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CONVERGENCE OF MATRIX MEANS OF MELLIN - FOURIER SERIES

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Abstract. Mellin analysis, a counterpart of the Fourier analysis, has been a field for growing interest for researchers in the last four decades [8, 9, 1, 2, 6, 7, 3, 4]. In this paper, we aim to study convergence of the Mellin - Fourier series of the recurrent functions through its matrix means. Our theorem generalizes some of the results of Butzer and Jansche [9].

1 Introduction

A function $f:\mathbb{R}_+\to\mathbb{C}$ is called recurrent if $f(e^{2\pi}x)=f(x)\forall x\in\mathbb{R}_+$, and $c-$recurrent for $c\in\mathbb{R}$, if $e^{2\pi c}f(e^{2\pi}x)=f(x)\forall x\in\mathbb{R}_+$. We denote the function spaces under consideration by Y_c, where $Y_c=\{f\in L^1_{loc}(\mathbb{R}_+) : f \text{ is } c-\text{recurrent}; \|f\|_{Y_c}:=\int_{e^{-\pi}}^{e^{\pi}}|f(u)|u^{c-1}du<\infty\}, c\in\mathbb{R}$. Note that $c=0$ corresponds to recurrent functions. Mellin-Fourier series of $f\in Y_c$ is defined as

$$f(x)\sim\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\mathcal{M}^c(f;k)x^{-c-ik},\quad x\in\mathbb{R}_+,$$

where $\mathcal{M}^c(f;k)$ is the finite Mellin transform of f at $k\in\mathbb{Z}$ defined by

$$\mathcal{M}^c(f;k)=\int_{e^{-\pi}}^{e^{\pi}}f(u)u^{c+ik-1}du.$$

Let $S_n^c(f;x)$ denote the partial sums of Mellin-Fourier series of f. Then

$$S_n^c(f;x)=\frac{x^{-c}}{2\pi}\sum_{k=-n}^{n}\mathcal{M}^c(f;k)x^{-ik}. \tag{1.1}$$

Using Mellin-Dirichlet kernel $D_n^c(x)$, which is given by

$$D_n^c(x)=\frac{x^{-c}}{2\pi}\sum_{k=-n}^{n}x^{-ik},\quad x\in\mathbb{R}_+,$$

we can write $S_n^c(f;x)$ as the finite Mellin convolution of f and D_n^c, i.e.,

$$S_n^c(f;x)=\int_{e^{-\pi}}^{e^{\pi}}D_n^c(u)f\left(\frac{x}{u}\right)\frac{du}{u}. \tag{1.2}$$

Using $x=\mathrm{e}^{\ln x}$, the relation $2\cos\theta=\mathrm{e}^{i\theta}-\mathrm{e}^{-i\theta}$ and properties of trigonometric cosine series, we can express $D_n^c(x)$ as

$$D_n^c(x)=\begin{cases}\dfrac{2n+1}{2\pi}, & x=1;\\ \dfrac{x^{-c}}{2\pi}\left(\frac{\sin((n+1/2)\ln x)}{\sin(\ln x/2)}\right), & x\neq 1,\end{cases}$$

and hence $S_n^c(f;x)$ can also be written as

$$S_n^c(f;x) = \int_{e^{-\pi}}^{e^{\pi}} \frac{u^{-c}}{2\pi} \frac{\sin\left((n+1/2)\ln u\right)}{\sin(\ln u/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \tag{1.3}$$

$$= \frac{1}{2\pi} \int_{e^{-\pi}}^{e^{\pi}} \frac{\sin\left((n+1/2)\ln u\right)}{\sin(\ln u/2)} \tau_{1/u}^c(f;x) \frac{du}{u}, \tag{1.4}$$

$\tau_{1/u}^c$ being the Mellin translation operator defined by

$$\tau_h^c(f;x) = h^c f(hx), \quad h \in \mathbb{R}_+.$$

For more details of the finite Mellin transform, finite Mellin convolution and Mellin translation operator one can refer to [9]. The arithmetic means of the Mellin - Fourier series of $f \in Y_c$, denoted by $\sigma_n^c(f;x)$, are given by

$$\sigma_n^c(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} S_k^c(f;x), \quad n = 0,1,2,\ldots, \tag{1.5}$$

which are known as the Cesàro means of order one, which are also referred as $(C,1)$ means.

In study of the Cesàro means, the Mellin-Fejér kernels play an important role. The Mellin-Fejér kernels $F_n^c(f;x)$ are defined as the average of Mellin-Dirichlet kernels, i.e.,

$$F_n^c(f;x) = \frac{1}{n+1} \sum_{k=0}^{n} D_k^c(x), \quad n = 0,1,2,\ldots.$$

Let $T = (a_{n,k}), n,k \in N_0$ be a lower triangular matrix and $f \in Y_c$. The sequence to sequence transform

$$T_n^c(f;x) = \sum_{k=0}^{n} a_{n,k} S_k^c(f;x), \quad n = 0,1,2,\ldots. \tag{1.6}$$

defines the matrix means or simply T-means of the Mellin-Fourier series of f. The Mellin-Fourier series of f is said to be T-summable to S, if $\lim_{n\to\infty} T_n^c(f;x) = S$. The T-summability is said to be regular if

$$\lim_{n\to\infty} S_n^c(f;x) = S \Rightarrow \lim_{n\to\infty} T_n^c(f;x) = S.$$

If we define $T = (a_{n,k}), n,k \in N_0$ as

$$a_{n,k} = \begin{cases} \dfrac{1}{n+1}, & k \le n; \\ 0, & k > n, \end{cases}$$

then 1.6 reduces to 1.5. Thus the $(C,1)$ summability is a particular case of T-summability.

The Fourier analysis, in general, and the Fourier series, in particular, has received attention of the researchers during last century as well as in the present century. Many variants of the Fourier series have been developed for the different type of functions. Mellin - Fourier series is one of these variants to handle the recurrent functions [8, 9, 1, 2, 6, 7, 3, 4]. The theory of recurrent functions with a counterpart of the Fourier series in Mellin settings has been discussed in [9], which has been further extended in [6] and [7]. Butzer and Jansche [9, p. 52] have proved that in general the Mellin-Fourier series of a recurrent function does not converge to the function itself whereas the arithmetic means (Cesaro means of order 1) of the series converge to the function. More precisely, they have proved the following:

Theorem 1.1. *If $f \in Y_c$ for $c \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \|\sigma_n^c(f;x) - f\|_{Y_c} = 0, \quad x \in \mathbb{R}_+.$$

For the proof one can see [9, pp. 52-53, Theorem 3.1]. We note that this theorem can be extended to a more general summability means, which we will discuss in the next section.

2 Main Result

In this paper, we extend the above theorem to matrix means of Mellin-Fourier series.

Theorem 2.1. *Let* $T = (a_{n,k});\ n, k \in \mathbb{N}_0$ *be a lower triangular regular matrix with non-negative entries which satisfies*

(i) $a_{n,k+1} \leq a_{n,k},\ 0 \leq k \leq n-1;\ \forall n \in N_0.$

(ii) $\sum_{k=0}^{n} a_{n,k} = 1,\ \forall n \in N_0.$

Then for any $f \in Y_c$

$$\lim_{n\to\infty} \|T_n^c(f;x) - f\|_{Y_c} = 0, \quad x \in \mathbb{R}_+.$$

We assume that $T_n^c : Y_c \to Y_c, \forall n \in \mathbb{N}_0$. Indeed this is true and trivial to verify, because for any $n \in \mathbb{N}_0, S_n^c(f;x) \in Y_c$ and $T_n^c(f;x)$ is linear combination of $S_n^c(f;x)$'s. So $T_n^c(f;x) \in Y_c$.

3 Lemmas

To prove the main result, we need following lemmas.

Lemma 3.1. *Let a function* $g : [-\pi, \pi] \to \mathbb{R}$ *be defined by*

$$g(t) = \begin{cases} \frac{\sin(n+1/2)t}{\sin(t/2)}, & t \neq 0; \\ 2n+1, & t = 0. \end{cases}$$

Then

$$\int_{-\pi}^{\pi} g(t)dt = 2\pi$$

For the proof one can see [5, p. 178].

Lemma 3.2. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a non-increasing sequence of non-negative numbers. Then*

$$\sum_{k=0}^{n} a_k \frac{\sin(k+1/2)t}{\sin(t/2)} \geq 0, \quad t \in (0, \pi].$$

Proof. For $t \in (0,\pi], \left\{\frac{\sin(n+1/2)t}{\sin(t/2)}\right\}_{n=0}^{\infty}$ is a sequence of real numbers whose partial sums s_n are given by

$$s_n = \sum_{k=0}^{n} \frac{\sin(k+1/2)t}{\sin(t/2)} = \left(\frac{\sin((n+1)t/2)}{\sin(t/2)}\right)^2,$$

so that $s_n \geq 0, \forall n \in \mathbb{N}_0$. Using Abel's lemma for lower bound of $\left\{\frac{\sin(n+1/2)t}{\sin(t/2)}\right\}_{n=0}^{\infty}$ we get

$$\sum_{k=0}^{n} a_k \frac{\sin(k+1/2)t}{\sin(t/2)} \geq a_0 \cdot 0 = 0.$$

□

Lemma 3.3. *For* $0 < \delta \leq \pi$

$$\lim_{n\to\infty} \int_{\delta}^{\pi} \frac{\sin(n+1/2)t}{\sin(t/2)} dt = 0.$$

Proof. To prove lemma we use generalized Riemann - Lebesgue lemma [5, pp. 170-171]. We see that $1 \in L_1(\delta, \pi]$ for $0 < \delta \leq \pi$ and $\sin(t/2) \geq \sin\delta/2 > 0, \forall \delta \leq t \leq \pi$. So $1/\sin(t/2) \in L_1(\delta, \pi]$.
Also

$$\frac{1}{c}\int_0^c \sin t\, dt = \frac{1}{c}(\cos c - 1),$$

and

$$\left|\frac{1}{c}(\cos c - 1)\right| \le \frac{2}{|c|},$$

so that

$$\lim_{c\to\pm\infty} \frac{1}{c}\int_0^c \sin t\, dt = 0.$$

Using generalized Riemann - Lebesgue lemma

$$\lim_{n\to\infty} \int_\delta^\pi \frac{\sin{(n+1/2)t}}{\sin(t/2)} dt = 0.$$

□

4 Proof of Theorem 2.1

Proof. We have

$$\begin{aligned} T_n^c(f;x) - f(x) &= \sum_{k=0}^n a_{n,k} \int_{e^{-\pi}}^{e^\pi} \frac{u^{-c}}{2\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} f\left(\frac{x}{u}\right) \frac{du}{u} - f(x) \\ &= \sum_{k=0}^n a_{n,k} \left(\int_{e^{-\pi}}^{e^\pi} \frac{u^{-c}}{2\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} f\left(\frac{x}{u}\right) \frac{du}{u} - f(x) \right), \end{aligned}$$

in view of $\sum_{k=0}^n a_{n,k} = 1.$

Using Lemma 3.1, we have

$$\begin{aligned} T_n^c(f;x) - f(x) &= \sum_{k=0}^n a_{n,k} \left(\int_{e^{-\pi}}^{e^\pi} \frac{u^{-c}}{2\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} f\left(\frac{x}{u}\right) \frac{du}{u} \right. \\ &\qquad \left. - \int_{e^{-\pi}}^{e^\pi} \frac{1}{2\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} f(x) \frac{du}{u} \right) \\ &= \frac{1}{2\pi} \sum_{k=0}^n a_{n,k} \int_{e^{-\pi}}^{e^\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} \left(u^{-c} f\left(\frac{x}{u}\right) - f(x) \right) \frac{du}{u} \\ &= \frac{1}{2\pi} \sum_{k=0}^n a_{n,k} \int_{e^{-\pi}}^{e^\pi} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)} \left(\tau_{1/u}^c f(x) - f(x) \right) \frac{du}{u}. \end{aligned}$$

Now

$$\|T_n^c(f;x)-f\|_{Y_c} = \int_{e^{-\pi}}^{e^{\pi}} |T_n^c(f;x)-f(x)|\, x^{c-1}dx$$

$$= \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}} \left|\sum_{k=0}^{n} a_{n,k}\int_{e^{-\pi}}^{e^{\pi}} \frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\left(\tau_{1/u}^c f(x)-f(x)\right)\frac{du}{u}\right| x^{c-1}dx$$

$$\leq \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}}\int_{e^{-\pi}}^{e^{\pi}} \left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right| \left|\tau_{1/u}^c f(x)-f(x)\right| \frac{du}{u}x^{c-1}dx.$$

Changing the order of integration

$$= \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}} \left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right| \int_{e^{-\pi}}^{e^{\pi}} \left|\tau_{1/u}^c f(x)-f(x)\right| x^{c-1}dx\frac{du}{u}$$

$$= \frac{1}{2\pi}\int_{e^{-\pi}}^{e^{\pi}} \left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right| \left\|\tau_{1/u}^c f(x)-f(x)\right\|_{Y_c}\frac{du}{u}.$$

Let $E_\delta := \{x\in[e^{-\pi},e^{\pi}] : |x-1|<\delta\}$, $0\leq\delta<1-e^{-\pi}$ and $CE_\delta=[e^{-\pi},e^{\pi}]-E_\delta$. Then

$$\|T_n^c(f;x)-f(x)\|_{Y_c} = \frac{1}{2\pi}\left\{\int_{E_\delta}+\int_{CE_\delta}\right\}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right| \left\|\tau_{1/u}^c f(x)-f\right\|_{Y_c}\frac{du}{u}.$$

Since $\lim_{h\to1}\|\tau_h^c f-f\|_{Y_c}=0$, so for a given $\varepsilon>0, \exists\ \delta\ (0\leq\delta<1-e^{-\pi})$ such that $\left\|\tau_{1/u}^c f-f\right\|_{Y_c}<\varepsilon\ \forall u\in E_\delta$. Also $\left\|\tau_{1/u}^c f-f\right\|_{Y_c}\leq 2\|f\|_{Y_c}$. So

$$\begin{aligned}
\|T_n^c(f;x)-f\|_{Y_c} &= \frac{1}{2\pi}\int_{E_\delta}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\cdot\varepsilon\cdot\frac{du}{u}\\
&\quad+\frac{\|f\|_{Y_c}}{2\pi}\int_{CE_\delta}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}\\
&\leq \frac{\varepsilon}{2\pi}\int_{e^{-\pi}}^{e^{\pi}}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}\\
&\quad+\frac{\|f\|_{Y_c}}{2\pi}\int_{CE_\delta}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}\\
&= \frac{\varepsilon}{2\pi}\times 2\pi+\frac{\|f\|_{Y_c}}{2\pi}\int_{CE_\delta}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}\\
&= \varepsilon+\frac{\|f\|_{Y_c}}{2\pi}\int_{CE_\delta}\left|\sum_{k=0}^{n} a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}.
\end{aligned}$$

Using Lemma 3.2, Lemma 3.3 and conditions on $a_{n,k}$, we have

$$\lim_{n\to\infty}\|T_n^c(f;x)-f\|_{Y_c}\leq\varepsilon+\lim_{n\to\infty}\frac{\|f\|_{Y_c}}{2\pi}\int_{CE_\delta}\left|\sum_{k=0}^{n}a_{n,k}\frac{\sin\left((k+1/2)\ln u\right)}{\sin(\ln u/2)}\right|\frac{du}{u}$$

$$=\varepsilon+\frac{\|f\|_{Y_c}}{2\pi}\times 0=\varepsilon,\ \text{for any } \varepsilon>0.$$

Hence

$$\lim_{n\to\infty}\|T_n^c(f;x)-f\|_{Y_c}=0.$$

This completes the proof.

□

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