An Interesting $q$-Continued Fractions of Ramanujan

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Abstract. In this paper, we establish an interesting $q$-identity and an integral representation of a $q$-continued fraction of Ramanujan. We also compute explicit evaluation of this continued fraction and derive its relation with Ramanujan Göllnitz -Gordon continued fraction.

1 Introduction

Ramanujan a pioneer in the theory of continued fraction has recorded several in the process rediscovered few continued fractions found earlier by Gauss, Eisenstein and Rogers in his notebook [10]. In fact Chapter 12 and Chapter 16 of his Second Notebook [10] is devoted to continued fractions. Proofs of these continued fractions over years are given by several mathematician, we mention here specially G.E. Andrews[3], C. Adiga, S. Bhargava and G.N. Watson [1] whose works have been compiled in [4] and [5].

The celebrated Roger Ramanujan continued fraction is defined by

$$R(q) := \frac{1}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}}}, \quad |q| < 1.$$  

On page 365 of his lost notebook [11], Ramanujan recorded five modular equations relating $R(q)$ with $R(-q), R(q^2), R(q^3), R(q^4)$ and $R(q^6)$.

The well known Ramanujan’s cubic continued fraction defined by

$$J(q) := \frac{q^{1/3}}{1 + q + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \ldots}}}}, \quad |q| < 1.$$  

is recorded on page 366 of his lost notebook [11]. Several new modular equation relating $J(q)$ with $J(-q), J(q^2)$ and $J(q^3)$ are established by H.H. Chan [8].

Similarly the Ramanujan Göllnitz-Gordon continued fraction $K(q)$ defined by

$$K(q) := \frac{q^{1/2}}{1 + q + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \ldots}}}}, \quad |q| < 1,$$

satisfies several beautiful modular relations. One may see traces of modular equation related to $K(q)$ on page 229 of Ramanujan’s lost notebook [11]. Further works related to $K(q)$ in recent years have been done by various authors including Chan and S.S Huang [9] and K.R. Vasuki and B.R. Srivatsa Kumar [12].

Motivated by these works in this paper we study the Ramanujan continued fraction

$$M(q) := \frac{q^{1/2}}{1 - q + \frac{q(1 - q^2)}{1 - (1 - q)(1 + q^2)} + \frac{q(1 - q^3)^2}{1 - (1 - q)(1 + q^3)} + \frac{q(1 - q^5)^2}{1 - (1 - q)(1 + q^5)} + \ldots, |q| < 1}$$

$$= q^{1/2} \frac{(q^4 - q^4 \infty)^2}{(q^2 - q^4 \infty)^2}. \tag{1.1}$$

In Chapter 16 Entry 12 of [5], Ramanujan has recorded the following continued fraction

$$\frac{(a^2q^3; q^4 \infty)(b^2q^3; q^4 \infty)}{(a^2q^3; q^4 \infty)(b^2q^3; q^4 \infty)} = \frac{1}{1 - ab} + \frac{(a - bq)(b - aq)}{(1 - ab)(1 + q^2)} + \frac{(a - bq^3)(b - aq^3)}{(1 - ab)(1 + q^3)} + \ldots, \quad |ab| < 1, |q| < 1. \tag{1.2}$$
In fact setting \( a = q^{1/2} \) and \( b = q^{1/2} \) in (1.2), we obtain (1.1).

In Section 2 we obtain an interesting \( q \)-identity related to \( M(q) \) using Ramanujan’s \( \psi \) summation formula [5, Ch. 16, Entry 17]

\[
\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_{\infty}(q/a)_{\infty}(b/a)_{\infty}}{(z)_{\infty}(b/az)_{\infty}(b/q)_{\infty}}, \quad |b/a| < |z| < 1, \tag{1.3}
\]

and Andrew’s identity [4, p. 57],

\[
\sum_{n=0}^{\infty} \frac{q^{kn}}{1 - q^{kn+k}} = \sum_{n=0}^{\infty} q^{n^2+2kn} - q^{n^{2}+k} = \frac{1 + q^{n+k}}{1 - q^{n+k}}. \tag{1.4}
\]

In Section 3 we obtain several relation of \( M(q) \) with theta function \( \varphi(q) \), \( \psi(q) \) and \( \chi(q) \). In Section 4 we obtain an integral representation of \( M(q) \). In Section 5 we derive a formula that help us to obtain relation among \( M(q^{1/2}) \), \( M(q) \), \( M(q^2) \) and \( M(q^4) \). We establish explicit formulas for the evaluation of \( \frac{M(e^{-\pi \sqrt{17}/2})}{M(e^{-\pi \sqrt{19}/2})} \) in Section 6.

We conclude this introduction with few customary definition we make use in the sequel. For \( a \) and \( q \) complex number with \( |q| < 1 \)

\[(a)_\infty := (a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) \]

and

\[(a)_{n} := (a; q)_{n} = \prod_{k=0}^{n-1} (1 - aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n : \text{any integer}. \]

\[
f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} q^{n(n-1)/2} \tag{1.5}\]

\[= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad |ab| < 1. \tag{1.6}\]

Identity (1.6) is the Jacobi’s triple product identity in Ramanujan’s notation [5, Ch. 16, Entry 19]. It follows from (1.5) and (1.6) that [5, Ch. 16, Entry 22],

\[
\varphi(q) := f(q; q) = \sum_{n=-\infty}^{\infty} q^n = \frac{(-q; q)_\infty}{(q; q)_\infty}, \tag{1.7}
\]

\[
\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.8}
\]

and

\[
\chi(q) := (-q; q^2)_\infty. \tag{1.9}
\]

2 \textbf{-} \textit{q-Identity related to } \textbf{M(q)}

\textbf{Theorem 2.1}

\[M(q) = \sum_{n=0}^{\infty} q^{n(8n+4)+1/2} \frac{1 + q^{8n+2}}{1 - q^{8n+2}} = \sum_{n=0}^{\infty} q^{n(8n+4)+1/2} \frac{1 + q^{8n+6}}{1 - q^{8n+6}} \tag{2.1}\]

\textbf{Proof:} Changing \( q \) to \( q^8 \), then setting \( a = q^2 \), \( b = q^{10} \) and \( z = q^2 \) in \( \psi \) summation formula (1.3) we obtain

\[
\frac{(q^4; q^4)_\infty}{(q^2; q^2)_\infty} = \sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{8n+2}} - \sum_{n=0}^{\infty} \frac{q^{8n+4}}{1 - q^{8n+6}} \tag{2.2}
\]
employing Andrews identity (1.4) with \( k = 2, l = 8 \) and \( k = 6, l = 8 \) in both the summations in right side of the identity (2.2) respectively and finally multiplying both sides of the resulting identity with \( q^{1/2} \) and using product represtation of \( M(q) \) (1.1), we complete the proof of Theorem 2.1.

### 3 Some Identities involving \( M(q) \)

We obtain relation of \( M(q) \) in terms of theta function \( \varphi(q) \), \( \psi(q) \) and \( \chi(q) \).

**Theorem 3.1**

\[
M(q) = q^{1/2} \frac{\psi^4(q)}{\varphi^2(q)},
\]

(3.1)

\[
8M(q^2) = \varphi^2(q) - \varphi^2(-q),
\]

(3.2)

\[
16M^2(q) = \varphi^4(q) - \varphi^4(-q),
\]

(3.3)

\[
\frac{M^2(q)}{M(q^2)} = \varphi^2(q^2),
\]

(3.4)

\[
4M(q^2) = \varphi^2(q) - \varphi^2(q^2),
\]

(3.5)

\[
\frac{M^{-1}(q) + M(q)}{M^{-1}(q) - M(q)} = \frac{1 + q\psi^4(q^2)}{1 - q\psi^4(q^2)},
\]

(3.6)

\[
8M(q^2) = \frac{\chi^2(q)}{\chi^2(-q)} \phi^2(-q^2) - \phi^2(-q).
\]

(3.7)

**Proof:** Using [5, Ch. 16, Entry 22(ii)] in (1.1), we obtain

\[
M(q) = q^{1/2} \psi^2(q^2).
\]

(3.8)

Employing [5, Ch. 16, Entry 25(iv)] in (3.8), we obtain (3.1). From (3.1) we have

\[
M(q^2) = q \frac{\psi^4(q^2)}{\varphi^2(q^2)}.
\]

(3.9)

Employing [5, Ch. 16, Entry 25(vii)] and [5, Ch. 16, Entry 25(vi)] in (3.9), we obtain (3.2). Identity (3.3) immediately follows from (3.8) and [5, Ch. 16, Entry 25(vii)]. Again from (3.1), we have

\[
\frac{M^2(q)}{M(q^2)} = \frac{\psi^2(q)\varphi^2(q^2)}{\psi^2(q^2)\psi^2(q)},
\]

(3.10)

employing [5, Ch. 16, Entry 25(iv)] in the identity (3.10) we obtain (3.4). From (3.2) and (3.3), we have

\[
64M^2(q^2) + 16M^2(q) = 16\varphi^2(q)M(q^2),
\]

(3.11)

dividing the identity (3.11) throughout by \( 16M(q^2) \) and using (3.4) we obtain (3.5). From (3.1) we deduce that

\[
M^{-1}(q) + M(q) = \frac{\varphi^4(q) + q\psi^8(q)}{q^{1/2}\varphi^2(q)\psi^4(q)},
\]

(3.12)

and

\[
M^{-1}(q) - M(q) = \frac{\varphi^4(q) - q\psi^8(q)}{q^{1/2}\varphi^2(q)\psi^4(q)}.
\]

(3.13)

On dividing (3.12) by (3.13) and using [5, Ch. 16, Entry 25(iv)] in the resulting identity, we complete the proof of (3.6). From (1.7) and (1.9) we have

\[
\varphi(-q) + \frac{\chi(q)}{\chi(-q)} \varphi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty} \left[ 1 + \frac{f(q, q)}{f(-q, -q)} \right],
\]
employing [5, Ch. 16, Entry 30(ii)] in right hand side of above identity we obtain
\[\varphi(-q) + \frac{\chi(q)}{\lambda(-q)} \varphi(-q^2) = \frac{2(q^8; q^8)_\infty (q^{32}; q^{32})_\infty}{(q^4; q^4)_\infty, (q^{16}; q^{16})_\infty, (q^{64}; q^{64})_\infty} \frac{M(q^{16})}{q^8}. \tag{3.14}\]

Again from (1.7) and (1.9) we have
\[\varphi(-q) - \frac{\chi(q)}{\lambda(-q)} \varphi(-q^2) = \frac{(q; q)_\infty}{(-q; q)_\infty} \left[ 1 - \frac{f(q, q)}{f(-q, -q)} \right], \]
employing [5, Ch. 16, Entry 30(ii)] and [5, Ch. 16, Entry 18(ii)] in right hand side of above identity we obtain
\[\varphi(-q) - \frac{\chi(q)}{\lambda(-q)} \varphi(-q^2) = \frac{-4(q; q)_\infty (q^{64}; q^{64})_\infty}{(-q^{16}; q^{32})_\infty} \frac{q^8}{M(q^{16})}. \tag{3.15}\]

Multiplying (3.14) and (3.15) we complete the proof of (3.7).

**Theorem 3.2.** Let \(u = M(q)\), \(v = M(-q)\) and \(w = M(q^2)\), then
\[u^2 - v^2 = 8w^2.\]

**Proof:** On substituting (3.4) in (3.5), we obtain
\[\varphi^2(q) = \frac{4M^2(q^2) + M^2(q)}{M(q^2)}. \tag{3.16}\]
Changing \(q\) to \(-q\) in (3.16), we have
\[\varphi^2(-q) = \frac{4M^2(q^2) + M^2(-q)}{M(q^2)}. \tag{3.17}\]
Subtracting (3.17) from (3.16) and using identity (3.2), we complete the proof of Theorem 3.2.

### 4 Integral Representation of \(M(q)\)

**Theorem 4.1.** For \(0 < |q| < 1\),
\[M(q) = \exp \int \left( \frac{1}{2q} + \frac{4}{q} \left[ \frac{\varphi^4(-q) - 1}{8} + \frac{q \varphi'(q)}{2 \varphi(q)} \right] \right) dq, \tag{4.1}\]
where \(\varphi(q)\) and \(\psi(q)\) are as defined in (1.7) and (1.8).

**Proof:** Taking log on both sides of (3.1), we have
\[\log M(q) = \frac{1}{2} \log q + 4 \log \psi(q) - 2 \log \varphi(q). \tag{4.2}\]
Employing [5, Ch. 16, Entry 23(ii)] and [5, Ch. 16, Entry 23(i)] on right hand side of (4.2), we obtain
\[\log M(q) = \frac{1}{2} \log q + 4 \sum_{n=1}^{\infty} \frac{q^{2n}}{2n(1 + q^{2n})}. \tag{4.3}\]
Differentiating (4.3) and simplifying, we have
\[\frac{d}{dq} \log M(q) = \frac{1}{2q} + \frac{4}{q} \left[ \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1 + q^{2n}} \right] \tag{4.4}\]
Using Jacobi’s identity [5, Ch. 16, Identity 33.5, p. 54] and [5, Ch. 16, Entry 23(i)] and integrating both sides and finally exponentiating both sides of identity (4.4), we complete the proof of Theorem 4.1.
5 Modular Equation of Degree \( n \) and Relation Between \( M(q) \) and \( M(q^n) \)

In the terminology of hypergeometric function, a modular equation of degree \( n \) is a relation between \( \alpha \) and \( \beta \) that is induced by

\[
n \frac{\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k}{\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k} = \frac{\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k}{\sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k},
\]

where

\[
2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k,
\]

and

\[
(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)}.
\]

Let \( Z_1(r) = 2F_1(1/r, r - 1/r; 1; \alpha) \) and \( Z_n(r) = 2F_1(1/r, r - 1/r; 1; \beta) \), where \( n \) is the degree of the modular equation. The multiplier \( m(r) \) is defined by the equation

\[
m(r) = \frac{Z_1(r)}{Z_n(r)}.
\]

**Theorem 5.1.** If

\[
q = \exp \left( -\pi \frac{2F_1(1/2, 1/2; 1; 1 - \alpha)}{2F_1(1/2, 1/2; 1; \alpha)} \right), \tag{5.1}
\]

then

\[
\alpha = 16 \frac{M^4(q)}{M^4(q^{1/2})} \tag{5.2}
\]

**Proof:** From (1.1) and (1.7), we have

\[
M(q)\varphi^2(q) = q^{1/2} \frac{(q^4; q^4)_{\infty}^2}{(q^2; q^2)_{\infty}^2} \frac{(-q; q^2)_{\infty}^2(q^4)^2}{(-q^2; q^2)_{\infty}^2(q^2)^2} = M^2(q^{1/2}). \tag{5.3}
\]

Substituting (5.3) in (3.3), we obtain

\[
16M^2(q) = \frac{M^4(q^{1/2})}{M^2(q)} \left[ 1 - \frac{\varphi^4(-q)}{\varphi(q)} \right]. \tag{5.4}
\]

From a known identity [5, Ch. 16, p. 100, Entry 5] and (5.1) it is implied that

\[
\alpha = 1 - \frac{\varphi^4(-q)}{\varphi(q)} \tag{5.5}
\]

Using (5.5) in (5.4), we complete the proof of (5.2).

Let \( \alpha \) and \( \beta \) be related by (5.1). If \( \beta \) has degree \( n \) over \( \alpha \) then from Theorem 5.1, we obtain

\[
\beta = 16 \frac{M^4(q^n)}{M^4(q^{n/2})}, \tag{5.6}
\]

**Corollary 5.2.** Let \( u = M(q^{1/2}), v = M(q), \ w = M(q^2) \) and \( x = M(q^4) \), then

\[
16x^4v^2 + 32x^3wv^2 - 4x^3wv^4 + 24x^2w^2v^2 + 8xw^3v^2 - xw^3u^4 + w^4v^2 = 0. \tag{5.7}
\]

**Proof:** From [5, Entry 24(v), p. 216], we have

\[
\sqrt{1 - \alpha} = \left( \frac{1 - \beta^{1/4}}{1 + \beta^{1/4}} \right)^2. \tag{5.8}
\]
On using (5.6) with \( n = 4 \) and (5.2) in (5.8), we obtain
\[
\sqrt{\frac{u^4 - 16v^4}{u^4}} = \left( \frac{w - 2x}{w + 2x} \right)^2.
\]  
(5.9)

Squaring both side of (5.9) and then simplifying, we obtain (5.7).

6 Evaluations of \( M(q) \)

As an application of Theorem 5.1, we establish few explicit evaluation of \( M(q) \).

Let \( q_n = e^{-\pi \sqrt{n}} \) and let \( \alpha_n \) denote the corresponding value of \( \alpha \) in (5.1). Then by Theorem 5.1, we have
\[
\frac{M(e^{-\pi \sqrt{n}})}{M(e^{-\pi \sqrt{n}/2})} = \frac{1}{2} \alpha_n^{1/4}.
\]  
(6.1)

From [5, Ch. 17, p. 97], we have \( \alpha_1 = \frac{1}{2}, \alpha_2 = (\sqrt{2} - 1)^2 \) and \( \alpha_4 = (\sqrt{2} - 1)^2 \).

Thus from (6.1), it immediately follows
\[
\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left( \frac{1}{2} \right)^{5/4},
\]  
(6.2)
\[
\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2} \sqrt{\sqrt{2} - 1},
\]  
(6.3)
\[
\frac{M(e^{-2\pi})}{M(e^{-\pi})} = \frac{\sqrt{2} - 1}{2}.
\]  
(6.4)

Ramanujan has recorded several modular equation in his notebook [10, p. 204-237] and [10, p. 156-160] which are very useful in the computation of class invariants and the values of theta function. Ramanujan has also recorded several values of theta function \( \varphi(q) \) and \( \psi(q) \) in his notebook. For example
\[
\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)};
\]  
(6.5)
\[
\psi(e^{-\pi}) = 2^{-5/8} e^{\pi/8} \frac{\pi^{1/4}}{\Gamma(3/4)},
\]  
(6.6)
\[
\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt{6\sqrt{3} - 9}.
\]  
(6.7)

From (3.8) and (6.6), we have
\[
M(e^{-\pi/2}) = 2^{-5/4} \frac{\sqrt{\pi}}{\Gamma^2(3/4)},
\]  
(6.8)

Using (6.8) and (6.2), we obtain
\[
M(e^{-\pi}) = \frac{\sqrt{\pi}}{\Gamma^2(3/2)}.
\]  
(6.9)

Setting (6.9) in (6.4), we obtain
\[
M(e^{-2\pi}) = \frac{\sqrt{2} - 1}{2} \frac{\sqrt{\pi}}{\Gamma^2(3/2)}.
\]  
(6.10)

J.M. Borwein and P.B. Borwein [7] are the first to observe that class invariant could be used
to evaluated certain values of $\varphi(e^{-n\pi})$. The Ramanujan Weber class invariants are defined by

$$G_n := 2^{-1/4} q_n^{-1/24} (-q_n; q_n^2)_{\infty}$$

and

$$g_n := 2^{-1/4} q_n^{-1/24} (q_n; q_n^2)_{\infty}, \quad (6.11)$$

where $q_n = e^{-\pi \sqrt{n}}$. Chan and Huang has derived few explicit formulas for evaluating $K(e^{-\pi \sqrt{n}/2})$ in the terms of Ramanujan Weber class. Similar works are done by Adiga et., al. Analogous to these works we obtain explicit formulas to evaluate $M(e^{-\pi \sqrt{n}})$.

**Theorem 6.1.** For Ramanujan Weber class invariant defined as in (6.11), let $p = G_n^{12}$ and $p_1 = g_n^{12}$, then

$$\frac{M(e^{-\pi \sqrt{n}})}{M(e^{-\pi \sqrt{n}/2})} = \frac{1}{2} \frac{1}{\sqrt{p(p+1) + \sqrt{p(p-1)}}}, \quad (6.12)$$

$$\frac{M(e^{-\pi \sqrt{n}})}{M(e^{-\pi \sqrt{n}/2})} = \frac{1}{2} \sqrt{\frac{p_1^2 + 1 - p_1}{2}}. \quad (6.13)$$

**Proof:** From [9], we have

$$G_n = [4\alpha_n(1 - \alpha_n)]^{-1/24}.$$

Hence

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)} + \sqrt{p(p-1)})^2}. \quad (6.14)$$

Using (6.14) in (6.1), we obtain (6.12).

Also from [9], we have

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}.$$

Hence

$$\sqrt{\alpha_n} = \sqrt{(p_1^2 + 1) - p_1}. \quad (6.15)$$

Using (6.15) in (6.1), we complete the proof of (6.13).

**Example:** Let $n = 1$. Since $G_1 = 1$, from Theorem 6.1 we have

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left(\frac{1}{2}\right)^{5/4}.$$ 

Let $n = 2$. Since $g_2 = 1$, from Theorem 6.1 we have

$$\frac{M(e^{-\sqrt{\pi}n})}{M(e^{-\sqrt{\pi}/2}n)} = \frac{1}{2} \sqrt{\sqrt{2} - 1}.$$ 

**Remark:** Using [10, p. 229] it is easily verified that $M(q)$ and $K(q)$ are related by the equation

$$M(q^2)K(q) + K(q)M(q) - M(q^2) = 0.$$ 

**References**


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