Palestine Journal of Mathematics

Vol. 4(2) (2015), 416–418

On Vector Valued Metric spaces

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 47H10; Secondary 54H25.

Keywords and phrases: Metric space, n-dimensional Euclidean space, Metrizability.

Abstract The notion of Vector valued metric (VVM) space was introduced by K.P.R. Sastry et al. (IJMA-3(7), 2012, 2680-2685.), by replacing real numbers with n-dimensional Euclidean space equipped with a partial order, in the definition of metric space and proved a couple of common fixed point theorem over such spaces. The purpose of this paper is to show that every Vector valued metric induces a metric on the underlying space, the induced metric topology coincides with VVM topology, thereby we prove that VVM topology is metrizable and some fixed point results over these spaces are derived.

1 Introduction

In what follows $\mathbb{R}$ and $\mathbb{N}$ denote the set of all real and natural numbers respectively and $\mathbb{R}^n$, the set of all n-tuples $\overline{a} = (a_1, a_2, ..., a_n)$ of reals with partial ordering $\leq$ defined on $\mathbb{R}^n$ by $\overline{a} \leq \overline{b}$ if and only if $a_i \leq b_i$ for each i, $i \in \{1, 2, ..., n\}$. For each $\overline{a}, \overline{b} \in \mathbb{R}^n$, we write $\overline{a} << \overline{b}$ if $a_i < b_i$ for each $i \in \{1, 2, ..., n\}$ and for each $a \in \mathbb{R}$, $\hat{a}$ stands for the element $(a, a, \ldots, a) \in \mathbb{R}^n$. For undefined terms and notations refer to ([1, 2, 4]). In [1], the concept of Vector Valued Metric space was introduced, as a generalization of metric space as well as of complex valued metric space as follows:

Definition 1.1. Let X be a nonempty set. Suppose that the mapping $d\colon X \times X \to \mathbb{R}^n$ satisfies:

(D1) $\hat{0} \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \hat{0}$ if and only if $x = y$;

(D2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(D3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called n-dimensional vector valued metric(for short VVM) on X and the pair (X, d) is called a n-dimensional vector valued metric space(for short VVM space).

Definition 1.2. [1] Let (x_n) be a sequence in a VVM space (X, d), $x \in X$. We say that

(i) $\{x_n\}$ converges to x if for every $\overline{r} \in \mathbb{R}^n$ with $\hat{0} << \overline{r}$ there is a $N \in \mathbb{N}$ such that for all $n > N,\ d(x_n, x) << \overline{r}$. In this case we write $\lim x_n = x$ in (X, d),

(ii) $\{x_n\}$ is a Cauchy sequence in X if for every $\overline{r} \in \mathbb{R}^n$ with $\hat{0} << \overline{r}$ there is a $N \in \mathbb{N}$ such that for all $k, m > N,\ d(x_k, x_m) << \overline{r}$,

(iii) (X, d) is a complete VVM space if every Cauchy sequence in (X, d) is convergent to an element in (X, d).

Let (X, d) be a VVM space and $A \subset X$. We declare A to be open if for each $a \in A$, there exists $\hat{0} << \overline{r}_a \in \mathbb{R}^n$ such that $A \subset B_d(a, \overline{r}_a)$ where $B_d(a, \overline{r}_a) = \{x \in X\colon d(x, a) << \overline{r}_a\}$. Then the family $F = \{B_d(x, \overline{r})\colon x \in X, \hat{0} << \overline{r} \in \mathbb{R}^n\}$ forms a subbase for a Hausdorff topology τ_d on X (See [1]).

Example 1.3. Let $D_1, D_2, ..., D_n$ be metrics on X. Then the mapping $d\colon X \times X \to \mathbb{R}^n$ defined by $d(x, y) = (D_1(x, y), D_2(x, y), ..., D_n(x, y))$ is a VVM on X. In particular, if $D_1 = D_2 = ... = D_n = D$ then $d(x, y) = D(x, y)\hat{1}$, is a VVM on X. More generally, if $\hat{0} << \overline{a} \in \mathbb{R}^n$ and D is a metric on X, then the function $d\colon X \times X \to \mathbb{R}^n$ defined by $d(x, y) = D(x, y)\overline{a}$ is a VVM on X.

2 MAIN RESULTS

Throughout this paper (X, d) will denote a VVM space.

Lemma 2.1. *If we define $D\colon X \times X \to \mathbb{R}$ by $D(x, y) = \max\{a_1, a_2, ..., a_n\}$ where $\overline{a} = d(x, y) \in \mathbb{R}^n$, then*

(i) *D is a metric on X,*

(ii) $\lim d(x_n, x) = \hat{0}$ *if and only if* $\lim D(x_n, x) = 0$,

(iii) *$\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, D),*

(iv) *(X, d) is complete if and only if (X, D) is complete.*

Proof. It is enough to verify Triangle inequality to prove (i).
Let $x, y, z \in X$. Then $\overline{a} = d(x, y)$, $\overline{b} = d(x, z)$, $\overline{c} = d(z, y)$ are in $\mathbb{R}^n$.
By (D3), $\overline{a} \leq \overline{b} + \overline{c}$ which implies $\max\{a_1, a_2, ..., a_3\} \leq \max\{b_1, b_2, ..., b_n\} + \max\{c_1, c_2, ..., c_n\}$ $\Rightarrow D(x, y) \leq D(x, z) + D(z, y)$. Hence (X, D) is a metric space. Proof of (ii),(iii) and (iv) are clear.□

Remark 2.2. It is clear from Lemma 2.1 that every VVM d on X induces a metric D on X. In fact, d induces several metrics on X. For instance, if $1 \leq p < \infty$ then the real function D_p on $X \times X$ defined by

$$D_p(x, y) = \{\sum_{i=0}^{n} a_i^p\}^{\frac{1}{p}} \tag{2.1}$$

for all $x, y \in X$, where $\overline{a} = d(x, y) \in \mathbb{R}^n$, is a metric on X.

Theorem 2.3. *Let D be an induced metric on X defined as in Lemma 2.1. Then the induced metric topology τ_D coincides with the VVM topology τ_d on X.*

Proof. Let $y \in B_d(x, \overline{r})$. Then $d(x, y) << \overline{r}$ which implies $\overline{a} << \overline{r}$ where $d(x, y) = \overline{a} \in \mathbb{R}^n$. Put $r = \min\{r_1 - a_1, r_2 - a_2, ..., r_n - a_n\}$. Then $r > 0$ and if $z \in B_D(y, r)$, then $\max\{b_1, b_2, ..., b_n\} < r$ where $\overline{b} = d(y, z) \in \mathbb{R}^n$. Now,
$\max\{b_1, b_2, ..., b_n\} < r = \min\{r_1 - a_1, r_2 - a_2, ..., r_n - a_n\}$
$\Rightarrow \overline{b} + \overline{a} << \overline{r}$
$\Rightarrow d(x, z) \leq d(x, y) + d(y, z) << \overline{r}$
$\Rightarrow z \in B_d(x, \overline{r})$.
Therefore $B_D(y, r) \subset B_d(x, \overline{r})$ and hence $B_d(x, \overline{r})$ is open in the usual sense.
Now, Let $y \in B_D(x, r)$ for some $r > 0$. Then $D(x, y) < r$.
$\Rightarrow \max\{a_1, a_2, ..., a_3\} < r$ where $\overline{a} = d(x, y) \in \mathbb{R}^n$.
$\Rightarrow \overline{a} << \hat{r}$.
Let $z \in B_d(x, \hat{r})$. Then $d(x, z) << \hat{r}$ imply $D(x, y) < r$. Therefore, $B_d(x, \hat{r}) \subset B_D(x, r)$ and hence $B_D(x, r)$ is an open set in (X, d). This completes the proof. □

In view of Theorem 2.3, we have

Corollary 2.4. *The Hausdorff topology τ_d induced by d on X is metrizable.*

Theorem 2.1 of [1] that every complex valued metric space (CVM space for short) is metrizable, can be proved as a consequence of Theorem 2.3, as every CVM space is a VVM space .

Theorem 2.5. *Let T be a contraction on a complete VVM space (X, d) with contracting constant λ. Then T has a unique fixed point in X.*

Proof. Let D be defined as in Lemma 2.1. Then (X, D) is a complete metric space. Let $x, y \in X$. Then $d(Tx, Ty) \leq \lambda d(x, y)$ implies $D(Tx, Ty) \leq \lambda D(x, y)$. Therefore, T is contraction on (X, D). Hence by Banach contraction principle for metric spaces, T has a unique fixed point x in X. □
The metric D induced by d on X is very useful in deriving several well known fixed point results for self maps on (X, d) from its metric counterparts. For instance, if T is a Kannan mapping on a complete VVM space (X, d), then T is also a Kannan map on (X, D) and hence T has a unique fixed point in X.

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Received: January 14, 2014.

Accepted: July 15, 2014.