each \( v_i \) to \( u_i \) and \( u_{i+1(\mod n)} \).

Theorem 2.18. The Lotus inside a circle \( LC_n \) is difference cordial, for all \( n \).

Proof. Define a map \( f \) from the vertex set of \( LC_n \) to the set \( \{1, 2, \ldots, 2n + 1\} \) as follows: \( f(v_0) = 1, f(v_i) = 2i, 1 \leq i \leq n, f(u_i) = 2i + 1, 1 \leq i \leq n \). Clearly \( f \) is a difference cordial labeling.

A Lotus inside a circle \( LC_4 \) with a difference cordial labeling is shown in figure 4.

![Figure 4](image)

The graph obtained by arranging vertices into a finite number of rows with \( i \) vertices in the \( i \)th row and in every row the \( j \)th vertex and \( j + 1 \)st vertex of the next row is called the Pyramid. We denote the Pyramid with \( n \) rows by \( P_{yn} \).

Theorem 2.19. All Pyramids are difference cordial.

Proof. Let \( a_{i,j} (1 \leq j \leq i \leq n) \) be the vertices in the \( i \)th row. Define an injective map \( f \) from the vertices of the Pyramid \( P_{yn} \) to the set \( \left\{1, 2, 3 \ldots \frac{n(n+1)}{2}\right\} \) by \( f(a_{i,j}) = \frac{1}{2}(j-1)(2n-j) + i \), \( j \leq i \leq n \). Clearly, \( e_f(0) = e_f(1) = \frac{n(n-1)}{2} \). Therefore, \( f \) is a difference cordial labeling of the Pyramid.

Example 2.20. The Pyramid \( P_{y6} \) with a difference cordial labeling is shown in figure 5.
The graph $P_n + 2K_1$ is called a double fan $DF_n$.

**Theorem 2.21.** The double fan $DF_n$ is difference cordial if $n \leq 4$.

**Proof.** Note that $DF_n$ is a $(n + 2, 3n - 1)$ graph. Suppose $DF_n$ is difference cordial, then by theorem 2.4, $3n - 1 \leq 2(n + 2) - 1$. It follows that $n \leq 4$. Since $DF_1 \cong P_3$, $DF_2 \cong K_4$, using theorem 2.2 and theorem 2.8, $DF_1$ and $DF_2$ are difference cordial. The difference cordial labeling of $DF_2$ and $DF_4$ is given in figure 6.

![Figure 6](Image 6)

**Theorem 2.22.** Books with $n$ pentagonal pages are difference cordial.

**Proof.** Let $G$ be a book with $n$ pentagonal pages. Let $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\} \cup \{u, v\}$ and $E(G) = \{u u_i, u_i w_i, w_i v_i, v_i v : 1 \leq i \leq n\}$. Define a map $f : V(G) \to \{1, 2, \ldots, 3n + 2\}$ by

$$ f(u_i) = 3i - 2 \quad 1 \leq i \leq n $$

$$ f(w_i) = 3i - 1 \quad 1 \leq i \leq n $$

$$ f(v_i) = 3i \quad 1 \leq i \leq n $$

$f(u) = 3n + 1$ and $f(v) = 3n + 2$. Since $e_f(1) = 2n + 1$ and $e_f(0) = 2n$, $f$ is a difference cordial labeling of $G$.

$H_{n,n}$ is a graph with vertex set $\{u_i, v_i : 1 \leq i \leq n\}$ and edge set $\{u_i v_j : 1 \leq i \leq j \leq n\}$.

**Theorem 2.23.** $H_{n,n}$ is difference cordial iff $n \leq 6$.

**Proof.** Suppose $H_{n,n}$ is difference cordial, then using theorem 2.4, $\frac{n(n+1)}{2} \leq 2(2n) - 1$. It follows that $n \leq 6$. For $n \leq 6$, the difference cordial labeling $f$ is given in table 1.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>3</td>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$v_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>7</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>9</td>
</tr>
<tr>
<td>11</td>
</tr>
</tbody>
</table>

Table 1.

Let $G$ be the graph derived from a wheel $W_n$ by duplicating the hub vertex one or more times. We call $G$ a $t$-fold wheel if there are $t$ hub vertices, each adjacent to all rim vertices and not adjacent to each other.

**Theorem 2.24.** A $t$-fold wheel $G$ is difference cordial iff $t \leq 2$ and $n = 3$. 
Proof. The 1-fold wheel is a wheel and is difference cordial by theorem 2.7. Let \( t \geq 1 \). Clearly \( G \) consists of \( n + t \) vertices and \( nt + n \) edges. Suppose \( G \) is difference cordial with \( t \geq 3 \), then by theorem 2.3, \( nt + n \leq 2(n + t) - 1 \). \( \Rightarrow 2n - 1 \geq (n - 2) t + n \geq (n - 2) 3 + n \). It follows that \( 2n \leq 5 \), a contradiction. Suppose \( G \) is difference cordial with \( t = 2 \), then by theorem 2.3, \( 2n + n \leq 2(n + 2) - 1 \). This implies \( n \leq 3 \). The difference cordial labeling of \( G \) with \( n = 3 \) and \( t = 2 \) is given in figure 7.

![Figure 7](image)

Proof. The order and size of \( P \) consists of \( G \) and \( v \) vertex of \( G \) in \( P \). Theorem 2.26. Suppose \( G \) is difference cordial, then by theorem 2

Next is the permutation graphs. For any permutation \( f \) on \( 1, 2, \ldots, n \), the \( f \)-permutation graph on a graph \( G \), \( P(G, f) \) consists of two disjoint copies of \( G \), say \( G_1 \) and \( G_2 \), each of which has vertices labeled \( v_1, v_2, \ldots, v_n \) with \( n \) edges obtained by joining each \( v_i \) in \( G_1 \) to \( v_f(i) \) in \( G_2 \). We shall denote the identity permutation by \( I \).

The product graph \( G_1 \times G_2 \) is defined as follows: Consider any two vertices \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) in \( V = V_1 \times V_2 \). Then \( u \) and \( v \) are adjacent in \( G_1 \times G_2 \) whenever \( [u_1 = v_1 \text{ and } u_2 \text{ adj } v_2] \) or \( [u_2 = v_2 \text{ and } u_1 \text{ adj } v_1] \).

Theorem 2.25. Let \( G \) be \((p, q)\) graph with \( q \geq p \). Then for any permutation \( f \), \( P(G \times K_2, f) \) is not difference cordial.

Proof. The order and size of \( P(G \times K_2, f) \) are \( 4p + 4q + 4p \) respectively. Suppose \( P(G \times K_2, f) \) is difference cordial with \( q \geq p \), then by theorem 2.4, \( 4q + 4p \leq 2(4p) - 1 \). \( \Rightarrow 8p - 1 \geq 4q + 4p \geq 8p \). \( \Rightarrow -1 \geq 0 \), a contradiction.

Theorem 2.26. For any permutation \( f \), \( P(W_n, f) \) is not difference cordial.

Proof. Obviously, the order and size of \( P(W_n, f) \) are \( 2n + 2 \) and \( 5n + 1 \) respectively. Suppose \( P(W_n, f) \) is difference cordial. Then by theorem 2.4, \( 5n + 1 \leq 2(2n + 2) - 1 \). \( \Rightarrow n \leq 2 \), a contradiction.

For a graph \( G \), the splitting graph of \( G \), \( S(G) \), is obtained from \( G \) by adding for each vertex \( v \) of \( G \) a new vertex \( v' \) so that \( v' \) is adjacent to every vertex that is adjacent to \( v \).

Theorem 2.27. If \( G \) is a \((p, q)\) graph and \( S(G) \) be the splitting graph of \( G \) with \( q \geq p \). Then for any permutation \( f \), \( P(S(G), f) \) is not difference cordial.

Proof. The order and size of \( P(S(G), f) \) are \( 4p + 6q + 2p \) respectively. Suppose \( P(S(G), f) \) is difference cordial, then by theorem 2.4, \( 6q + 2p \leq 2(4p) - 1 \). \( \Rightarrow 6p - 1 \geq 6q \geq 6p \). \( \Rightarrow -1 \geq 0 \), a contradiction.

The helm \( H_n \) is the graph obtained from a wheel by attaching a pendant edge at each vertex of the \( n \)-cycle. A flower \( Fl_n \) is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

Theorem 2.28. For any permutation \( f \), \( P(Fl_n, f) \) is not difference cordial.

Proof. The order and size of \( P(Fl_n, f) \) are \( 4n + 2 \) and \( 10n + 1 \) respectively. Suppose \( P(Fl_n, f) \) is difference cordial. Then by theorem 2.4, \( 10n + 1 \leq 2(4n + 2) - 1 \). \( \Rightarrow n \leq 1 \), a contradiction.

Theorem 2.29. \( P(P_{2k}, f) \) is difference cordial where \( f = (1 2)(3 4) \ldots (k k+1) \ldots (2k - 1 2k) \).
Proof. Let \( u_i \) and \( v_i \) be the vertices in the first and second copies of \( P_{2k} \). Define, \( f : V(P(P_{2k}, f)) \rightarrow \{1, 2, \ldots, 4k\} \) by \( f(u_i) = i, \; 1 \leq i \leq 2k \),
\[
\begin{align*}
    f(v_{4i-3}) &= 2k + 4i - 3 & 1 \leq i \leq \frac{k+1}{2} & \text{if } 2k \equiv 2 \pmod{4} \\
    f(v_{4i-2}) &= 2k + 4i - 2 & 1 \leq i \leq \frac{k+1}{2} & \text{if } 2k \equiv 2 \pmod{4} \\
    f(v_{4i-1}) &= 2k + 4i & 1 \leq i \leq \frac{k+1}{2} & \text{if } 2k \equiv 2 \pmod{4} \\
    f(v_{4i}) &= 2k + 4i - 1 & 1 \leq i \leq \frac{k+1}{2} & \text{if } 2k \equiv 2 \pmod{4} \\
    f(u_1) &= 4i - 3 & 1 \leq i \leq n \\
    f(v_1) &= 4i - 2 & 1 \leq i \leq n \\
    f(w_i) &= 4i - 1 & 1 \leq i \leq n - 1 \\
    f(x_i) &= 4i & 1 \leq i \leq n - 1 \\
    f(w_n) &= 4n & f(x_n) = 4n - 1.
\end{align*}
\]
Since, \( e_f(0) = e_f(1) = 3k - 1, f \) is a difference cordial labeling. □

**Theorem 2.30.** \( P(P_n, I) \) is difference cordial.

**Proof.** Since \( P(P_n, I) \cong L_n \), proof follows from theorem 2.9. □

**Theorem 2.31.** \( P(C_n, I) \) is difference cordial.

**Proof.** Since \( P(C_n, I) \cong C_n \times P_2 \), proof follows from theorem 2.10. □

**Theorem 2.32.** \( P(K_n, I) \) is difference cordial iff \( n \leq 3 \).

**Proof.** Since \( P(K_1, I) \cong K_2, P(K_2, I) \cong C_4 \) and \( P(K_n, I) \equiv C_3 \times P_2, P(K_n, I), n \leq 3 \) is difference cordial. The order and size of \( P(K_n, I) \) are \( 2n \) and \( n^2 \) respectively. Suppose \( P(K_n, I) \) is difference cordial, then by theorem 2.4, \( n^2 \leq 4n - 1 \). It follows that \( n \leq 3 \). □

The corona of \( G \) with \( H, G \odot H \) is the graph obtained by taking one copy of \( G \) and \( p \) copies of \( H \) and joining the \( i^{th} \) vertex of \( G \) with an edge to every vertex in the \( i^{th} \) copy of \( H \). \( P_n \odot K_1 \) is called the comb and \( P_n \odot 2K_1 \) is called the double comb.

**Theorem 2.33.** \( P(P_n \odot K_1, I) \) is difference cordial.

**Proof.** Let \( V(P(P_n \odot K_1, I)) = \{u_i, v_i, w_i, x_i : 1 \leq i \leq n\} \) and \( E(P(P_n \odot K_1, I)) = \{u_iw_i, v_iw_i, u_iw_i, w_i, u_iw_i, u_iw_i, x_iw_i, x_iw_i : 1 \leq i \leq n\} \). Define an injective map \( f : V(P(P_n \odot K_1, I)) \rightarrow \{1, 2, \ldots, 4n\} \) by
\[
\begin{align*}
    f(u_i) &= 6i - 5 & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
    f(v_i) &= 6i - 4 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_i) &= 6i & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_i) &= 6i - 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(u_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 5 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(v_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 4 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 3 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(u_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 1 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(v_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i + 1 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i + 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor.
\end{align*}
\]
\( f(u_n) = 6n - 5 \) and \( f(x_n) = 4n - 1 \). Since \( e_f(0) = e_f(1) = 3n - 1, f \) is a difference cordial labeling of \( P(P_n \odot K_1, I) \). □

**Theorem 2.34.** \( P(P_n \odot 2K_1, I) \) is difference cordial.

**Proof.** Let \( V(P(P_n \odot 2K_1, I)) = \{u_i, v_i, w_i, x_i, y_i, z_i : 1 \leq i \leq n\} \) and \( E(P(P_n \odot 2K_1, I)) = \{u_iw_i, u_iw_i, u_iw_i, v_iw_i, u_iw_i, x_iw_i, x_iw_i, y_iw_i, z_iw_i : 1 \leq i \leq n\} \cup \{u_iw_i, v_iw_i : 1 \leq i \leq n - 1\} \). Define a one to one map \( f : V(P(P_n \odot 2K_1, I)) \rightarrow \{1, 2, \ldots, 6n\} \) as follows:
\[
\begin{align*}
    f(u_i) &= 6i - 5 & 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \\
    f(v_i) &= 6i - 4 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_i) &= 6i & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_i) &= 6i - 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(y_i) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 5 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(z_i) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 4 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 3 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(y_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i - 1 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(z_1) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(w_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i + 1 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor \\
    f(x_2) &= 6 \left\lfloor \frac{n}{4} \right\rfloor + 5i + 2 & 1 \leq i \leq \lfloor \frac{n}{4} \rfloor.
\end{align*}
\]
The following table 2 shows that \( f \) is a difference cordial labeling.

<table>
<thead>
<tr>
<th>Nature of ( n )</th>
<th>( e_f(0) )</th>
<th>( e_f(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \equiv 0 \pmod{2} )</td>
<td>( \frac{2n-2}{2} )</td>
<td>( \frac{2n-2}{2} )</td>
</tr>
<tr>
<td>( n \equiv 1 \pmod{2} )</td>
<td>( \frac{2n-3}{2} )</td>
<td>( \frac{n-1}{2} )</td>
</tr>
</tbody>
</table>

Table 2.

**Theorem 2.35.** \( P(K_2 + mK_1, I) \) is difference cordial iff \( m \leq 3 \).

**Proof.** The order and size of \( P(K_2 + mK_1, I) \) are \( 2m + 4 \) and \( 5m + 4 \) respectively. By theorem 2.4, \( 5m + 4 \leq 2(2m + 4) - 1 \). It follows that \( m \leq 3 \). The difference cordial of \( K_2 + K_1, K_2 + 2K_1 \) and \( K_2 + 3K_1 \) are shown in figure 8.

![Figure 8](image)

**Theorem 2.36.** Let \( C_n \) be the cycle \( u_1u_2 \ldots u nu_1 \). Let \( G^* \) be the graph with \( V(G^*) = V(C_n) \cup \{v_i, w_i : 1 \leq i \leq n\} \) and \( E(G^*) = E(C_n) \cup \{u_iu_{i+1(mod n)}w_i, v_iw_i : 1 \leq i \leq n\} \). Then \( G^* \) is difference cordial.

**Proof.** Define a function \( f : V(G^*) \to \{1, 2 \ldots 3n\} \) as follows:

\[
\begin{align*}
  f(u_i) &= i & 1 \leq i \leq n \\
  f(v_i) &= n + 2i + 1 & 1 \leq i \leq n - 1 \\
  f(w_i) &= n + 2i + 2 & 1 \leq i \leq n - 1 \\
\end{align*}
\]

\( f(v_n) = n + 1 \) and \( f(w_n) = n + 2 \). Since \( e_f(0) = e_f(1) = 2n \), \( f \) is a difference cordial labeling of \( G^* \).

**Theorem 2.37.** \( P(K_{m,n}, I) \) \( (m, n > 1) \) is difference cordial iff \( m = n = 2 \) and \( n = 3, 4, 5 \).

**Proof.** The order and size of \( P(K_{m,n}, I) \) are \( 2m + 2n \) and \( 2mn + m + n \) respectively. Suppose \( P(K_{m,n}, I) \) is difference cordial, then by theorem 2.4, \( 2mn + m + n \leq 2(2m + 2n) - 1 \).

\( \Rightarrow 2mn \leq 3m + 3n - 1 \quad \Rightarrow (1) \)

**Case 1.** \( m = n \).

(1) \( \Rightarrow 2m^2 \leq 6m - 1 \). \( \Rightarrow m = n = 2 \).

**Case 2.** \( m \neq n \).

Assume \( m > n \geq 2 \). (1) \( \Rightarrow 0 \leq -2mn + 3m + 3n - 1 < -2mn + 6m - 1 \). \( \Rightarrow 6m - 1 > 2mn \quad \Rightarrow (2) \).

**Subcase 1.** \( n \geq 3 \).

(2) \( \Rightarrow 6m - 1 \geq 6m \). \( \Rightarrow -1 \geq 0 \), a contradiction.

**Subcase 2.** \( n = 2 \).

Here \( p = 2m + 4 \) and \( q = 5m + 2 \). Suppose \( f \) is difference cordial, then by theorem 2.4, \( 5m + 2 \leq 2(2m + 4) - 1 \). This implies \( m \leq 5 \). Since \( P(K_{5,2}, I) \equiv C_4 \times P_2 \), by theorem 2.10, \( P(K_{2,5}, I) \) is difference cordial. The difference cordial labeling of \( P(K_{2,3}, I) \), \( P(K_{2,4}, I) \) and \( P(K_{2,5}, I) \) is shown in figure 9.
Finally we investigate the difference cordial labeling behavior of special graphs which are generated from cycle.

**Theorem 2.38.** Let $C_n$ be the cycle $u_1u_2\ldots u_nu_1$. Let $G$ be a graph with $V(G) = V(C_n) \cup \{w_i : 1 \leq i \leq n\}$ and $E(G) = E(C_n) \cup \{u_iw_i, u_{i+1(mod\ n)}w_i : 1 \leq i \leq n\}$. Then $G$ is difference cordial.

**Proof.** Define a map $f : V(G) \rightarrow \{1, 2, \ldots, 2n\}$ as follows:

**Case 1.** $n$ is even.

- $f(u_i) = 2i - 1, \quad 1 \leq i \leq \frac{n}{2}$
- $f(u_{i+\frac{n}{2}}) = \frac{3n}{4} + i, \quad 1 \leq i \leq \frac{n}{4}$
- $f(w_i) = 2i, \quad 1 \leq i \leq \frac{n+2}{2}$
- $f(w_{n-i}) = n + 2 + i, \quad 1 \leq i \leq \frac{n}{2}$

**Case 2.** $n$ is odd.

- $f(u_i) = 2i - 1, \quad 1 \leq i \leq \frac{n-1}{2}$
- $f(u_{i+\frac{n+1}{2}}) = \frac{3n+5}{4} + i, \quad 1 \leq i \leq \frac{n+5}{4}$
- $f(w_i) = 2i, \quad 1 \leq i \leq \frac{n+3}{2}$
- $f(w_{n-i}) = n + 4 + i, \quad 1 \leq i \leq \frac{n-3}{2}$

The table 3 gives the nature of the edge condition of the above labeling $f$. It follows that $f$ is a difference cordial labeling.

<table>
<thead>
<tr>
<th>Values of $n$</th>
<th>$ef(0)$</th>
<th>$ef(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \equiv 0 (mod\ 2)$</td>
<td>$\frac{3n}{4}$</td>
<td>$\frac{3n}{4}$</td>
</tr>
<tr>
<td>$n \equiv 1 (mod\ 2)$</td>
<td>$\frac{3n-1}{4}$</td>
<td>$\frac{3n+1}{4}$</td>
</tr>
</tbody>
</table>

The table 3.

References


Author information

R. Ponraj, S. Sathish Narayanan, Department of Mathematics, Sri Paramakalyani College, Alwarkurichi-627 412, Tamil Nadu, India.

E-mail: ponrajmath@gmail.com; sathishrvss@gmail.com

R. Kala, Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli-627 012, Tamil Nadu, India.

E-mail: karthipy1991@yahoo.co.in

Received: July 26, 2013.

Accepted: January 23, 2014.