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SOME NEW FIXED POINT THEOREMS IN DISLOCATED QUASI-METRICSPACES

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Abstract. In this paper, we establish some new type of fixed point theorems for a single self-mapping using expanding and comparison function in the setting of dislocated quasi-metric space. Our establish results improve and modify some existing results in the literature.

1 Introduction and Preliminaries

Banach contraction principle [1] is one of pivotal results in functional analysis. There are a number of generalizations of Banach contraction principle in different type of spaces. Dass and Gupta [2] generalized Banach contraction principle through rational expression in metric spaces. Mathews [3] introduced the concept of partial metric space (pms) as a part of study of denotational semantics and dataflow networks. The most interesting property in partial metric spaces is that the self distance between points may not be zero. In 2001, Hitzler [4] generalized the idea of partial metric spaces and initiated the concept of dislocated metric (d-metric) space. Dislocated metric play a vital role in logic programming semantics, computer science and electronic engineering etc. Hitzler [4] showed that Banach contraction principle is valid in dislocated metric space.

Zeyada et al. [5] further generalized the concept of dislocated metric space and introduced the idea of complete dislocated quasi-metric space. In this new notion the symmetric property is also omitted. Sevaral papers have been published containing fixed point results for a single and a pair of self-mappings with different contraction conditions in dislocated quasi-metric space (see [6, 7, 8]).

The purpose of this article is to obtain some new fixed point theorems in dislocated quasi-metric space using the concepts of expanding and comparison mappings. Examples are given in the support of our establish results.

We begin with the following definitions.

Definition [5]. Let X be a non-empty set and let $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ be a distance function satisfying the following conditions

d_1) $d(x, y) = d(y, x) = 0$ implies $x = y$;

d_2) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called dislocated quasi-metric on X and the pair (X, d) is called dislocated quasi-metric (dq-metric) space.

Example. Let X=$\mathbb{R}$ define the distance function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ by

$$d(x, y) = |x| \quad \text{for all } x, y \in X.$$

In the main work we will use the following definitions which can be found in [5].

Definition. A sequence $\{x_n\}$ in dislocated quasi-metric space (X, d) is called dislocated quasi convergent (dq-convergent) if for $n \in N$ we have

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

In such a case x is called dislocated quasi limit (dq-limit) of the sequence $\{x_n\}$.

Definition. A sequence $\{x_n\}$ in dislocate quasi-metric space (X, d) is called Cauchy sequence if for $\epsilon > 0$ there exists a positive integer n_0 such that for $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$ i.e

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Definition. A dislocated quasi-metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a point in X.
Definition. Let (X,d) be a dislocated quasi-metric space. A mapping $T : X \to X$ is called contraction if there exist $0 \leq \alpha < 1$ such that

$$d(Tx,Ty) \leq \alpha d(x,y) \text{ for all } x,y \in X \text{ and } \alpha \in [0,1).$$

The following well-known results can be seen in [5].

Lemma 1.1. *Limit in dislocated quasi-metric space (X,d) is unique.*

Theorem 1.2. *Let (X,d) be a complete dislocated quasi-metric space $T : X \to X$ be a contraction. Then T has a unique fixed point.*

Definition[10]. A map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies:

(i) φ is monotonic increasing;

(ii) The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converge to zero for all $t \in \mathbb{R}_+$ where φ^n stand for nth iterate of φ.
If φ satisfies:

(iii) $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converge for all $t \in \mathbb{R}_+$.

Then φ is called (c)-comparison function.
Thus every comparison function is (c)-comparison function. A prototype example for comparison function is

$$\varphi(t) = \alpha t \quad t \in \mathbb{R}_+ \;\; 0 \leq \alpha < 1.$$

Some more examples and properties of comparison and (c)-comparison function can be found in [10].

Lemma 1.3. *[10]. For every comparison function for* t > 0 *implies that*

$$\varphi(t) < t$$

and $\varphi(t) = 0$ iff $t = 0$.

Definition[11]. Let (X,d) be a metric space. Let $T : X \to X$ be a self-mapping. Then T is called Kannan mapping if

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)] \text{ for all } x,y \in X \text{ and } \alpha \in [0,1/2). \tag{1.1}$$

Definition. Let (X,d) be a metric space. Let $T : X \to X$ be a self-mapping. Then T is called conjugate Kannan mapping if

$$d(Tx,Ty) \leq \alpha[d(Tx,x) + d(Ty,y)] \text{ for all } x,y \in X \text{ and } \alpha \in [0,1/2). \tag{1.2}$$

2 Main Results

Now we use conjugate type of mapping to establish fixed point theorem for expanding mapping in dislocated quasi-metric space.

Theorem 2.1. *Let (X,d) be a complete dislocated quasi-metric space. $T : X \to X$ be a self-mapping satisfying*

$$d(Tx,Ty) \geq ad(x,y) + b\frac{d(Tx,x)[1+d(Ty,y)]}{1+d(x,y)} + c\frac{[d(Tx,x)+d(Ty,y)]d(x,y)}{d(Tx,y)} \tag{2.1}$$

for all $x,y \in X$ and $a,b,c \geq 0$ with $a > 1$ and $b \leq 1$ also $d(Tx,y) \neq 0$. Then T has a unique fixed point.

Proof. Since $a > 1$ and $b,c \geq 0$ then obviously $a+b+c > 1$. For $x_0 \in X$ we define a sequence $\{x_n\}$ in X by the following way

$$x_n = Tx_{n+1} \text{ for } n = 0,1,2,3,........$$

Consider

$$d(x_{n-1},x_n)=d(Tx_n,Tx_{n+1}).$$

Using (2.1) and the defined construction of the sequence we have

$$\geq ad(x_n,x_{n+1})+b\frac{d(Tx_n,x_n)[1+d(Tx_{n+1},x_{n+1})]}{1+d(x_n,x_{n+1})}+$$

$$c\frac{[d(Tx_n,x_n)+d(Tx_{n+1},x_{n+1})]d(x_n,x_{n+1})}{d(Tx_n,x_{n+1})}$$

$$=ad(x_n,x_{n+1})+b\frac{d(x_{n-1},x_n)[1+d(x_n,x_{n+1})]}{1+d(x_n,x_{n+1})}+c\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]d(x_n,x_{n+1})}{d(x_{n-1},x_{n+1})}.$$

By simplification and using the fact that

$$d(x_n,x_{n+1})\leq\frac{[d(x_{n-1},x_n)+d(x_n,x_{n+1})]d(x_n,x_{n+1})}{d(x_{n-1},x_{n+1})}.$$

We have

$$(a+c)d(x_n,x_{n+1})\leq(1-b)d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1})\leq\left(\frac{1-b}{a+c}\right)d(x_{n-1},x_n).$$

Let

$$\frac{1-b}{a+c}=h<1.$$

So the above inequality take the form

$$d(x_n,x_{n+1})\leq hd(x_{n-1},x_n).$$

Also

$$d(x_{n-1},x_n)\leq hd(x_{n-2},x_{n-1}).$$

Hence

$$d(x_n,x_{n+1})\leq h^2d(x_{n-2},x_{n-1}).$$

Continuing the same procedure we have

$$d(x_n,x_{n+1})\leq h^nd(x_0,x_1).$$

Since $h<1$ and taking $n\to\infty$, $h^n\to 0$. Hence

$$\lim_{n\to\infty}d(x_n,x_{n+1})=0$$

which proves that $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi-metric space X. So there must exists $u\in X$ such that $\lim_{n\to\infty}x_n=u$. Also since T is continuous so

$$T\lim_{n\to\infty}x_n=Tu\;\Rightarrow\;\lim_{n\to\infty}Tx_n=Tu\;\Rightarrow\;\lim_{n\to\infty}x_{n-1}=Tu\;\Rightarrow\;Tu=u.$$

Hence u is fixed point of T.

Uniqueness. To show that

$$d(u,u)=d(v,v)=0.$$

Where u and v are two distinct fixed points of T for this consider

$$d(u,u)=d(Tu,Tu)$$

$$\geq ad(u,u)+b\frac{d(Tu,u)[1+d(Tu,u)]}{1+d(u,u)}+c\frac{[d(Tu,u)+d(Tu,u)]d(u,u)}{d(Tu,u)}.$$

Simplification yields

$$d(u,u)\geq(a+b+c)d(u,u)$$

which is a contradiction therefore $d(u,u)=0$. Similarly we can show that $d(v,v)=0$.

Now consider

$$d(u,v)=d(Tu,Tv)$$

$$\geq ad(u,v) + b\frac{d(Tu,u)[1+d(Tv,v)]}{1+d(u,v)} + c\frac{[d(Tu,u)+d(Tv,v)]d(u,v)}{d(Tu,v)}.$$

Using the above proved facts we have

$$d(u,v) \geq ad(u,v)$$

which is again contradiction thus $d(u,v) = 0$ similarly we can show that $d(v,u) = 0$ implies $u = v$. Thus fixed point of T is unique.

Corollary 2.2. *Let (X,d) be a complete dislocated quasi-metric space. $T : X \to X$ be a self-mapping satisfying*

$$d(Tx,Ty) \geq ad(x,y)$$

for all $x, y \in X$ with $a > 1$. Then T has a unique fixed point.

Example. Let $X = \mathbb{R}$ with complete dislocated quasi-metric on X is defined by

$$d(x,y) = |x| \text{ for all } x, y \in X$$

and $Tx = 2x$ for all $x \in X$. Then

$$d(Tx,Ty) = |2x| \geq \frac{3}{2}|x| = ad(x,y).$$

Satisfy all the conditions of Corollary 2.2 having $x = 0$ is the unique fixed point of T.

Theorem 2.3. *Let (X,d) be a complete dislocated quasi-metric space. $T : X \to X$ be a self-mapping satisfying*

$$d(Tx,Ty) \leq a\varphi d(x,y) + b\varphi \max\left\{d(x,Tx), d(x,y)\right\} + c\varphi\left(\frac{d(x,y)[1+\sqrt{d(x,y)d(x,Tx)}]^2}{(1+d(x,y))^2}\right) \quad (2.2)$$

for all $x, y \in X$, $a, b, c \geq 0$ with $a + b + c < 1$ and φ is a comparison function as defined in Definition 1. Then T has a unique fixed point.

Proof. Let x_0 be arbitrary point in X. Define a sequence $\{x_n\}$ in X by the rule

$$x_{n+1} = Tx_n \quad n = 0, 1, 2, ...$$

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Now using (2.2) we have

$$\leq a\varphi d(x_{n-1}, x_n) + b\varphi \max\left\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, x_n)\right\} + c\varphi\left(\frac{d(x_{n-1},x_n)[1+\sqrt{d(x_{n-1},x_n)d(x_{n-1},Tx_{n-1})}]^2}{(1+d(x_{n-1},x_n))^2}\right).$$

Now using the defined construction and then simplifying we have

$$= a\varphi d(x_{n-1}, x_n) + b\varphi \max\left\{d(x_{n-1}, x_n), d(x_{n-1}, x_n)\right\} + c\varphi\left(\frac{d(x_{n-1},x_n)[1+\sqrt{d(x_{n-1},x_n)d(x_{n-1},x_n)}]^2}{(1+d(x_{n-1},x_n))^2}\right)$$
$$= (a+b+c)\varphi d(x_n, x_{n+1}).$$

Since $\varphi(t) \leq t \; \forall \; t \geq 0$ so

$$d(x_n, x_{n+1}) \leq (a+b+c)d(x_{n-1}, x_n).$$

Let $h = a + b + c < 1$. Hence

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n).$$

Continuing the same procedure we have

$$d(x_n, x_{n+1}) \leq h^n d(x_0, x_1).$$

Taking limit $n \to \infty$ so $h^n \to 0$. Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

which prove that $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi-metric space X. So there must exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Also since T is continuous function so we have

$$Tu = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n + \lim_{n\to\infty} x_{n+1} = u.$$

Therefore u is the fixed point of T.

Uniqueness. Let $u \neq v$ are two distinct fixed points of T then consider

$$d(u, v) = d(Tu, Tv).$$

Using (2.2) we have

$$\leq a\varphi d(u, v) + b\varphi \max\left\{d(u, Tu), d(u, v)\right\} + c\varphi\left(\frac{d(u, v)[1 + \sqrt{d(u, v)d(u, Tu)}]^2}{(1 + d(u, v))^2}\right)$$

$$= a\varphi d(u, v) + b\varphi \max\left\{d(u, u), d(u, v)\right\} + c\varphi\left(\frac{d(u, v)[1 + \sqrt{d(u, v)d(u, u)}]^2}{(1 + d(u, v))^2}\right). \quad (2.3)$$

It is easy to show that $d(u, u) = 0$ by putting $x = y = u$ in (2.2) and similarly we can show that $d(v, v) = 0$ by putting $x = y = v$ in (2.2). Putting these information in (2.3) we get

$$d(u, v) \leq a\varphi d(u, v) + b\varphi d(u, v) + c\varphi \frac{d(u, v)}{(1 + d(u, v))^2}. \quad (2.4)$$

Since

$$1 \leq 1 + d(u, v) \text{ so } 1 \leq [1 + d(u, v)]^2$$
$$d(u, v) \leq [1 + d(u, v)]^2 d(u, v)$$
$$\frac{d(u, v)}{[1 + d(u, v)]^2} \leq d(u, v).$$

Thus (2.4) becomes

$$d(u, v) \leq (a + b + c)\varphi d(u, v).$$

Again since $\varphi(t) \leq t$ for all $t \geq 0$. So

$$d(u, v) \leq (a + b + c)d(u, v).$$

Since $a + b + c < 1$ so the above inequality is possible only if $d(u, v) = 0$. Similarly we can show that $d(v, u) = 0$ which implies that $u = v$. Hence fixed point of T is unique.

Corollary 2.4. *Let (X, d) be a complete dislocated quasi-metric space. $T : X \to X$ be a self-mapping satisfying*

$$d(Tx, Ty) \leq a\varphi d(x, y)$$

for all $x, y \in X$ with $0 \leq a < 1$ and φ is a comparison function as defined in Definition 1. Then T has a unique fixed point.

Remark 2.5. In Corollary 2.4 if $\varphi = I$(identity). Then we get the result of Zeyada et al. [5] as a corollary of our Theorem 2.3.

Example. Let $X = \mathbb{R}$ and the complete dislocated quasi-metric on X is defined by

$$d(x, y) = |x| \text{ for all } x, y \in X$$

and $Tx = \frac{x}{8}$ for all $x \in X$ then

$$d(Tx, Ty) = |\frac{x}{8}| \leq |\frac{x}{4}| = \frac{1}{4}|x| = \frac{1}{2}\frac{1}{2}|x| = a\varphi(d(x, y)).$$

Thus for $a = \frac{1}{2}$ and $\varphi(t) = \frac{t}{2}$ for all $t \geq 0$ satisfy all the conditions of Corollary 2.4 having $x = 0$ is the unique fixed point of T.

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