Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 518–530

A Survey of some generalized extending conditions

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 16D50, 16D70; Secondary 16D40.

Keywords and phrases: Extending module, fully invariant submodule, projection invariant submodule, C_{11}-module, FI-extending module.

The first-named author is grateful for the hospitality provided by Hacettepe University.

Abstract. A module is called an extending (or CS) module if every submodule is essential in a direct summand of the module. In this survey, we consider various generalizations of the extending property that the authors have developed over the last 25 years. To this end, we provide results, examples, applications, and open problems to motivate further interest and research on the extending condition and related concepts. Our generalizations make it clear how much of the extending condition is maintained in various closure operations.

Introduction

All rings are associative with unity and all modules are unital.

Recall a module is called an **extending** module if every submodule is essential in a direct summand of the module. This condition was identified by Utumi as the C_1 condition in his work on continuous and self-injective rings in the 1960's [36] . The importance of this condition is due, at least in part, to the fact that it is a common generalization of the injective condition and the semisimple condition on modules. The extending condition is also known as the CS-condition. Since the 1960's this condition has been developed in numerous papers and in at least 3 research monographs [24], [20], and [15].

In this survey, we focus on various generalizations of the extending property that the authors have developed over the last 25 years. We give results, examples, applications, and open problems. Unfortunately, the class of extending modules lacks closure with respect to direct sums, direct products, submodules, homomorphic images, extensions and essential extensions. The class of right extending rings lacks closure with respect to full and upper triangular matrix rings, trivial extensions, generalized triangular matrix rings, homomorphic images, and polynomial rings. There are at least two ways to overcome these obstructions to the usefulness of the extending property:

(1) adjoin additional conditions on the module to obtain the desired closure properties; or

(2) generalize or weaken the extending condition to obtain the desired closure properties.

Herein we consider way (2) to obtain closure properties. Thus our generalizations answer the question : "How much of the extending condition is maintained in various closure operations?" The generalized extending properties of interest in this paper are :

(1) M is $\mathcal{G}$**-extending** if for each X a submodule of M, there exists a direct summand D of M such that $X \cap D$ is essential in both X and D.

(2) M is C_{11} if each submodule has at least one complement which is a direct summand.

(3) M is PI**-extending** if each projection invariant submodule is essential in a direct summand of M.

(4) M is **(strongly)** FI**-extending** if each fully invariant submodule is essential in a (fully invariant) direct summand of M.

These conditions range from stronger to weaker:

$$extending \Longrightarrow \mathcal{G}-extending \Longrightarrow C_{11} \Longrightarrow PI-extending$$

$$\Longrightarrow FI-extending \quad and \quad strongly \quad FI-extending \Longrightarrow FI-extending.$$

These implications are, in general, irreversible:

(a) $\mathcal{G}-extending \nRightarrow extending$: The ring $T_2(\mathbb{Z}_4)$ is right $\mathcal{G}-extending$ but not right extending.

(b) $C_{11} \nRightarrow \mathcal{G}-extending$: The ring $T_2(\mathbb{Z})$ is right C_{11} but not right $\mathcal{G}$-extending.

(c) $PI-extending \nRightarrow C_{11}$: This is only a conjecture by the authors at the present time.

(d) $FI-extending \nRightarrow PI-extending$: Every prime ring is strongly FI-extending, however the free ring, $\mathbb{Z} < x_1, x_2, \cdots >$, in 2 or more indeterminates is right strongly FI-extending domain but not right PI-extending.

There are at least two conditions on a module M for which

$$extending \Longleftrightarrow \mathcal{G}-extending \Longleftrightarrow C_{11} \Longleftrightarrow PI-extending:$$

(1) M is indecomposable.

(2) $End(M_R)$ is Abelian (i.e., every idempotent is central) and for each submodule X of M, $X = \sum_{i \in I} h_i(M)$ where $h_i \in End(M_R)$.

At the end of the paper, we briefly discuss a recent paper which investigates the $\mathcal{C}-extending$ condition. This condition targets a designated subset $\mathcal{C}$ of submodules (e.g., the projective submodules) of a module for the extending condition. Moreover, this condition abstracts several of the aforementioned generalized extending conditions.

This survey is not a comprehensive listing of results, examples, and applications of generalized extending properties. Its purpose is to generate further interest in various generalizations of the extending condition. Therefore readers are encouraged to read the papers in the references as well as the forthcoming book [32], which contains a more comprehensive and detailed coverage of this topic.

Let R be a ring and M a right R-module. If $X \subseteq M$, then $X \leq M$, $X \leq^{ess} M$, $X \trianglelefteq M$, $Soc(M)$, $Z(M)$, $Z_2(M)$, $J(M)$, $E(M)$, $\tilde{E}(M)$ and $End(M_R)$ denote X is a submodule of M, X is an essential submodule of M, X is a fully invariant submodule of M, the socle of M, the singular submodule of M, the second singular submodule of M, the Jacobson radical of M, the injective hull of M, the rational hull of M, and the ring of endomorphisms of M, respectively. For R, $T_m(R)$ and $Mat_m(R)$ symbolize the ring of m-by-m upper triangular matrices over R, and the ring of m-by-m matrices over R. $\mathbb{Z}$ and $\mathbb{Z}_m$ will stand for ring of integers and the quotient module $\mathbb{Z}/\mathbb{Z}m$, respectively. A module is called UC if every submodule has a unique closure [28]. A module is called $polyform$ if every essential submodule is dense [20]. A module is called *strongly bounded* if every nonzero submodule contains a nonzero fully invariant submodule [10]. A ring is called *quasi-Baer* if the left annihilator of every ideal is generated by an idempotent of the ring [15]. Other terminology and notation can be found in [24], [20] and [15].

1 Goldie Extending Modules

To the authors there are at least two sources for the motivation of the Goldie extending condition. To understand the first source we need the following definition.

Definition 1.1. Let M be a module. On the set of submodules of M, we define the β relation by $X\beta Y$ if $X \cap Y \leq^{ess} X$ and $X \cap Y \leq^{ess} Y$. Equivalently, $X\beta Y$ if $X \cap A = 0$ implies $Y \cap A = 0$ and $Y \cap B = 0$ implies $X \cap B = 0$, for all $A, B \leq M$. Note β is an equivalence relation.

This relation is defined in [22] for right ideals of a ring and used later in [28].

So the $\mathcal{G}$-extending condition answers the question: Can one combine the β relation and the extending condition in a meaningful and fruitful manner?

Another source is the somewhat strange condition in the characterization of the extending Abelian bounded p-groups as direct sums of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_{p^{n+1}}$. Thus even though $A = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is a finite direct sum of uniform (hence extending) Abelian groups, A is not extending. So one may ask: Is there a generalization of the extending condition which is satisfied by all finitely generated Abelian groups and all bounded p-groups? In the following results we see that the aforementioned question has an affirmative answer in the $\mathcal{G}$-extending condition.

Proposition 1.2. [1, Proposition 1.5] Let M be a module. The following conditions are equivalent.

(i) M is $\mathcal{G}$-extending;

(ii) For each $Y \leq M$, there exists $X \leq M$ and a direct summand D of M such that $X \leq^{ess} Y$ and $X \leq^{ess} D$;

(iii) For each $Y \leq M$ there exists a complement L of Y and a complement K of L such that $Y \beta K$ and every homomorphism $f : K \oplus L \to M$ extends to a homomorphism.

Proposition 1.3. [1, Proposition 1.8(ii)] Let M be a UC-module (e.g., M is nonsingular). Then M is $\mathcal{G}$-extending if and only if M is extending.

The following definition generalizes the notion of a module N being M-injective. This generalization is extremely useful in analyzing the structure of $\mathcal{G}$-extending modules.

Proposition 1.4. [1, Proposition 1.6] Let M be a module and consider the following conditions:

(i) M is extending;

(ii) M is $\mathcal{G}$-extending;

(iii) M has C_{11};

(iv) M is FI-extending.
Then $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. In general, the reverse implications do not hold.

Definition 1.5. [1, Definition 2.1] Let N and M be modules. We say that N is M**-ejective** if, for each $K \leq M$ and each homomorphism $f : K \to N$, there exists a homomorphism $\bar{f} : M \to N$ and a $X \leq^{ess} K$ such that $\bar{f}(x) = f(x)$, for all $x \in X$.

Proposition 1.6. [1, Proposition 2.2] Let ρ be a left exact preradical and $M = M_1 \oplus M_2$, where $\rho(M) \leq^{ess} M_2$. Then M_1 is M_2-ejective.

Theorem 1.7. [1, Theorem 2.7] Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2-ejective if and only if for every $K \leq M$ such that $K \cap M_1 = 0$, there exists $M_3 \leq M$ such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq^{ess} K$.

From Proposition 1.3, the extending and $\mathcal{G}$-extending conditions coincide on nonsingular modules. Since the ring $R = T_2(\mathbb{Z})$ is nonsingular but neither left nor right extending, it is neither left nor right $\mathcal{G}$-extending. However R_R is a direct sum of two uniform submodules. So the class of $\mathcal{G}$-extending modules is not closed under direct sums. Moreover, it is an open question whether or not the class of $\mathcal{G}$-extending modules is closed under direct summands. The next two theorems and corollaries provide some conditions which ensure closure of the $\mathcal{G}$-extending class of modules under direct sums and/or direct summands.

Theorem 1.8. [1, Corollary 3.2] Let $M = \bigoplus_{i=1}^{n} M_i$ be a finite direct sum.

(i) If M_i is M_j-ejective for all $j > i$ and each M_i is $\mathcal{G}$-extending, then M is $\mathcal{G}$-extending.

(ii) If all the M_i are relatively injective and M is $\mathcal{G}$-extending, then each M_i is $\mathcal{G}$-extending.

Theorem 1.9. [1, Theorem 3.7] Let K be a projection invariant submodule of M.

(i) If M is $\mathcal{G}$-extending, there exists $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$ and $K \leq^{ess} M_2$.

(ii) If M is $\mathcal{G}$-extending and K has a unique essential closure, then there exists $M_1, M_2 \leq M$ such that $M = M_1 \oplus M_2$, $K \leq^{ess} M_2$, and M_1 and M_2 are $\mathcal{G}$-extending.

(iii) If $M = M_1 \oplus M_2$, where M_1 and M_2 are $\mathcal{G}$-extending and $\rho(M) \leq^{ess} M_2$, for some left exact preradical ρ, then M is $\mathcal{G}$-extending.

Corollary 1.10. [1, Corollary 3.8] If M is $\mathcal{G}$-extending and $M = \bigoplus_{i \in I} M_i$, where each M_i is projection invariant, then each M_i is $\mathcal{G}$-extending.

Corollary 1.11. [1, Corollary 3.10] Let ρ be the radical for a stable hereditary torsion theory (e.g., $\rho = Z_2$). Then a module M is $\mathcal{G}$-extending if and only if $M = M_1 \oplus M_2$, where M_1 and M_2 are $\mathcal{G}$-extending and $M_2 = \rho(M)$.

The following result characterizes the $\mathcal{G}$-extending Abelian groups. This result has been extended to principal ideal domains and partially extended to Dedekind domains in [2].

Theorem 1.12. [1, Theorem 3.15] The following conditions are equivalent for an Abelian group A:

(i) A is $\mathcal{G}$-extending;

(ii) Every pure subgroup of A is a direct summand;

(iii) $A = D \oplus T \oplus F$, where D is a divisible group, T is a reduced torsion group each of whose p-components, T_p, is a bounded p-group and F is a direct sum of a finite number of copies of a fixed subgroup X of rational numbers.

Thus every finitely generated Abelian group and every bounded Abelian group are $\mathcal{G}$-extending.

Theorem 1.13. [38, Theorem 2.1] Let M be a $\mathcal{G}$-extending module satisfying C_3. Then every direct summand of M is $\mathcal{G}$-extending.

The next theorem and corollary consider conditions which ensure that a $\mathcal{G}$-extending module is a direct sum of uniform modules.

Theorem 1.14. [1, Theorem 4.2] Let R be a ring and let M be an R-module such that R satisfies the ACC on right ideals of the form $r(m)$, where $m \in M$. Then:

(i) If M is $\mathcal{G}$-extending, then for each $m \in M$ such that $r(m)$ is maximal in $\{r(x) \mid 0 \neq x \in M\}$, there exists a primitive idempotent $e \in End(M_R)$ with $mR\beta eM$;

(ii) If every direct summand of M is $\mathcal{G}$-extending, then mR is uniform for each $m \in M$ such that $r(m)$ is maximal in $\{r(x) \mid 0 \neq x \in M\}$. Hence any direct summand of M contains a uniform direct summand;

(iii) If M is $\mathcal{G}$-extending and has SIP or satisfies C_3, then M is a direct sum of uniform submodules.

Corollary 1.15. [1, Corollary 4.4] For any ring R, any locally Noetherian $\mathcal{G}$-extending module which has SIP or satisfies C_3 is a direct sum of uniform submodules.

Our next result and example show that $\mathcal{M}od - R$ has a minimal cogenerator which is $\mathcal{G}$-extending and strongly bounded, but, in general, it is not extending. Recall that a module is *strongly bounded* if every nonzero submodule contains a nonzero fully invariant submodule.

Theorem 1.16. [1, Theorem 4.6 (ii)] Let $\{X_i \mid i \in I\}$ be a set of representatives of the isomorphism classes of all simple R-modules, and $C = \bigoplus_{i \in I} E(X_i)$. Then C is $\mathcal{G}$-extending. Moreover, every minimal cogenerator and every minimal injective cogenerator is strongly bounded.

Example 1.17. [1, Example 4.7] There exists rings R such that C is not extending, where C is as in Theorem 1.16. Osofsky [27] indicates that if C is quasi-injective, then C is the unique (up to isomorphism) minimal cogenerator of $Mod - R$. She then produces several rings for which C is not the unique (up to isomorphism) cogenerator. Hence for such rings C is not quasi-injective. By $[20, Corollary\ 8.10]$, C is not uniform-extending (hence not extending) for these rings.

The remaining results of this section focus on $\mathcal{G}$-extending rings.

Theorem 1.18. [4, Theorem 1] Let S be a right essential overring of R.

(i) If R_R is $\mathcal{G}$-extending, then S_R and S_S are $\mathcal{G}$-extending;

(ii) If S is a subring of $Q(R)$, then S_R is $\mathcal{G}$-extending if and only if S_S is $\mathcal{G}$-extending.

For the remaining results in this section, T denotes the generalized triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$, where R and S are rings and M is an (R, S)-bimodule.

Theorem 1.19. [3, Theorem 2.6] T_T is $\mathcal{G}$-extending if and only if all of the following conditions are satisfied.

1. If D_S is a direct summand of M_S, then there is an $f = f^2 \in R$ such that $D_S \beta f M_S$;
2. M_S and S_S are $\mathcal{G}$-extending;
3. M_S is S_S-ejective;
4. Let $A = \ell_R(M)$. There exists $a = a^2 \in R$ such that $aR = A$ and aR_R is $\mathcal{G}$-extending.

The following corollary generalizes one of the main results of [31, Theorem 3.4]

Corollary 1.20. [3, Corollary 2.7] T_T is nonsingular and extending if and only if all of the following conditions are satisfied.

1. For every complement K in M_S, there exists $f = f^2 \in R$ such that $K = fM$;
2. S_S is nonsingular and extending;
3. M_S is nonsingular and injective;
4. Let $A = \ell_R(M)$. There exists $a = a^2 \in R$ such that $aR = A$ and aR_R is extending;
5. $\{c \in R \mid cJ = 0$, for some $J_S \leq^{ess} M_S\} \cap \{c \in R \mid cK = 0$, for some $K_R \leq^{ess} A_R\} = 0$

Corollary 1.21. [3, Corollary 2.8] Let W be a ring with unity.

1. $T_2(W)$ is right $\mathcal{G}$-extending if and only if W is right G-extending and W_W is W_W-ejective.
2. For $n > 2$, $T_n(W)$ is right $\mathcal{G}$-extending if and only if W is right $\mathcal{G}$-extending and $cT_{n-1}(W)$ is $T_{n-1}(W)$-ejective, where $c \in T_{n-1}(W)$ with 1 in the $(1, 1)$-position and zero elsewhere.
3. If W is right selfinjective, then $T_n(W)$ is right $\mathcal{G}$-extending for all $n > 0$.

Example 1.22. [3, Example 2.9]

(i) Let S be any commutative subdirectly ring. Then S is extending and by $[1, Corollary$ 2.5], S-ejective. Therefore, $T_2(S)$ is right $\mathcal{G}$-extending by Corollary 1.21(1). Since there exists commutative subdirectly irreducible rings that are not selfinjective [1, Example 2.6 (ii)], the converse of Corollary 1.21(3) does not hold.

(ii) There exists selfinjective rings S such that $T_2(S)$ is not right extending. Let $S = D/M^n$, where D is a Dedekind domain and M is a maximal ideal of D. By Corollary 1.21, $T_2(S)$ is right $\mathcal{G}$-extending. For $S = \mathbb{Z}_{p^n}$, $T_2(S)$ is not extending for $n > 1$. To see this, note that the right ideal generated by $\begin{bmatrix} 0 & \bar{q} \\ 0 & \bar{p} \end{bmatrix}$, where q is a prime such that $q \neq p$ and $\bar{q}, \bar{p} \in S$, is not essential in an idempotent generated right ideal of $T_2(S)$. Note that D/M^n is also a commutative subdirectly irreducible ring.

(iii) Let $R = A \oplus E$, where A and E are rings such that A_A is $\mathcal{G}$-extending, E_E is injective, $M = E \trianglelefteq R$, and $S = E$(the ring). Then, by Theorem 1.19, T_T is $\mathcal{G}$-extending and $\ell_R(M) = A$.

Corollary 1.23. [3, Corollary 2.12] Let S be a Dedekind domain that is not a field, M_S a finitely generated torsion module, and R be a subring of $End(M_S)$ such that for each direct summand D_S of M_S, there is an $f = f^2 \in R$ such that $D_S \beta f M_S$ (e.g., $R = End(M_S)$). Then T_T is $\mathcal{G}$-extending, but T_T is not extending.

Our last example of this section makes a connection with Operator theory.

Example 1.24. [3, Example 2.21] Let S be a commutative AW^*-algebra, M_S a simple module, and R the field of complex numbers. Then T is a Banach algebra that is right $\mathcal{G}$-extending. If S is (von Neumann) regular, then M_S is injective (since S is a V-ring); so T_T is extending. In fact, T_T is extending for all simple M in $Mod - S$ if and only if S is regular.

We note that the β relation can be dualized to a β^* relation and this allows a dualization of the $\mathcal{G}$-extending condition to a $\mathcal{G}^*$-lifting condition (equivalently, the H-supplemented condition). For more details see [8].

2 C_{11}-modules

This section is devoted to deal with the class of modules which satisfy the C_{11} property. Recall that C_{11}-modules were defined and developed as a generalization of C_1-modules in [29, 33], and then investigated in [17, 30, 34, 35].

Definition 2.1. A module M satisfies C_{11} if every submodule of M has a complement which is a direct summand of M (i.e., for each submodule N of M there exists a direct summand K of M which is maximal with respect to having zero intersection with N in M).

The next lemma characterizes the C_{11} condition in terms of complement submodules and also shows that any module with C_1 (i.e., extending) satisfies C_{11}.

Lemma 2.2. [29, Proposition 2.3] The following statements are equivalent for a module M.

(i) M has C_{11}.

(ii) For any complement submodule L in M, there exists a direct summand K of M such that K is a complement of L in M.

(iii) For any submodule N of M, there exists a direct summand K of M such that $N \cap K = 0$ and $N \oplus K$ is an essential submodule of M.

(iv) For any complement submodule L in M, there exists a direct summand K of M such that $L \cap K = 0$ and $L \oplus K$ is essential submodule of M.

Any indecomposable module with C_{11} is uniform. The next theorem and corollary show that every module which is a direct sum of uniform modules satisfies C_{11}. In particular, for any prime p, the $\mathbb{Z}$-module $M = (\mathbb{Z}/\mathbb{Z}p) \oplus (\mathbb{Z}/\mathbb{Z}p^3)$ satisfies C_{11}. However M does not satisfy C_1.

Theorem 2.3. [29, Theorem 2.4] Any direct sum of modules with C_{11} satisfies C_{11}.

Corollary 2.4. [29, Corollary 2.6] Any direct sum of uniform modules satisfies C_{11}.

Our next result shows that the study of modules with C_{11} reduces to the case of Goldie torsion modules and nonsingular modules. It is the analogue of [23, Theorem 1].

Theorem 2.5. [29, Theorem 2.7] A module M satisfies C_{11} if and only if $M = Z_2(M) \oplus K$ for some (nonsingular) submodule K of M and $Z_2(M)$ and K both satisfy C_{11}.

The following two results consider nonsingular C_{11}-modules. The first result shows that the study of nonsingular modules satisfying C_{11} reduces to the case of modules with essential socle and modules with zero socle.

Lemma 2.6. [29, Lemma 2.8] Let M be a module which satisfies C_{11}. Then $M = M_1 \oplus M_2$ where M_1 is a submodule of M with essential socle and M_2 a submodule of M with zero socle.

Theorem 2.7. [29, Theorem 2.9] A nonsingular module M satisfies C_{11} if and only if $M = M_1 \oplus M_2$ where M_1 is a module satisfying C_{11} and having essential socle and M_2 is a module satisfying C_{11} and having zero socle.

Theorem 2.5 and 2.7 raise the following natural question: Let M be a module which satisfies C_{11}. Does any direct summand of M satisfy C_{11}? We answer this question negatively by providing the following counterexample. Incidentally, for more examples, refer to [35].

Example 2.8. [30, Example 4] Let $n \geq 3$ be any odd integer. Let $\mathbb{R}$ be the real field and S the polynomial ring $\mathbb{R}[x_1, x_2, \dots, x_n]$. Then the ring R/sS, where $s = (\sum\limits_{i=1}^{n} x_i^2) - 1$, is a commutative Noetherian domain. Let $M_R = \bigoplus\limits_{i=1}^{n} R$ be a module. Then M_R is a C_{11}-module which contains a direct summand that is not a C_{11}-module.

Our next objective is to give some special cases such that the aforementioned question has a positive answer. The first result is based on Abelian groups (i.e., $\mathbb{Z}$-modules).

Theorem 2.9. [29, Theorem 5.5] Let M be a $\mathbb{Z}$-module such that M is a direct sum of uniform modules. Then any direct summand of M is a direct sum of uniform modules.

Proposition 2.10. [29, Proposition 2.11] Let M be a module which satisfies C_{11}. Let N be a direct summand of M such that M/N is an injective module. Then N satisfies C_{11}.

Theorem 2.11. [29, Theorem 4.3] Let M be a module such that M satisfies C_{11} and C_3. Then every direct summand of M satisfies C_{11}.

Proposition 2.12. [17, Lemma 2.1] Let $M = M_1 \oplus M_2$, where $M_1 \trianglelefteq M$. Then M has C_{11} if and only if both M_1 and M_2 have C_{11}.

Theorem 2.13. [30, Theorem 7] Let R be a ring, r a left exact preradical for the category of right R-modules, and M a right R-module such that $r(M)$ has a unique closure in M. Then M is a C_{11}-module if and only if $M = M_1 \oplus M_2$ is a direct sum of C_{11}-modules M_1 and M_2 such that $r(M_1)$ is essential in M_1 and $r(M_2) = 0$.

Note that Theorem 2.9 is a corollary of Theorem 2.13 since the socle is a left exact preradical and nonsingular modules are UC.

Theorem 2.14. [30, Theorem 10] Let $M = M_1 \oplus M_2$ be a C_{11}-module such that for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is a C_{11}-module.

In the rest of this section, we focus on the transference of the C_{11} condition from a given ring or module to various ring or module extensions. To this end, our following results show that if R is a right C_{11}-ring (i.e., R_R is a C_{11}-module), then the ring of column and row finite matrices of size Γ over R, the ring of m-by-m upper triangular matrices over R, and any right essential overring T of R (i.e., T is an essential extension of R as an R-module) are all right C_{11}-rings. For a module M, we obtain that all essential extensions of M satisfying C_{11} are essential extensions of C_{11}-modules constructed from M and certain subsets of idempotents of $End(E(M))$. Moreover, we show that if M is a C_{11}-module, then the rational hull of M is a C_{11}-module.

Theorem 2.15. [17, Theorem 3.1] Let R be a right C_{11}-ring. Then

(i) the ring of column and row finite matrices of size Γ over R is a right C_{11}-ring;

(ii) $End(F_R)$ is a right C_{11}-ring, where F_R is a free right R-module.

Corollary 2.16. [17, Corollary 3.3] Let R be a ring. Then R is a right C_{11}-ring if and only if $T_m(R)$ is a right C_{11}-ring.

Theorem 2.17. [17, Theorem 3.5] If R is a right C_{11}-ring and T is a right essential overring of R, then T is a right C_{11}-ring and T has C_{11} as a right R-module.

Corollary 2.18. [17, Corollary 3.6], [31, Corollary 3.8] If R is a right C_{11}-ring, then $Mat_n(R)$ is a right C_{11}-ring, for each positive integer n.

Proposition 2.19. [17, Proposition 3.9] Let $K \subseteq \{f = f^2 \in End(E(M_R))\}$ such that for each $X \leq M_R$ there exists $k \in K$ such that $kE(M_R)$ is a complement of X in $E(M_R)$. Let $\langle K \rangle$ denote the subring of $End(E(M_R))$ generated by K. Then $\langle K \rangle M_R$ is a C_{11} submodule of $E(M_R)$ which contains M_R.

Proposition 2.20. [17, Proposition 3.11] Assume that $M_R \leq N_R \leq E(M_R)$. If N_R is C_{11}-module, then there exists $K \subseteq \{f = f^2 \in End(E(M_R))\}$ such that for each $X \leq M_R$ there exists $k \in K$ such that $kE(M_R)$ is a complement of X in $E(M_R)$ and $\langle K \rangle M$ is C_{11} submodule of N_R which contains M_R.

Corollary 2.21. [17, Corollary 3.14] Let K be a right R-module. If K has C_{11}, then so does the rational hull, $\tilde{E}(K)$, of K.

In Osofsky [25, 26], the author poses the open question: If $E(R_R)$ has a ring multiplication which extends its right R-module scalar multiplication, must $E(R_R)$ be right self-injective. Our next result shows that if R_R has C_{11} and $E(R_R)$ has such a compatible ring structure, then $E(R_R)_{E(R_R)}$ at least satisfies C_{11}.

Corollary 2.22. [17, Corollary 3.7] Let R be a right C_{11}-ring. If $E(R_R)$ has a ring multiplication which extends its right R-module scalar multiplication, then $E(R_R)$ is a right C_{11}-ring.

Finally we provide necessary and sufficient conditions to make a generalized triangular matrix ring and a trivial extension have the C_{11} property in our following two results.

Theorem 2.23. [17, Theorem 3.2] For rings S and R, assume that ${}_SM_R$ is an (S, R)-bimodule. Let $T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ be the corresponding generalized triangular matrix ring. Then T is a right C_{11}-ring if and only if the following conditions hold.

(i) R is a right C_{11}-ring,

(ii) For any $X_S \leq S_S$ and $N_R \leq M_R$ such that $XM \subseteq N$ there exists $e = e^2 \in S$ such that eM_R is a complement of N_R in M_R and $eS \cap X = 0$; and if Y_S is a complement of X_S in S_S such that $eS \subsetneq Y$, then $YM \nsubseteq eM$.

Proposition 2.24. [17, Proposition 3.4] Assume R is a ring, M is an ideal of R and S = S(R,M) (i.e., the trivial extension of M by R). If ${}_RM$ is faithful and R_R is a C_{11}-module, then S_S is a C_{11}-module.

3 PI-extending Modules

Observe that the invariance of certain submodules with respect to some subset of endomorphisms of a module is useful and often related to the extending property. Recall from [21] a submodule X of M is *projection invariant* if for each $e = e^2 \in End(M_R)$, $eX \subseteq X$. This motivates us to think of extending condition for projection invariant submodules. So we define:

Definition 3.1. A module M is PI-extending if each projection invariant submodule of M is essential in a direct summand of M.

Obviously extending modules are PI-extending. The next result provides a characterization of a PI-extending module in terms of lifting homomorphisms and also shows that C_{11} modules are PI-extending.

Proposition 3.2. [18, Corollary 3.2] Let M be a module. The following conditions are equivalent.

1. M is PI-extending;
2. Each projection invariant submodule X of M has a complement which is a direct summand of M;
3. For each projection invariant submodule X of M, there exists $e = e^2 \in End(E(M))$ such that $X \leq^{ess} e(E(M))$ and $e(M) \leq M$;
4. For each projection invariant submodule X of M, there exists a closed submodule K of M and a complement L of K in M such that $X \leq^{ess} K$ and every homomorphism $f : L \oplus K \to M$ can be lifted to an endomorphism $g : M \to M$.

Proposition 3.3. [18, Proposition 3.8]

(i) Let M be an indecomposable module. Then the following conditions are equivalent.

1. M is uniform,
2. M is extending,
3. M is PI-extending.

(ii) Let M be a module such that $End(M_R)$ is Abelian and $X \leq M$ implies that $X = \sum_{i \in I} h_i(M)$, where each $h_i \in End(M_R)$. Then M is extending if and only if M is PI-extending. In particular, if $M = R$ is Abelian, then R_R is extending if and only if R_R is PI-extending.

(iii) Let M be a distributive module. Then M is extending if and only if M is PI-extending.

From [18, Proposition 3.7], we have that for a module M: $C_{11} \Rightarrow PI - extending \Rightarrow FI - extending$. The next two results show that the PI-extending condition behaves like the C_{11}-condition in terms of direct sums and Goldie torsion submodule, respectively.

Theorem 3.4. [18, Corollary 4.11] Let $M = \bigoplus_{j \in J} M_j$. If each M_j is a PI-extending module, then M is a PI-extending module.

Theorem 3.5. [18, Corollary 4.15] M is PI-extending if and only if $M = M_1 \oplus M_2$ where each M_i is PI-extending and $M_1 = Z_2(M)$.

Our next result provides a characterization of the PI-extending Abelian groups.

Theorem 3.6. [18, Theorem 4.18] Let M be an Abelian group. Then M is PI-extending if and only if $M = D \oplus T \oplus F$, where D is a divisible group, T is a direct sum of seperable p-groups, and F is a torsion free group such that each of its projection invariant pure subgroups is a direct summand.

Note that an indecomposable module is PI-extending if and only if it is uniform. Using Theorem 3.3, Example 2.9 provides a PI-extending module M which has a direct summand D such that D is not PI-extending. On the other hand, D_R is an essential extension of a direct sum of uniform modules which is PI-extending by Theorem 3.3. Hence the class of PI-extending modules is not closed under direct summands and essential extensions. However, the following results focus on some special cases such that being PI-extending is inherited by direct summands.

Proposition 3.7. [18, Corollary 4.14] Let K be a projection invariant submodule of M (resp., $K \trianglelefteq M$) such that K is essentially closed in M. Then M is PI-extending if and only if $M = K \oplus N$ where K and N are PI-extending (resp., FI-extending).

Proposition 3.8. [18, Corollary 4.15] M is PI-extending (resp., FI-extending) if and only if $M = M_1 \oplus M_2$ where each M_i is PI-extending (resp., FI-extending) and $M_1 = Z_2(M)$.

Proposition 3.9. [18, Corollary 4.16] Assume that M is polyform and K is a projection invariant submodule of M. Then M is PI-extending if and only if $M = M_1 \oplus M_2$, where each M_i is PI-extending and $K \leq^{ess} M$.

We conclude this section with the following fact which shows that if a module has PI-extending property so too does its rational hull.

Theorem 3.10. [19, Corollary 2.7] Let M be a module. If M satisfies any of the following conditions, then so does $\tilde{E}(M)$
(1) extending; (2) PI-extending; (3) FI-extending.

4 FI-Extending Modules

An important aspect of the extending condition is that it provides a means to "essentially split-off" certain types of submodules (i.e., the certain types of submodules are essential in direct summands). Since many of the most important types of submodules (e.g., $Soc(M)$, $J(M)$, $Z(M)$, $Z_2(M)$, etc.) are fully invariant, it seems that one may be able to obtain many of the benefits of the extending condition by targeting only the fully invariant submodules with the extending condition as in the definition of the FI-extending condition. Further support for this "targeting" comes from the realization that all preradicals, as well as, all submodules of the form MX where M is a right R-module and X is a right ideal of R are fully invariant. Moreover, the fully invariant submodules of R_R are exactly the ideals of R.

Another motivation for the FI-extending condition is that the class of FI-extending modules is closed under direct sums. However, it is an open question whether this class is closed under direct summands, see [7, p.1414]

In this section, we also consider the class of strongly FI-extending modules. This proper subclass of the class of FI-extending modules includes all polyform (hence nonsingular) FI-extending modules, it is closed under direct summands, and the strongly FI-extending condition is a Morita-invariant. Moreover every semiprime ring has a strongly FI-extending hull [12]

Proposition 4.1. [18, Corollary 3.2] Let M be a module. The following conditions are equivalent.

1. M is FI-extending;
2. Each $X \trianglelefteq M$ has a complement which is a direct summand of M;
3. For each $X \trianglelefteq M$, there exists $e = e^2 \in End(E(M))$ such that $X \leq^{ess} e(E(M))$ and $e(M) \leq M$;
4. For each $X \trianglelefteq M$, there exists a closed submodule K of M and a complement L of K in M such that $X \leq^{ess} K$ and every homomorphism $f : L \oplus K \to M$ can be lifted to an endomorphism $G : M \to M$.

Proposition 4.2. [7, Proposition 1.2] Let M be a module and X a fully invariant submodule of M. If M is FI-extending, then X is FI-extending.

Theorem 4.3. [5, Lemma 1.1], [7, Theorem 1.3] Let $M = \bigoplus_{i \in I} X_i$. If each X_i is an FI-extending module, then M is an FI-extending module.

As mentioned above, the closure of the FI-extending class of modules under direct summands is an open question; however this question has an affirmative answer for FI-extending Abelian groups.

Proposition 4.4. [5, Theorem 3.2] Every direct summand of a group with the FI-extending property enjoys the FI-extending property.

Proposition 4.5. [7, Proposition 1.10] Let $H = End(M_R)$, $e = e^2 \in H$, and $A \trianglelefteq M$ such that $A \leq^{ess} eM$. Then

(i) $(1-e)H(eM) \subseteq Z(M)$;

(ii) $eM + Z(M) \trianglelefteq M$;

(iii) If $Z_2(M) = fM$ for some $f = f^2$, then $eM + Z_2(M) = (e + f - fe)M \trianglelefteq M$ and $e + f - fe \in S_\ell(H)$;

(iv) If $Z(M) \subseteq eM$, then $eM \trianglelefteq M$. Moreover, if $A \leq^{ess} X$, then $X \subseteq eM$. In particular, $Z_2(M) \subseteq eM$.

Lemma 4.6. [5, Lemma 1.2] If the module $M = B \oplus C$ has the FI-extending property and B is a fully invariant direct summand, then both B and C have the FI-extending property.

Our next result generalizes [5, Proposition 3.1] and [7, Propositin 1.11]

Theorem 4.7. [18, Corollary 4.15] M is FI-extending if and only if $M = M_1 \oplus M_2$ where each M_i is FI-extending and $M_1 = Z_2(M)$.

For Abelian groups our next two results characterize the FI-extending torsion group and the torsion-free groups of finite rank.

Theorem 4.8. [5, Theorem 2.3] A torsion group has the FI-extending property if and only if it is a direct sum of a divisible group and separable p-groups.

Theorem 4.9. [5, Theorem 4.1] A torsion-free group whose quasiendomorphism ring is left or right Artinian has the FI-extending property if and only if it is a finite direct sum of irreducible groups.

Proposition 4.10. [10, Proposition 1.5] Let M be a polyform module, then the following conditions are equivalent:

(i) M is FI-extending.

(ii) M is strongly FI-extending.

(iii) Every fully invariant essentially closed submodule of M is a direct summand.

Theorem 4.11. [10, Theorem 2.4] Every direct summand of a strongly FI-extending module is strongly FI-extending.

Unfortunately, the class of strongly FI-extending modules is not closed under direct sums as is indicated in the next example.

Example 4.12. [10, Example 3.1] Let $R = (\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}, \mathbb{Z})$ denote the Dorroh extension of $\begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ by $\mathbb{Z}$. This ring has $Z(R_R) \neq 0$, $Z({}_RR) = 0$, R is strongly right bounded and right FI-extending, but R is neither right extending nor quasi-Baer, nor left FI-extending. Through calculation it can be shown that every proper direct summand is strongly FI-extending, but R is not right FI-extending. For more details see [10, pp. 1846-1847].

The next two results provide conditions for a direct sum of modules to be strongly FI-extending.

Theorem 4.13. [10, Theorem 3.2] Let $M = \bigoplus_{i \in I} N_i$ and let $N_i \trianglelefteq M$ for all i. Then M is strongly FI-extending if and only if N_i is strongly FI-extending for all $i \in I$.

Theorem 4.14. [10, Theorem 3.3] Let $M = \bigoplus_{i \in I} M_i$, where $M_i \cong M_j$, and M_i is strongly FI-extending for all $i, j \in I$. Then M is strongly FI-extending.

The following result and corollary show that the strongly FI-extending property is a Morita invariant.

Theorem 4.15. [10, Theorem 4.2] Let R be a right strongly FI-extending ring. Then for any projective generator P in $Mod - R$, $End(P_R)$ is a right strongly FI-extending.

Corollary 4.16. [10, Corollary 4.3] The right strongly FI-extending property is a Morita invariant.

The remaining results are on right (and/or left) FI-extending rings. Recall a module is *complement bounded* if every nonzero complement contains a nonzero fully invariant submodule.

Theorem 4.17. [7, Theorem 4.7] Consider the following conditions on a ring R.

a) R_R is FI-extending;

b) ${}_RR$ is FI-extending;

c) R is quasi-Baer;

d) every ideal is right (left) essential in a direct summand;

e) every ideal which is right (left) essentially closed is a direct summand;

f) R_R is quasi-extending;

g) R_R is extending;

h) R is Baer.

The following statements hold true for R:

(i) If R is semiprime, then a) through f) are equivalent.

(ii) If R is semiprime and R_R is complement bounded, then a) through g) are equivalent.

(iii) If R_R is nonsingular and complement bounded, then a) through h) are equivalent.

Proposition 4.18. [7, Proposition 2.3] If R is right FI-extending, then $Mat_n(R)$ is right FI-extending, for all positive integers n.

Theorem 4.19. [6, Theorem 1.16] Let S and R be rings, M an (S, R)-bimodule, and $T = \begin{bmatrix} S & M \\ 0 & R \end{bmatrix}$ the 2×2 generalized triangular matrix ring. Then T_T is FI-extending if and only if all of the following conditions hold.

(i) $\ell_S(M) = eS$, where $e \in S_\ell(S)$, and eS_S is FI-extending;

(ii) For ${}_SN_R \leq_S M_R$, there is $f = f^2 \in S$ such that $N_R \leq^{ess} fM_R$;

(iii) R_R is FI-extending.

For other characterizations of generalized triangular matrix rings which are FI-extending, strongly FI-extending, or quasi-Baer see [16].

From Theorem 4.17, one can see that the next result on group algebras applies to (strongly) FI-extending rings.

Theorem 4.20. [9, Proposition 1.7] Let $R = F[G]$ be a semiprime group algebra over a field F. Then R is quasi-Baer if and only if each annihilator ideal is finitely generated.

Corollary 4.21. Let $R = F[G]$ be a semiprime group algebra where G is Abelian. Then R is extending if and only if each annihilator ideal is finitely generated.

In a series of papers [11], [14], [12], and [13] a theory of ring and module hulls is developed. In [12, Theorem 3.3], it is shown that every semiprime ring has a (strongly) FI-extending hull. In [13], this result is extended to finitely generated projective modules over a semiprime ring. In [14] and [12] applications are made to C^*-algebras. Some further results on the FI-extending condition appear in [37].

Finally we mention that many of the basic results in this paper have been generalized to modules satisfying the C-extending condition in [18]. Also from [17, Corollary 3.14] and [19, Corollary 2.8], we have that if a module M satisfies any of the conditions: extending, $\mathcal{G}$-extending, C_{11}, PI-extending, FI-extending, then so does its rational hull.

OPEN QUESTIONS AND PROBLEMS

1. Find necessary and sufficient conditions to characterize (finite) direct sums of $\mathcal{G}$-extending or strongly FI-extending modules, respectively.

2. Find necessary and sufficient conditions to characterize when direct summands of $\mathcal{G}$-extending, C_{11}, PI-extending, or FI-extending modules are $\mathcal{G}$-extending, C_{11}, PI-extending, or FI-extending, respectively.

3. Characterize when a generalized extending module (i.e., a module which is $\mathcal{G}$-extending, C_{11}, PI-extending, FI-extending, or strongly FI-extending) is a (finite) direct sum of uniform modules.
4. When does a module have a generalized extending hull?
5. How do generalized extending modules behave with respect to the tensor product or Hom functors?
6. Investigate how generalized extending conditions interact with torsion theory.
7. Characterize the rings such that every (cyclic, finitely generated, singular, etc.) module is a generalized extending module for some fixed generalized extending condition.
8. Characterize when $R[x]$ is a generalized extending ring.
9. Determine necessary and/or sufficient conditions when the homomorphic image of a module is generalized extending.
10. Determine necessary and/or sufficient conditions when a submodule is generalized extending.

References

[1] Evrim Akalan, Gary F. Birkenmeier, and Adnan Tercan. Goldie extending modules. *Comm. Algebra*, 37(2):663–683, 2009. Corrigendum, 38(2010), 4747-4748. Corrigendum, 41(2013), 2005.

[2] Evrim Akalan, Gary F. Birkenmeier, and Adnan Tercan. A characterization of Goldie extending modules over Dedekind domains. *J. Algebra and Its App.*, 10(06):1291–1299, 2011.

[3] Evrim Akalan, Gary F. Birkenmeier, and Adnan Tercan. Characterizations of extending and $\mathcal{G}$-extending generalized triangular matrix rings. *Comm. Algebra*, 40(3):1069–1085, 2012.

[4] Evrim Akalan, Gary F. Birkenmeier, and Adnan Tercan. Goldie extending rings. *Comm. Algebra*, 40(2):423–428, 2012. Corrigendum (2014), to appear.

[5] Gary F. Birkenmeier, Girgore Călugăreanu, Laszlo Fuchs, and H. Pat Goeters. The fully invariant extending property for abelian groups. *Comm. Algebra*, 29:673–685, 2001.

[6] Gary F. Birkenmeier and Matthew J. Lennon. Extending sets of idempotents to ring extensions. *Comm. Algebra*, 42(12):5134–5151, 2014.

[7] Gary F. Birkenmeier, Bruno J. Müller, and S. Tariq Rizvi. Modules in which every fully invariant submodule is essential in a direct summand. *Comm. Algebra*, 30(3):1395–1415, 2002.

[8] Gary F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan. Goldie*-supplemented modules. *Glasgow Math. J.*, 52(A):41–52, 2010. Corrigendum, 54(2012), 479-480.

[9] Gary F. Birkenmeier and Jae Keol Park. Triangular matrix representations of ring extensions. *J. Algebra*, 265(2):457–477, 2003.

[10] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. Modules with fully invariant submodules essential in fully invariant summands. *Comm. Algebra*, 30:1833–1852, 2002.

[11] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. Ring hulls and applications. *J. Algebra*, 304(2):633–665, 2006.

[12] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. Hulls of semiprime rings with applications to C*-algebras. *J. Algebra*, 322(2):327–352, 2009.

[13] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. Modules with FI-extending hulls. *Glasgow Math. J.*, 51(02):347–357, 2009.

[14] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. The structure of rings of quotients. *J. Algebra*, 321(9):2545–2566, 2009.

[15] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. *Extensions of Rings and Modules*. Springer New York, 2013.

[16] Gary F. Birkenmeier, Jae Keol Park, and S. Tariq Rizvi. Generalized triangular matrix rings and the fully invariant extending property. *Rocky Mount. J. of Math.*, 32(4):1299–1319, 2002.

[17] Gary F. Birkenmeier and Adnan Tercan. When some complement of a submodule is a summand. *Comm. Algebra*, 35(2):597–611, 2007. Addendum and Corrigendum.

[18] Gary F. Birkenmeier, Adnan Tercan, and Canan C. Yücel. The extending condition relative to sets of submodules. *Comm. Algebra*, 42(2):764–778, 2014.

[19] Gary F. Birkenmeier, Adnan Tercan, and Canan C. Yücel. Essential extensions with the $\mathcal{C}$-extending property. Submitted.

[20] Nguyen Viet Dung, Dinh Van Huynh, Patrick F. Smith, and Robert Wisbauer. *Extending modules*, volume 313. CRC Press, 1994.

[21] László Fuchs. *Infinite Abelian Groups*, volume 1. Academic press, 1970.

[22] Alfred W. Goldie. Semi-prime rings with maximum condition. *Proc. of the London Math. Soc.*, 3(1):201–220, 1960.

[23] Mahmoud A. Kamal and Bruno J. Müller. Extending modules over commutative domains. *Osaka J. of Math.*, 25(3):531–538, 1988.

[24] Saad H. Mohamed and Bruno J.M. Müller. *Continuous and discrete modules*. Number 147. Cambridge Univ. Press, 1990.

[25] Barbara L. Osofsky. On ring properties of injective hulls. *Canad. Math. Bull.*, 7(3):405–413, 1964.

[26] Barbara L. Osofsky. *Homological properties of rings and modules*. PhD thesis, 1964 Univ. of Rutgers.

[27] Barbara L. Osofsky. Minimal cogenerators need not be unique. *Comm. Algebra*, 19(7):2071–2080, 1991.

[28] P.F. Smith. Modules for which every submodule has a unique closure. *Ring theory*, pages 302–313, 1993.

[29] P.F. Smith and A. Tercan. Generalizations of CS-modules. *Comm. Algebra*, 21(6):1809–1847, 1993.

[30] P.F. Smith and A. Tercan. Direct summands of modules which satisfy (C_{11}). *Algebra Colloq.*, 11:231–237, 2004.

[31] A. Tercan. On certain CS-rings. *Comm. Algebra*, 23(2):405–419, 1995.

[32] A. Tercan and C.C. Yücel. *Module Theory, Extending Modules and Generalizations*. (Birkhauser, Basel, Frontiers in Mathematics, Monograph Series). To appear (2015).

[33] Adnan Tercan. *Generalizations of CS-modules*. PhD thesis, 1992 Univ. of Glasgow.

[34] Adnan Tercan. Eventually weak (C_{11}) modules and matrix (C_{11}) rings. *Southeast Asian Bull. of Math.*, 27(4), 2003.

[35] Adnan Tercan. Weak (C_{11}) modules and algebraic topology type examples. *Rocky Mount. J. of Math.*, 34(2), 2004.

[36] Yuzo Utumi. On continuous rings and self injective rings. *Trans. of the Amer. Math. Soc.*, 118:158–173, 1965.

[37] Xiao-Rong Wang and Jian-Long Chen. On FI-extending rings and modules. *Northeast. Math. J.*, 24(1):77–84, 2008.

[38] Dejun Wu and Yongduo Wang. Two open questions on Goldie extending modules. *Comm. Algebra*, 40(8):2685–2692, 2012.

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Received : September 15, 2014.

Accepted: September 23, 2014.