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An Interesting q-Continued Fractions of Ramanujan

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Abstract. In this paper, we establish an interesting q-identity and an integral representation of a q-continued fraction of Ramanujan. We also compute explicit evaluation of this continued fraction and derive its relation with Ramanujan Göllnitz -Gordon continued fraction.

1 Introduction

Ramanujan a pioneer in the theory of continued fraction has recorded several in the process rediscovered few continued fractions found earlier by Gauss, Eisenstein and Rogers in his notebook [10]. In fact Chapter 12 and Chapter 16 of his Second Notebook [10] is devoted to continued fractions. Proofs of these continued fractions over years are given by several mathematician, we mention here specially G.E. Andrews[3], C. Adiga, S. Bhargava and G.N. Watson [1] whose works have been compiled in [4] and [5].

The celebrated Roger Ramanujan continued fraction is defined by

$$R(q) := \frac{1}{1} \, + \, \frac{q}{1} \, + \, \frac{q^2}{1} \, + \, \frac{q^3}{1} \, + \cdots, \qquad |q| < 1.$$

On page 365 of his lost notebook [11], Ramanujan recorded five modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$ $R(q^4)$ and $R(q^5)$.

The well known Ramanujan's cubic continued fraction defined by

$$J(q) := \frac{q^{1/3}}{1} \, + \, \frac{q+q^2}{1} \, + \, \frac{q^2+q^4}{1} \, + \, \frac{q^3+q^6}{1} \, + \cdots, \qquad |q| < 1.$$

is recorded on page 366 of his lost notebook [11]. Several new modular equation relating $J(q)$ with $J(-q)$, $J(q^2)$ and $J(q^3)$ are established by H.H. Chan [8].

Similarly the Ramanujan Göllnitz-Gordon continued fraction K(q) defined by

$$K(q) := \frac{q^{1/2}}{1+q} \, + \, \frac{q^2}{1+q^3} \, + \, \frac{q^4}{1+q^5} \, + \, \frac{q^6}{1+q^7} \, + \cdots, \qquad |q| < 1,$$

satisfies several beautiful modular relations. One may see traces of modular equation related to $K(q)$ on page 229 of Ramanujan's lost notebook [11]. Further works related to $K(q)$ in recent years have been done by various authors including Chan and S.S Huang [9] and K.R. Vasuki and B.R. Srivatsa Kumar [12].

Motivated by these works in this paper we study the Ramanujan continued fraction

$$\begin{aligned} M(q) \;&:=\; \frac{q^{1/2}}{1-q} \, + \, \frac{q(1-q)^2}{(1-q)(1+q^2)} \, + \, \frac{q(1-q^3)^2}{(1-q)(1+q^4)} \, + \, \frac{q(1-q^5)^2}{(1-q)(1+q^6)} \, + \cdots, |q| < 1 \\ &=\; q^{1/2}\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}. \end{aligned} \tag{1.1}$$

In Chapter 16 Entry 12 of [5], Ramanujan has recorded the following continued fraction

$$\begin{aligned} \frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} \;&=\; \frac{1}{1-ab} \, + \, \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} \, + \\ &\quad \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} \, + \cdots, \qquad |ab| < 1, |q| < 1. \end{aligned} \tag{1.2}$$

In fact setting $a = q^{1/2}$ and $b = q^{1/2}$ in (1.2), we obtain (1.1).

In Section 2 we obtain an interesting q-identity related to $M(q)$ using Ramanujan's ${}_1\psi_1$ summation formula [5, Ch. 16, Entry 17]

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/az)_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/az)_\infty (b)_\infty (q/a)_\infty}, \qquad |b/a| < |z| < 1, \tag{1.3}$$

and Andrew's identity [4, p. 57],

$$\sum_{n=0}^{\infty} \frac{q^{kn}}{1 - q^{ln+k}} = \sum_{n=0}^{\infty} q^{ln^2+2kn} \frac{1 + q^{ln+k}}{1 - q^{ln+k}}. \tag{1.4}$$

In Section 3 we obtain several relation of $M(q)$ with theta function $\varphi(q)$, $\psi(q)$ and $\chi(q)$. In Section 4 we obtain an integral representation of $M(q)$. In Section 5 we derive a formula that help us to obtain relation among $M(q^{1/2})$, $M(q)$, $M(q^2)$ and $M(q^4)$. We establish explicit formulas for the evaluation of $\dfrac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})}$ in Section 6.

We conclude this introduction with few customary definition we make use in the sequel. For a and q complex number with $|q| < 1$

$$(a)_\infty := (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n)$$

and

$$(a)_n := (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) = \frac{(a)_\infty}{(aq^n)_\infty}, \quad n: \text{ any integer.}$$

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{1.5}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \qquad |ab| < 1. \tag{1.6}$$

Identity (1.6) is the Jacobi's triple product identity in Ramanujan's notation [5, Ch. 16, Entry 19]. It follows from (1.5) and (1.6) that [5, Ch. 16, Entry 22],

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.7}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.8}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{1.9}$$

2 q-Identity related to $M(q)$

Theorem 2.1

$$M(q) = \sum_{n=0}^{\infty} q^{n(8n+4)+1/2} \frac{1 + q^{8n+2}}{1 - q^{8n+2}} - \sum_{n=0}^{\infty} q^{(n+1)(8n+4)+1/2} \frac{1 + q^{8n+6}}{1 - q^{8n+6}} \tag{2.1}$$

Proof: Changing q to q^8, then setting $a = q^2$, $b = q^{10}$ and $z = q^2$ in ${}_1\psi_1$ summation formula (1.3) we obtain

$$\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2} = \sum_{n=0}^{\infty} \frac{q^{2n}}{1 - q^{8n+2}} - \sum_{n=0}^{\infty} \frac{q^{6n+4}}{1 - q^{8n+6}} \tag{2.2}$$

employing Andrews identity (1.4) with $k = 2, l = 8$ and $k = 6, l = 8$ in both the summations in right side of the identity (2.2) respectively and finally multiplying both sides of the resulting identity with $q^{1/2}$ and using product represtation of $M(q)$ (1.1), we complete the proof of Theorem 2.1.

3 Some Identities involving $M(q)$

We obtain relation of $M(q)$ in terms of theta function $\varphi(q)$, $\psi(q)$ and $\chi(q)$.
Theorem 3.1

$$M(q) = q^{1/2}\,\frac{\psi^4(q)}{\varphi^2(q)}\,, \tag{3.1}$$

$$8M(q^2) = \varphi^2(q) - \varphi^2(-q)\,, \tag{3.2}$$

$$16M^2(q) = \varphi^4(q) - \varphi^4(-q)\,, \tag{3.3}$$

$$\frac{M^2(q)}{M(q^2)} = \varphi^2(q^2)\,, \tag{3.4}$$

$$4M(q^2) = \varphi^2(q) - \varphi^2(q^2)\,, \tag{3.5}$$

$$\frac{M^{-1}(q) + M(q)}{M^{-1}(q) - M(q)} = \frac{1 + q\psi^4(q^2)}{1 - q\psi^4(q^2)}\,, \tag{3.6}$$

$$8M(q^2) = \frac{\chi^2(q)}{\chi^2(-q)}\,\phi^2(-q^2) - \phi^2(-q)\,. \tag{3.7}$$

Proof: Using [5, Ch. 16, Entry 22(ii)] in (1.1), we obtain

$$M(q) = q^{1/2}\psi^2(q^2)\,. \tag{3.8}$$

Employing [5, Ch. 16, Entry 25(iv)] in (3.8), we obtain (3.1).
From (3.1) we have

$$M(q^2) = q\,\frac{\psi^4(q^2)}{\varphi^2(q^2)}\,. \tag{3.9}$$

Employing [5, Ch. 16, Entry 25(vii)] and [5, Ch. 16, Entry 25(vi)] in (3.9), we obtain (3.2). Identity (3.3) immediately follows from (3.8) and [5, Ch. 16, Entry 25(vii)]. Again from (3.1), we have

$$\frac{M^2(q)}{M(q^2)} = \frac{\psi^8(q)\varphi^2(q^2)}{\psi^4(q^2)\varphi^4(q)}\,, \tag{3.10}$$

employing [5, Ch. 16, Entry 25(iv)] in the identity (3.10) we obtain (3.4).
From (3.2) and (3.3), we have

$$64M^2(q^2) + 16M^2(q) = 16\varphi^2(q)M(q^2)\,, \tag{3.11}$$

dividing the identity (3.11) throughout by $16M(q^2)$ and using (3.4) we obtain (3.5).
From (3.1) we deduce that

$$M^{-1}(q) + M(q) = \frac{\varphi^4(q) + q\,\psi^8(q)}{q^{1/2}\varphi^2(q)\psi^4(q)}\,, \tag{3.12}$$

and

$$M^{-1}(q) - M(q) = \frac{\varphi^4(q) - q\,\psi^8(q)}{q^{1/2}\varphi^2(q)\psi^4(q)}\,. \tag{3.13}$$

On dividing (3.12) by (3.13) and using [5, Ch. 16, Entry 25(iv)] in the resulting identity, we complete the proof of (3.6).
From (1.7) and (1.9) we have

$$\varphi(-q) + \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 + \frac{f(q,q)}{f(-q,-q)}\right]\,,$$

employing [5, Ch. 16, Entry 30(ii)] in right hand side of above identity we obtain

$$\varphi(-q) + \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{2(q^8;q^8)_\infty^5(q^{32};q^{32})_\infty^2}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2(q^{64};q^{64})_\infty^4}\,\frac{M(q^{16})}{q^8}\,. \tag{3.14}$$

Again from (1.7) and (1.9) we have

$$\varphi(-q) - \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{(q;q)_\infty}{(-q;q)_\infty}\left[1 - \frac{f(q,q)}{f(-q,-q)}\right],$$

employing [5, Ch. 16, Entry 30(ii)] and [5, Ch. 16, Entry 18(ii)] in right hand side of above identity we obtain

$$\varphi(-q) - \frac{\chi(q)}{\chi(-q)}\varphi(-q^2) = \frac{-4q(-q^8;q^8)_\infty(q^{64};q^{64})_\infty^3}{(-q^{16};q^{32})_\infty}\,\frac{q^8}{M(q^{16})}\,. \tag{3.15}$$

Multiplying (3.14) and (3.15) we complete the proof of (3.7).

Theorem 3.2. Let $u = M(q),\ v = M(-q)$ and $w = M(q^2)$, then

$$u^2 - v^2 = 8w^2$$

Proof: On substituting (3.4) in (3.5), we obtain

$$\varphi^2(q) = \frac{4M^2(q^2) + M^2(q)}{M(q^2)}\,. \tag{3.16}$$

Changing q to $-q$ in (3.16), we have

$$\varphi^2(-q) = \frac{4M^2(q^2) + M^2(-q)}{M(q^2)}\,. \tag{3.17}$$

Subtracting (3.17) from (3.16) and using identity (3.2), we complete the proof of Theorem 3.2.

4 Integral Representation of $M(q)$

Theorem 4.1. For $0 < |q| < 1$,

$$M(q) = \exp\int\left(\frac{1}{2q} + \frac{4}{q}\left[\frac{\varphi^4(-q)-1}{8} + \frac{q\,\varphi'(q)}{2\,\varphi(q)}\right]\right)dq\,, \tag{4.1}$$

where $\varphi(q)$ and $\psi(q)$ are as defined in (1.7) and (1.8).

Proof: Taking log on both sides of (3.1), we have

$$\log M(q) = \frac{1}{2}\log q + 4\log\psi(q) - 2\log\varphi(q)\,. \tag{4.2}$$

Employing [5, Ch. 16, Entry 23(ii)] and [5, Ch. 16, Entry 23(i)] on right hand side of (4.2), we obtain

$$\log M(q) = \frac{1}{2}\log q + 4\sum_{n=1}^{\infty}\frac{q^{2n}}{2n(1+q^{2n})}\,. \tag{4.3}$$

Differentiating (4.3) and simplifying, we have

$$\frac{d}{dq}\log M(q) = \frac{1}{2q} + \frac{4}{q}\left[\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{(1+q^n)^2} + \sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}\right]. \tag{4.4}$$

Using Jacobi's identity [5, Ch. 16, Identity 33.5, p. 54)] and [5, Ch. 16, Entry 23(i)] and integrating both sides and finally exponentiating both sides of identity (4.4), we complete the proof of Theorem 4.1.

5 Modular Equation of Degree n and Relation Between $M(q)$ and $M(q^n)$

In the terminology of hypergeometric function, a modular equation of degree n is a relation between α and β that is induced by

$$n\,\frac{{}_2F_1(1/2,1/2;1;1-\alpha)}{{}_2F_1(1/2,1/2;1;\alpha)}=\frac{{}_2F_1(1/2,1/2;1;1-\beta)}{{}_2F_1(1/2,1/2;1;\beta)}\,,$$

where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}\,x^k\,,$$

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}\,.$$

Let $Z_1(r) ={}_2 F_1(1/r, r-1/r;1;\alpha)$ and $Z_n(r) ={}_2 F_1(1/r, r-1/r;1;\beta)$, where n is the degree of the modular equation. The multiplier $m(r)$ is defined by the equation

$$m(r)=\frac{Z_1(r)}{Z_n(r)}\,.$$

Theorem 5.1. If

$$q = \exp\left(-\pi\,\frac{{}_2F_1(1/2,1/2;1;1-\alpha)}{{}_2F_1(1/2,1/2;1;\alpha)}\right), \tag{5.1}$$

then

$$\alpha = 16\,\frac{M^4(q)}{M^4(q^{1/2})} \tag{5.2}$$

Proof: From (1.1) and (1.7), we have

$$\begin{aligned} M(q)\varphi^2(q) &= q^{1/2}\,\frac{(q^4;q^4)_\infty^2}{(q^2;q^4)_\infty^2}\,\frac{(-q;q^2)_\infty^4(q^4;q^4)_\infty^2}{(-q^2;q^2)_\infty^4(q^2;q^4)_\infty^2} \\ &= M^2(q^{1/2})\,. \end{aligned} \tag{5.3}$$

Substituting (5.3) in (3.3), we obtain

$$16M^2(q)=\frac{M^4(q^{1/2})}{M^2(q)}\left[1-\frac{\varphi^4(-q)}{\varphi^4(q)}\right]. \tag{5.4}$$

From a known identity [5, Ch. 16, p. 100, Entry 5] and (5.1) it is implied that

$$\alpha=1-\frac{\varphi^4(-q)}{\varphi^4(q)}\,. \tag{5.5}$$

Using (5.5) in (5.4), we complete the proof of (5.2).

Let α and β be related by (5.1). If β has degree n over α then from Theorem 5.1, we obtain

$$\beta=16\,\frac{M^4(q^n)}{M^4(q^{n/2})}\,. \tag{5.6}$$

Corollary 5.2. Let $u=M(q^{1/2})$, $v=M(q)$, $w=M(q^2)$ and $x=M(q^4)$, then

$$16x^4v^2+32x^3wv^2-4x^3wu^4+24x^2w^2v^2+8xw^3v^2-xw^3u^4+w^4v^2=0\,. \tag{5.7}$$

Proof: From [5, Entry 24(v), p. 216], we have

$$\sqrt{1-\alpha}=\left(\frac{1-\beta^{1/4}}{1+\beta^{1/4}}\right)^2. \tag{5.8}$$

On using (5.6) with $n = 4$ and (5.2) in (5.8), we obtain

$$\sqrt{\frac{u^4 - 16v^4}{u^4}} = \left(\frac{w - 2x}{w + 2x}\right)^2. \tag{5.9}$$

Squaring both side of (5.9) and then simplifying, we obtain (5.7).

6 Evaluations of $M(q)$

As an application of Theorem 5.1, we establish few explicit evaluation of $M(q)$.

Let $q_n = e^{-\pi\sqrt{n}}$ and let α_n denote the corresponding value of α in (5.1). Then by Theorem 5.1, we have

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} = \frac{1}{2}\alpha_n^{1/4} \tag{6.1}$$

From [5, Ch. 17, p. 97], we have $\alpha_1 = \dfrac{1}{2}$, $\alpha_2 = (\sqrt{2}-1)^2$ and $\alpha_4 = (\sqrt{2}-1)^4$.

Thus from (6.1), it immediately follows

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left(\frac{1}{2}\right)^{5/4}, \tag{6.2}$$

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2}-1}, \tag{6.3}$$

$$\frac{M(e^{-2\pi})}{M(e^{-\pi})} = \frac{\sqrt{2}-1}{2}. \tag{6.4}$$

Ramanujan has recorded several modular equation in his notebook [10, p. 204-237] and [10, p. 156-160] which are very useful in the computation of class invariants and the values of theta function. Ramanujan has also recorded several values of theta function $\varphi(q)$ and $\psi(q)$ in his notebook. For example

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)}, \tag{6.5}$$

$$\psi(e^{-\pi}) = 2^{-5/8}e^{\pi/8}\frac{\pi^{1/4}}{\Gamma(3/4)}, \tag{6.6}$$

$$\frac{\varphi(e^{-\pi})}{\varphi(e^{-3\pi})} = \sqrt[4]{6\sqrt{3}-9}. \tag{6.7}$$

From (3.8) and (6.6), we have

$$M(e^{-\pi/2}) = 2^{-5/4}\frac{\sqrt{\pi}}{\Gamma^2(3/4)}, \tag{6.8}$$

Using (6.8) and (6.2), we obtain

$$M(e^{-\pi}) = \frac{\sqrt{\pi}}{\Gamma^2(3/2)}. \tag{6.9}$$

Setting (6.9) in (6.4), we obtain

$$M(e^{-2\pi}) = \frac{\sqrt{2}-1}{2}\frac{\sqrt{\pi}}{\Gamma^2(3/2)}. \tag{6.10}$$

J.M. Borwein and P.B. Borwein [7] are the first to observe that class invariant could be used to evaluated certain values of $\varphi(e^{-n\pi})$. The Ramanujan Weber class invariants are defined by

$$G_n \quad := \quad 2^{-1/4}\, q_n^{-1/24}\, (-q_n; q_n^2)_\infty$$

and

$$g_n \quad := \quad 2^{-1/4}\, q_n^{-1/24}\, (q_n; q_n^2)_\infty\,, \tag{6.11}$$

where $q_n = e^{-\pi\sqrt{n}}$. Chan and Huang has derived few explicit formulas for evaluating $K(e^{-\pi\sqrt{n}/2})$ in the terms of Ramanujan Weber class. Similar works are done by Adiga et., al. Analogoues to these works we obtain explicit formulas to evaluate $\dfrac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})}$.

Theorem 6.1. For Ramanujan Weber class invariant defined as in (6.11), let $p = G_n^{12}$ and $p_1 = g_n^{12}$, then

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} \quad = \quad \frac{1}{2}\,\frac{1}{\sqrt{\sqrt{p\,(p+1)}+\sqrt{p\,(p-1)}}}\,, \tag{6.12}$$

$$\frac{M(e^{-\pi\sqrt{n}})}{M(e^{-\pi\sqrt{n}/2})} \quad = \quad \frac{1}{2}\sqrt{\sqrt{p_1^2+1}-p_1}\,. \tag{6.13}$$

Proof: From [9], we have

$$G_n = [4\alpha_n(1-\alpha_n)]^{-1/24}\,.$$

Hence

$$\alpha_n = \frac{1}{(\sqrt{p(p+1)}+\sqrt{p(p-1)})^2}\,. \tag{6.14}$$

Using (6.14) in (6.1), we obtain (6.12).

Also from [9], we have

$$2g_n^{12} = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}\,.$$

Hence

$$\sqrt{\alpha_n} = \sqrt{(p_1^2+1)} - p_1\,. \tag{6.15}$$

Using (6.15) in (6.1), we complete the proof of (6.13).

Example: Let $n = 1$. Since $G_1 = 1$, from Theorem 6.1 we have

$$\frac{M(e^{-\pi})}{M(e^{-\pi/2})} = \left(\frac{1}{2}\right)^{5/4}.$$

Let $n = 2$. Since $g_2 = 1$, from Theorem 6.1 we have

$$\frac{M(e^{-\sqrt{2}\pi})}{M(e^{-\pi/\sqrt{2}})} = \frac{1}{2}\sqrt{\sqrt{2}-1}\,.$$

Remark: Using [10, p. 229] it is easily verified that $M(q)$ and $K(q)$ are related by the equation

$$M(q^2)K(q) + K(q)M(q) - M(q^2) = 0.$$

References

[1] C. Adiga, B. C. Berndt, S.Bhargava and G. N. Watson, *Chapter 16 of Ramanujan's Second Notebook: Theta-Functions and q-Series*, Mem. Amer. Math. Soc., No. 315, 53(1985), 1-85, Amer. Math. Soc., Providence, 1985.

[2] C. Adiga, T. Kim, M. S. Mahadeva Naika and H. S. Madhusudhan, *On Ramanujan's Cubic Continued Fraction and Explicit Evaluations of Theta Functions*,Indian J. Pure and App. Math., No. 55, 9(2004), 1047-1062.

[3] G. E. Andrews, *An introduction to Ramanujan's 'lost' notebook*, Amer. Math. Monthly, 86(1979), 89-108.

[4] G. E. Andrews, *The Lost Notebook Part I*, Springer Verlang, New York, 2005.

[5] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1991.

[6] B. C. Berndt, *Ramanujan's Notebooks*, Part V, Springer-Verlag, New York, 1998.

[7] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Wiley, New York, 1987.

[8] H.H. Chan, *On Ramanujan's Cubic Continued Fraction*, Acta Arith. 73(1995), 343-355.

[9] H.H. Chan and S.S. Huang, *On the Ramanujan Göllnitz Gordon Continued Fraction*, Ramanujan J., 1(1997), 75-90.

[10] S. Ramanujan, *Notebooks*, 2 volumes, Tata Institute of Fundamental Research, Bombay, 1957.

[11] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[12] K.R. Vasuki and B.R. Srivatsa Kumar, *Certain identities for the Ramanujan Gollnitz Gordon Continued fraction*, J. of Comp. and Applied Mathematics. 187(2006),87-95.

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