Palestine Journal of Mathematics

Vol. 4(2) (2015) , 419–430

Convergence theorems for a finite family of T-Ciric quasi-contractive operators in CAT(0) spaces

G. S. Saluja

Communicated by Ayman Badawi

MSC 2010 Classifications: 54H25, 54E40.

Keywords and phrases: T-Ciric quasi-contractive operator, two-step implicit iteration process, common fixed point, strong convergence, CAT(0) space.

Abstract. The aim of this paper is to study an implicit iterative process for a finite family of T-Ciric quasi-contractive operators and also establish the strong convergence of above said iteration process using the common fixed points of a finite family of above said operators in the framework of CAT(0) spaces. Our results improve and extend some corresponding recent results from the current existing literature.

1 Introduction and Preliminaries

A metric space X is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces (see [4]), $\mathbb{R}$-trees (see [21]), Euclidean buildings (see [5]), the complex Hilbert ball with a hyperbolic metric (see [14]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [4].

Fixed point theory in a CAT(0) space was first studied by Kirk (see [22, 23]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since, then the fixed point theory for single-valued and multi-valued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [1], [8], [11]-[13], [16], [19]-[20], [24]-[25], [30], [38], [41] and the references therein). It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(k') space for every $k' \geq k$ (see,e.g., [4]).

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say X is (i) a *geodesic space* if any two points of X are joined by a geodesic and (ii) a *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$, which we will denoted by $[x, y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Such a triangle always exists (see [4]).

A geodesic metric space is said to be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following $CAT(0)$ comparison axiom.

CAT(0) space

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the $CAT(0)$ inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}). \tag{1.1}$$

Complete $CAT(0)$ spaces are often called *Hadamard spaces* (see [18]). If x, y_1, y_2 are points of a $CAT(0)$ space and y_0 is the midpoint of the segment $[y_1, y_2]$ which we will denote by $(y_1 \oplus y_2)/2$, then the $CAT(0)$ inequality implies

$$d^2\Big(x, \frac{y_1 \oplus y_2}{2}\Big) \leq \frac{1}{2}\, d^2(x, y_1) + \frac{1}{2}\, d^2(x, y_2) - \frac{1}{4}\, d^2(y_1, y_2). \tag{1.2}$$

The inequality (1.2) is the (CN) inequality of Bruhat and Tits [6]. The above inequality has been extended in [12] as

$$\begin{aligned} d^2(z, \alpha x \oplus (1-\alpha)y) &\leq \alpha d^2(z, x) + (1-\alpha) d^2(z, y) \\ &\quad -\alpha(1-\alpha) d^2(x, y). \end{aligned} \tag{1.3}$$

for any $\alpha \in [0, 1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (see [4, page 163]). Moreover, if X is a $CAT(0)$ metric space and $x, y \in X$, then for any $\alpha \in [0, 1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z, x) + (1-\alpha) d(z, y), \tag{1.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0, 1]\}$.

A subset C of a $CAT(0)$ space X is convex if for any $x, y \in C$, we have $[x, y] \subset C$.

Since last 30 years, the convergence problems of implicit or non-implicit iteration process to a common fixed point for a finite family of nonexpansive mappings, asymptotically nonexpansive mappings, pseudocontractive mappings, and Zamfirescu operators in arbitrary Banach spaces, Hilbert spaces, uniformly convex Banach spaces or normed linear spaces have been considered by several authors (see, for example [2, 10, 15], [31]-[35], [39]-[40], [42]-[45]) and many others.

In 2001, Xu and Ori [45] introduced the following implicit iteration process for a finite family of nonexpansive mappings.

Let C be a nonempty closed convex subset of a normed linear space E. Let $\{T_i : i \in I\}$ $(I = \{1, 2, \ldots, N\})$ be a finite family of nonexpansive mappings. For an initial point $x_0 \in C$, define the sequence $\{x_n\}$ as follows:

$$x_n = \alpha_n\, x_{n-1} + (1-\alpha_n) T_n x_n, \quad n \geq 1,$$

where $T_n = T_{n\,(mod\,N)}$ (here the mod N function takes values in I) and $\{\alpha_n\}_{n=1}^{\infty}$ a real sequence in $(0, 1)$. They proved the weak convergence of this process to a common fixed point for a finite family of nonexpansive mappings defined in a Hilbert space.

In 2006, Rafiq [35] studied the following implicit iteration process with errors for a finite family of Z-operators.

Let C be a nonempty closed convex subset of a normed linear space E and $x_0 \in C$. Define the sequence $\{x_n\}$ as follows:

$$x_n = \alpha_n\, x_{n-1} + (1-\alpha_n) T_n x_n + u_n, \quad n \geq 1,$$

where $T_n = T_{n\,(mod\,N)}$ (here the mod N function takes values in I), $\{\alpha_n\}_{n=1}^{\infty}$ a real sequence in $(0, 1)$ and $\{u_n\}$ is a summable sequence in C. He established the strong convergence of this iteration process to a common fixed point for a finite family of Z-operators in normed linear spaces.

We recall the following definitions in a metric space (X, d). A mapping $T\colon X \to X$ is called an a-contraction if

$$d(Tx, Ty) \leq a\, d(x, y) \text{ for all } x,\, y \in X, \tag{1.5}$$

where $a \in (0, 1)$.

The mapping T is called Kannan mapping [17] if there exists $b \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)] \text{ for all } x,\, y \in X. \tag{1.6}$$

The following definition is due to Chatterjea [9]: there exists $c \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \quad \leq \quad c\,[d(x, Ty) + d(y, Tx)] \text{ for all } x,\ y\ \in X. \tag{1.7}$$

In 1972, combining these three definitions, Zamfirescu [46] proved the following important result.

Theorem Z. Let (X, d) be a complete metric space and $T\colon X \to X$ a mapping for which there exists the real number $a,\ b$ and c satisfying $a \in (0,1),\ b,\ c \in (0, \frac{1}{2})$ such that for any pair $x,\ y \in X$, at least one of the following conditions holds:

(Z_1) $d(Tx, Ty) \leq a\, d(x, y)$,

(Z_2) $d(Tx, Ty) \leq b\,[d(x, Tx) + d(y, Ty)]$,

(Z_3) $d(Tx, Ty) \leq c\,[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n,\ n = 0, 1, 2, \ldots$$

converges to p for any arbitrary but fixed $x_0 \in X$.

The conditions $(Z_1) - (Z_3)$ can be written in the following equivalent form

$$d(Tx, Ty) \quad \leq \quad h \max\Big\{d(x, y),\ \frac{d(x, Tx) + d(y, Ty)}{2},\ \frac{d(x, Ty) + d(y, Tx)}{2}\Big\}, \quad (QC)$$

for all $x,\ y \in X$ and $0 < h < 1$, has been obtained by Ciric [7] in 1974.

A mapping satisfying (QC) is called Ciric quasi-contraction. It is obvious that each of the conditions $(Z_1) - (Z_3)$ implies (QC).

An operator T satisfying the contractive conditions $(Z_1) - (Z_3)$ in the theorem Z is called Z-operator.

In 2009, Beiranvand et al. [3] introduced the concept of T-Banach contraction and T-contractive mappings and they extended Banach's contraction principle and Edelstein fixed point theorem. Followed by this, Moradi [26] introduced T-Kannan contractive mappings, extending in the way, the well-known Kannan fixed point theorem [17].

Recently, Morales and Rojas ([28], [29]) have extended the concept of T-contraction mappings to cone metric space by proving fixed point theorems for T-Kannan, T-Zamfirescu and T-weakly contraction mappings. In [27], they studied the existence of fixed point for T-Zamfirescu operators in complete metric spaces and proved a convergence theorem of T-Picard iteration for the class of T-Zamfirescu operators. The result is as follows:

Theorem 1.1. *(See [27]) Let (M, d) be a complete metric space and $T, S\colon M \to M$ be two mappings such that T is continuous, one-to-one and subsequentially convergent. If S is a TZ operator, S has a unique fixed point. Moreover, if T is sequentially convergent, then for every $x_0 \in M$ the T-Picard iteration associated to S, $TS^n x_0$ converges to Tx^*, where x^* is the fixed point of S.*

Here we recall the definitions of the following classes of generalized T-contraction type mappings as given by Morales and Rojas [27].

Definition 1.2. (See [27]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-contraction, if there exists a real number $a \in [0, 1)$ such that for all $x, y \in X$,

$$d(TSx, TSy) \leq a\, d(Tx, Ty).$$

If we take $T = I$ (the identity map) in definition 1.2, then we obtain the definition of Banach's contraction.

The following example shows that a T-contraction mapping need not be a contraction mapping.

Example 1.3. Let $X = [1, \infty)$ be with the usual metric. Define two mappings $T, S\colon X \to X$ as $Tx = \frac{1}{2x} + 2$ and $Sx = 3x$. Obviously, S is not contraction but S is T-contraction which is seen from the following:

$$|TSx - TSy| = \Big|\frac{1}{6x} - \frac{1}{6y}\Big| = \frac{1}{3}|Tx - Ty|.$$

Definition 1.4. (See [27]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Kannan contraction, if there exists a real number $b \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(TSx, TSy) \le b\,[d(Tx, TSx) + d(Ty, TSy)].$$

If we take $T = I$ (the identity map) in definition 1.4, then we obtain the definition of Kannan contraction [17].

Definition 1.5. (See [27]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Chatterjea contraction, if there exists a real number $c \in [0, \frac{1}{2})$ such that for all $x, y \in X$,

$$d(TSx, TSy) \le c\,[d(Tx, TSy) + d(Ty, TSx)].$$

If we take $T = I$ (the identity map) in definition 1.5, then we obtain the definition of Chatterjea contraction [9].

Definition 1.6. (See [27]) Let (X, d) be a metric space and $S, T\colon X \to X$ be two mappings. A mapping S is said be T-Zamfirescu operator (TZ-operator), if there are real numbers $0 \le a < 1$, $0 \le b < \frac{1}{2}$, $0 \le c < \frac{1}{2}$ such that for all $x, y \in X$ at least one of the following conditions holds:

(TZ_1) $d(TSx, TSy) \le a\, d(Tx, Ty)$,

(TZ_2) $d(TSx, TSy) \le b\,[d(Tx, TSx) + d(Ty, TSy)]$,

(TZ_3) $d(TSx, TSy) \le c\,[d(Tx, TSy) + d(Ty, TSx)]$.

If we take $T = I$ (the identity map) in definition 1.6, then we obtain the definition of Zamfirescu operator [46].

In this paper, inspired and motivated by [27, 35, 45, 46], we study the following iteration scheme and prove strong convergence theorem to approximate the common fixed point for (T, C)-quasi contractive operator in the framework of CAT(0) spaces.

Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-Ciric quasi-contractive operators with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence in $(0, 1)$. Define the sequence $\{Tx_n\}$ as follows:

$$Tx_n \;=\; \alpha_n Tx_{n-1} \oplus (1 - \alpha_n) TS_n x_n, \quad \forall\, n \ge 1 \tag{1.8}$$

where $S_n = S_{n\,(mod\,N)}$ (here the mod N function takes values in $I = \{1, 2, \dots, N\}$).

We also study the following two-step implicit iterative process for a finite family of T-Ciric quasi-contractive operators $\{S_i\}_{i=1}^{N}\colon C \to C$ with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ in the framework of CAT(0) spaces, where $T\colon C \to C$ and define the sequence $\{Tx_n\}$ as follows:

$$\begin{aligned} Tx_n &= \alpha_n Tx_{n-1} \oplus (1 - \alpha_n) S_n Ty_n, \\ Ty_n &= \beta_n Tx_{n-1} \oplus (1 - \beta_n) TS_n x_n \quad \forall\, n \ge 1 \end{aligned} \tag{1.9}$$

where $S_n = S_{n\,(mod\,N)}$ (here the mod N function takes values in $I = \{1, 2, \dots, N\}$), $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are real sequences in $[0, 1]$ and T and S_n $(n = 1, 2, \dots, N)$ are commuting mappings.

If we take $T = I$ (the identity map) in (1.8) we get the following iteration process for a finite family of Ciric quasi- contractive operators $\{S_i\}_{i=1}^N : C \to C$ with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$ in the framework of CAT(0) spaces as follows:

$$x_n = \alpha_n x_{n-1} \oplus (1-\alpha_n) S_n x_n, \quad \forall\, n \geq 1 \tag{1.10}$$

where $S_n = S_{n\,(mod\,N)}$ (here the mod N function takes values in $I = \{1, 2, \ldots, N\}$).

If we take $T = I$ (the identity map) in (1.9) we get the following iteration process for a finite family of Ciric quasi- contractive operators $\{S_i\}_{i=1}^N : C \to C$ with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$ in the framework of CAT(0) spaces as follows:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} \oplus (1-\alpha_n) S_n y_n, \\ y_n &= \beta_n x_{n-1} \oplus (1-\beta_n) S_n x_n, \quad \forall\, n \geq 1 \end{aligned} \tag{1.11}$$

where $S_n = S_{n\,(mod\,N)}$ (here the mod N function takes values in $I = \{1, 2, \ldots, N\}$), $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ are real sequences in $[0, 1]$.

We need the following useful lemmas to prove our main results in this paper.

Lemma 1.7. *(See [30]) Let X be a CAT(0) space.*

(i) For $x,\, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x,\, y]$ such that

$$d(x,\, z) = t\, d(x,\, y) \text{ and } d(y,\, z) = (1-t)\, d(x,\, y). \quad (A)$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) For $x,\, y \in X$ and $t \in [0, 1]$, we have

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z).$$

Lemma 1.8. *(See [36]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ and $\{s_n\}$ be sequences of nonnegative numbers satisfying the following conditions:*

$$p_{n+1} \leq (1-q_n)p_n + q_n r_n + s_n,\ n \geq 1.$$

If $\sum_{n=1}^\infty q_n = \infty$, $\lim_{n\to\infty} r_n = 0$ and $\sum_{n=1}^\infty s_n < \infty$ hold, then $\lim_{n\to\infty} p_n = 0$.

2 Strong convergence theorems in CAT(0) Spaces

In this section, we establish some strong convergence results of an implicit iterative process (1.8) and a two-step implicit iteration scheme (1.9) to approximate common fixed point for a finite family of T-Ciric quasi-contractive operators in the framework of $CAT(0)$ spaces.

Theorem 2.1. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T: C \to C$ and let $\{S_i\}_{i=1}^N : C \to C$ be N, T-Ciric quasi-contractive operators with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$ satisfying the condition:*

$$\begin{aligned} d(TS_i x, TS_i y) \quad \leq \quad h \max\Big\{ & d(Tx, Ty),\ \frac{d(Tx, TS_i x) + d(Ty, TS_i y)}{2}, \\ & \frac{d(Tx, TS_i y) + d(Ty, TS_i x)}{2} \Big\} \end{aligned} \tag{2.1}$$

$\forall\, i = 1, 2, \ldots, N$; $\forall\, x,\, y \in C$ and $0 < h < 1$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^\infty (1-\alpha_n) = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in C.

Proof. From the assumption of the Theorem $F = \cap_{i=1}^N F(S_i) \neq \emptyset$, it follows that operators $\{S_i\}_{i=1}^N$ have a common fixed point in C, say u. Consider $x,\, y \in C$. Since each S_i, $i = 1, 2, \ldots, N$ is a T-Ciric quasi-contractive operator satisfying (2.1), then if

$$\begin{aligned} d(TS_i x, TS_i y) &\leq \frac{h}{2}[d(Tx, TS_i x) + d(Ty, TS_i y)] \\ &\leq \frac{h}{2}[d(Tx, TS_i x) + d(Ty, Tx) + d(Tx, TS_i x) + d(TS_i x, TS_i y)], \end{aligned}$$

implies

$$\Big(1-\frac{h}{2}\Big)d(TS_ix,TS_iy) \leq \frac{h}{2}d(Tx,Ty)+hd(Tx,TS_ix),$$

which yields (using the fact that $0<h<1$)

$$d(TS_ix,TS_iy) \leq \Big(\frac{h/2}{1-h/2}\Big)d(Tx,Ty)+\Big(\frac{h}{1-h/2}\Big)d(Tx,TS_ix). \tag{2.2}$$

If

$$\begin{aligned} d(TS_ix,TS_iy) &\leq \frac{h}{2}\,[d(Tx,TS_iy)+d(Ty,TS_ix)] \\ &\leq \frac{h}{2}\,[d(Tx,TS_ix)+d(TS_ix,TS_iy)+d(Ty,Tx)+d(Tx,TS_ix)], \end{aligned}$$

implies

$$\Big(1-\frac{h}{2}\Big)\,d(TS_ix,TS_iy) \leq \frac{h}{2}\,d(Tx,Ty)+h\,d(Tx,TS_ix),$$

which also yields (using the fact that $0<h<1$)

$$d(TS_ix,TS_iy) \leq \Big(\frac{h/2}{1-h/2}\Big)\,d(Tx,Ty)+\Big(\frac{h}{1-h/2}\Big)\,d(Tx,TS_ix). \tag{2.3}$$

Denote

$$\delta = \max\Big\{h,\ \frac{h/2}{1-h/2}\Big\} = h,$$

$$L = \frac{h}{1-h/2}.$$

Thus, in all cases,

$$\begin{aligned} d(TS_ix,TS_iy) &\leq \delta\,d(Tx,Ty)+L\,d(Tx,TS_ix) \\ &= h\,d(Tx,Ty)+\Big(\frac{h}{1-h/2}\Big)d(Tx,TS_ix) \end{aligned} \tag{2.4}$$

holds for all $x,\ y \in C$.

Also from (2.1) with $y=u=S_iu$ for all $i=1,2,\dots,N$, we have

$$\begin{aligned} d(TS_ix,TS_iu) &\leq h\max\Big\{d(Tx,Tu),\ \frac{d(Tx,TS_ix)}{2},\ \frac{d(Tx,TS_iu)+d(Tu,TS_ix)}{2}\Big\} \\ &\leq h\max\Big\{d(Tx,Tu),\ \frac{d(Tx,Tu)+d(Tu,TS_ix)}{2},\ \frac{d(Tx,Tu)+d(Tu,TS_ix)}{2}\Big\} \\ &= h\max\Big\{d(Tx,Tu),\ \frac{d(Tx,Tu)+d(Tu,TS_ix)}{2}\Big\} \\ &\leq h\,d(Tx,Tu). \end{aligned} \tag{2.5}$$

Since $S_iu=u$, $S_n=S_{n\,(mod\,N)}$ and the $mod\,N$ function takes values in $\{1,2,\dots,N\}$, taking $x=x_n$ in (2.5), we obtain

$$d(TS_nx_n,Tu) \leq h\,d(Tx_n,Tu). \tag{2.6}$$

Using (1.8), (2.6) and Lemma 1.7(ii), we have

$$\begin{aligned} d(Tx_n,Tu) &= d(\alpha_nTx_{n-1}\oplus(1-\alpha_n)TS_nx_n,Tu) \\ &\leq \alpha_nd(Tx_{n-1},Tu)+(1-\alpha_n)d(TSx_n,Tu) \\ &\leq \alpha_nd(Tx_{n-1},Tu)+(1-\alpha_n)h\,d(Tx_n,Tu), \end{aligned} \tag{2.7}$$

which gives

$$[1-(1-\alpha_n)h]d(Tx_n,Tu) \leq \alpha_nd(Tx_{n-1},Tu).$$

Thus we get

$$d(Tx_n, Tu) \leq \Big(\frac{\alpha_n}{1-(1-\alpha_n)h}\Big)\, d(Tx_{n-1}, Tu). \tag{2.8}$$

Let $A_n = \alpha_n$, $B_n = 1-(1-\alpha_n)h$.

Now, consider

$$\begin{aligned} 1-\frac{A_n}{B_n} &= 1-\frac{\alpha_n}{1-(1-\alpha_n)h} \\ &= \frac{1-(1-\alpha_n)h-\alpha_n}{1-(1-\alpha_n)h} \\ &= \frac{(1-h)(1-\alpha_n)}{1-(1-\alpha_n)h} \\ &\geq (1-h)(1-\alpha_n). \end{aligned} \tag{2.9}$$

From (2.9), we get

$$\frac{A_n}{B_n} \leq 1-(1-h)(1-\alpha_n). \tag{2.10}$$

From (2.8), we have the following inequality

$$d(Tx_n, Tu) \leq \frac{A_n}{B_n}\, d(Tx_{n-1}, Tu). \tag{2.11}$$

Using (2.10) in inequality (2.11), we get

$$\begin{aligned} d(Tx_n, Tu) &\leq [1-(1-h)(1-\alpha_n)]\, d(Tx_{n-1}, Tu) \\ &= (1-m_n)\, d(Tx_{n-1}, Tu), \end{aligned} \tag{2.12}$$

where $m_n = (1-h)(1-\alpha_n)$, since $0 < h < 1$ and by assumption of the theorem $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, it follows that $\sum_{n=1}^{\infty} m_n = \infty$, therefore by Lemma 1.8, we get that $\lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0$. Thus, we conclude that $\{Tx_n\}$ converges strongly to Tu, where u is the common point of the operators $\{S_i\}_{i=1}^{N}$ in C. This completes the proof.

Since T-Kannan contraction and T-Chatterjea contraction are both included in the T-Ciric quasi-contractive operators, by Theorem 2.1, we obtain the corresponding convergence results of the iteration process defined by (1.8) for the above said class of operators as corollaries:

Corollary 2.2. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-contractive operators with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ satisfying the condition:*

$$d(TS_ix, TS_iy) \leq b\Big[\frac{d(Tx, TS_ix) + d(Ty, TS_iy)}{2}\Big],$$

$\forall\, i = 1, 2, \ldots, N$; $\forall\, x, y \in C$ and $b \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^{N}$ in C.

Corollary 2.3. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-contractive operators with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ satisfying the condition:*

$$d(TS_ix, TS_iy) \leq c\Big[\frac{d(Tx, TS_ix) + d(Ty, TS_iy)}{2}\Big],$$

$\forall\, i = 1, 2, \ldots, N$; $\forall\, x, y \in C$ and $c \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^{N}$ in C.

Corollary 2.4. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^N \colon C \to C$ be N, T-Zamfirescu operators with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.8). If $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in C.*

Theorem 2.5. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^N \colon C \to C$ be N, T-Ciric quasi-contractive operators with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$ where T and S_n, $(n = 1, 2, \dots, N)$ are commuting mappings satisfying the condition:*

$$
\begin{aligned}
d(TS_ix, TS_iy) \quad \leq \quad & h \max\Big\{d(Tx,Ty),\ \frac{d(Tx,TS_ix)+d(Ty,TS_iy)}{2}, \\
& \frac{d(Tx,TS_iy)+d(Ty,TS_ix)}{2}\Big\} \quad (TCQC)
\end{aligned}
$$

$\forall\, i = 1, 2, \dots, N$; $\forall\, x, y \in C$ and $0 < h < 1$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.9). If $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^N$ in C.

Proof. From the assumption of the theorem $F = \cap_{i=1}^N F(S_i) \neq \emptyset$, it follows that operators $\{S_i\}_{i=1}^N$ have a common fixed point in C, say u. Consider $x, y \in C$. Since each S_i, $i = 1, 2, \dots, N$ is a T-Ciric quasi-contractive operator satisfying $(TCQC)$, then if

$$
\begin{aligned}
d(TS_ix, TS_iy) \quad &\leq \quad \frac{h}{2}\,[d(Tx,TS_ix)+d(Ty,TS_iy)] \\
&\leq \quad \frac{h}{2}\,[d(Tx,TS_ix)+d(Ty,Tx)+d(Tx,TS_ix)+d(TS_ix,TS_iy)]
\end{aligned}
$$

implies

$$\Big(1-\frac{h}{2}\Big)d(TS_ix,TS_iy) \leq \frac{h}{2}\,d(Tx,Ty) + h\,d(Tx,TS_ix),$$

which yields (using the fact that $0 < h < 1$)

$$d(TS_ix,TS_iy) \quad \leq \quad \Big(\frac{h/2}{1-h/2}\Big)\,d(Tx,Ty) + \Big(\frac{h}{1-h/2}\Big)\,d(Tx,TS_ix).$$

If

$$
\begin{aligned}
d(TS_ix, TS_iy) \quad &\leq \quad \frac{h}{2}\,[d(Tx,TS_iy)+d(Ty,TS_ix)] \\
&\leq \quad \frac{h}{2}\,[d(Tx,TS_ix)+d(TS_ix,TS_iy)+d(Ty,Tx)+d(Tx,TS_ix)],
\end{aligned}
$$

implies

$$\Big(1-\frac{h}{2}\Big)\,d(TS_ix,TS_iy) \leq \frac{h}{2}\,d(Tx,Ty) + h\,d(Tx,TS_ix),$$

which also yields (using the fact that $0 < h < 1$)

$$d(TS_ix,TS_iy) \quad \leq \quad \Big(\frac{h/2}{1-h/2}\Big)\,d(Tx,Ty) + \Big(\frac{h}{1-h/2}\Big)\,d(Tx,TS_ix).$$

Denote

$$\delta = \max\Big\{h,\ \frac{h/2}{1-h/2}\Big\} = h,$$

$$L = \frac{h}{1-h/2}.$$

Thus, in all cases,

$$
\begin{aligned}
d(TS_ix, TS_iy) \quad &\leq \quad \delta\,d(Tx,Ty) + L\,d(Tx,TS_ix) \\
&= \quad h\,d(Tx,Ty) + \Big(\frac{h}{1-h/2}\Big)d(Tx,TS_ix)
\end{aligned}
$$

holds for all $x,\, y \in C$.

Also from $(TCQC)$ with $y = u = S_i u$ for all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
d(TS_i x, TS_i u) &\leq h \max\Big\{d(Tx, Tu),\, \frac{d(Tx, TS_i x)}{2},\, \frac{d(Tx, TS_i u) + d(Tu, TS_i x)}{2}\Big\} \\
&\leq h \max\Big\{d(Tx, Tu),\, \frac{d(Tx, Tu) + d(Tu, TS_i x)}{2},\, \frac{d(Tx, Tu) + d(Tu, TS_i x)}{2}\Big\} \\
&= h \max\Big\{d(Tx, Tu),\, \frac{d(Tx, Tu) + d(Tu, TS_i x)}{2}\Big\} \\
&\leq h\, d(Tx, Tu). \qquad (2.13)
\end{aligned}$$

Since $S_i u = u$, $S_n = S_{n\,(mod\,N)}$ and the $mod\,N$ function takes values in $\{1, 2, \ldots, N\}$, taking $x = x_n$ in (2.13), we obtain

$$d(TS_n x_n, Tu) \leq h\, d(Tx_n, Tu) \qquad (2.14)$$

and

$$d(TS_n y_n, Tu) \leq h\, d(Ty_n, Tu). \qquad (2.15)$$

Using (1.9), (2.14) and Lemma 1.7(ii), we have

$$\begin{aligned}
d(Ty_n, Tu) &= d(\beta_n Tx_{n-1} \oplus (1-\beta_n) TS_n x_n, Tu) \\
&\leq \beta_n\, d(Tx_{n-1}, Tu) + (1-\beta_n)\, d(TS_n x_n, Tu) \\
&\leq \beta_n\, d(Tx_{n-1}, Tu) + (1-\beta_n) h\, d(Tx_n, Tu). \qquad (2.16)
\end{aligned}$$

Again using (1.9), (2.15), (2.16), $S_n T = TS_n$ (by assumption of the theorem) and Lemma 1.7(ii), we have

$$\begin{aligned}
d(Tx_n, Tu) &= d(\alpha_n Tx_{n-1} \oplus (1-\alpha_n) S_n Ty_n, Tu) \\
&\leq \alpha_n d(Tx_{n-1}, Tu) + (1-\alpha_n) d(S_n Ty_n, Tu) \\
&= \alpha_n d(Tx_{n-1}, Tu) + (1-\alpha_n)\, d(TS_n y_n, Tu) \\
&\leq \alpha_n d(Tx_{n-1}, Tu) + (1-\alpha_n) h\, d(Ty_n, Tu) \\
&\leq \alpha_n d(Tx_{n-1}, Tu) + (1-\alpha_n) h\, [\beta_n\, d(Tx_{n-1}, Tu) \\
&\quad + (1-\beta_n) h\, d(Tx_n, Tu)] \\
&= [\alpha_n + (1-\alpha_n)\beta_n h] d(Tx_{n-1}, Tu) \\
&\quad + (1-\alpha_n)(1-\beta_n) h^2\, d(Tx_n, Tu).
\end{aligned}$$

Thus we get that the inequality

$$[1 - (1-\alpha_n)(1-\beta_n)h^2] d(Tx_n, Tu) \leq [\alpha_n + (1-\alpha_n)\beta_n h]\, d(Tx_{n-1}, Tu),$$

which implies that

$$d(Tx_n, Tu) \leq \Big(\frac{\alpha_n + (1-\alpha_n)\beta_n h}{1 - (1-\alpha_n)(1-\beta_n)h^2}\Big) d(Tx_{n-1}, Tu). \qquad (2.17)$$

Now, let $M_n = \alpha_n + (1-\alpha_n)\beta_n h$ and $N_n = 1 - (1-\alpha_n)(1-\beta_n)h^2$. Then

$$\begin{aligned}
1 - \frac{M_n}{N_n} &= 1 - \Big(\frac{\alpha_n + (1-\alpha_n)\beta_n h}{1 - (1-\alpha_n)(1-\beta_n)h^2}\Big) \\
&= \frac{1 - [(1-\alpha_n)(1-\beta_n)h^2 + \alpha_n + (1-\alpha_n)\beta_n h]}{1 - (1-\alpha_n)(1-\beta_n)h^2}.
\end{aligned}$$

Since $1 - (1-\alpha_n)(1-\beta_n)h^2 \leq 1$, then from the above inequality, we get that

$$1 - \frac{M_n}{N_n} \geq 1 - [(1-\alpha_n)(1-\beta_n)h^2 + \alpha_n + (1-\alpha_n)\beta_n h],$$

which implies

$$\frac{M_n}{N_n} \le (1-\alpha_n)(1-\beta_n)h^2 + \alpha_n + (1-\alpha_n)\beta_n h.$$

Since $0 < h < 1$ and $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$, this gives

$$\begin{aligned} \frac{M_n}{N_n} &\le (1-\alpha_n)(1-\beta_n) + \alpha_n + (1-\alpha_n)\beta_n h \\ &= 1 - \beta_n(1-\alpha_n) + (1-\alpha_n)\beta_n h \\ &= 1 - \beta_n(1-\alpha_n)(1-h). \end{aligned} \tag{2.18}$$

From (2.17), we have the following inequality

$$d(Tx_n, Tu) \le \frac{M_n}{N_n}\, d(Tx_{n-1}, Tu).$$

Using equation (2.18) in the above inequality, we obtain

$$\begin{aligned} d(Tx_n, Tu) &\le [1 - \beta_n(1-\alpha_n)(1-h)]\, d(Tx_{n-1}, Tu) \\ &\le (1 - R_n)\, d(Tx_{n-1}, Tu), \end{aligned} \tag{2.19}$$

where $R_n = \beta_n(1-\alpha_n)(1-h)$, since $0 < h < 1$ and by assumption of the theorem $\sum_{n=1}^{\infty} (1-\alpha_n)\beta_n = \infty$, it follows that $\sum_{n=1}^{\infty} R_n = \infty$, therefore by Lemma 1.8, we get that $\lim_{n\to\infty} d(Tx_{n-1}, Tu) = 0$. Thus, we conclude that $\{Tx_n\}$ converges strongly to Tu, where u is the common point of the operator $\{S_i\}_{i=1}^{N}$ in C. This completes the proof.

By Theorem 2.5, we get the following convergence results of the iterative process defined by (1.9) for a finite family of T-Kannan contraction, T-Chatterjea contraction and T-Zamfirescu operator as corollaries to our result:

Corollary 2.6. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-Kannan contraction with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ where T and S_n are commuting mappings satisfying the condition*

$$d(TS_ix, TS_iy) \le b\Big[\frac{d(Tx, TS_ix) + d(Ty, TS_iy)}{2}\Big],$$

$\forall\, i = 1, 2, \dots, N$; $\forall\, x, y \in C$ and $b \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.9). If $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^{N}$ in C.

Corollary 2.7. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-Chatterjea contraction with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ where T and S_n are commuting mappings satisfying the condition*

$$d(TS_ix, TS_iy) \le b\Big[\frac{d(Tx, TS_ix) + d(Ty, TS_iy)}{2}\Big],$$

$\forall\, i = 1, 2, \dots, N$; $\forall\, x, y \in C$ and $b \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.9). If $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^{N}$ in C.

Corollary 2.8. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T\colon C \to C$ and let $\{S_i\}_{i=1}^{N}\colon C \to C$ be N, T-Zamfirescu operator with $F = \cap_{i=1}^{N} F(S_i) \neq \emptyset$ where T and S_n are commuting mappings satisfying the condition*

$$d(TS_ix, TS_iy) \le b\Big[\frac{d(Tx, TS_ix) + d(Ty, TS_iy)}{2}\Big],$$

$\forall\, i = 1, 2, \dots, N$; $\forall\, x, y \in C$ and $b \in (0, \frac{1}{2})$. Let $\{Tx_n\}$ be defined by the iteration scheme (1.9). If $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$, then $\{Tx_n\}$ converges strongly to Tu, where u is the common fixed point of the operators $\{S_i\}_{i=1}^{N}$ in C.

Theorem 2.9. *Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $\{S_i\}_{i=1}^N: C \to C$ be N operators with $F = \cap_{i=1}^N F(S_i) \neq \emptyset$ satisfying the condition*

$$d(S_i x, S_i y) \quad \leq \quad h \max\Big\{d(x,y),\, \frac{d(x,S_i x)+d(y,S_i y)}{2},\, d(x,S_i y),\, d(y,S_i x)\Big\}, \quad (2.20)$$

$\forall\, i = 1,2,\ldots,N$; $\forall\, x,\, y \in C$ and $0 < h < 1$. Let $\{x_n\}$ be defined by the iteration scheme (1.11). If $\sum_{n=1}^{\infty}(1-\alpha_n)\beta_n = \infty$, then $\{x_n\}$ converges strongly to a common fixed point of the operators $\{S_i\}_{i=1}^N$ in C.

Proof. The proof of Theorem 2.9 is similar to that of Theorem 2.5 by taking $T = I$. This completes the proof.

Remark 2.10. Theorem 2.1 and 2.5 extend Theorem 2.1 and 2.5 of Raphael and Pulickakunnel [37] (Functional Analysis, Approximation and Computation 5(2) (2013), 1-9) to the case of T-Ciric quasi-contractive operator and from normed linear space to $CAT(0)$ spaces considered in this paper.

Remark 2.11. Our results generalize, improve and extend some recent results from the current existing literature.

References

[1] A. Abkar and M. Eslamian, Common fixed point results in CAT(0) spaces, *Nonlinear Anal.: TMA* **74(5)**, 1835–1840 (2011).

[2] H. H. Bauschke, The approximation of fixed points of compositions of nonexpansive mappings in Hilbert spaces, *J. Math. Anal. Appl.* **202**, 150–159 (1996).

[3] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, Two fixed point theorems for spcial mappings, arXiv:0903.1504v1 [math.FA].

[4] M. R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Vol. **319** of Grundlehren der Mathematischen Wissenschaften, Springer, Berlin, Germany, (1999).

[5] K. S. Brown, Buildings, Springer, New York, NY, USA, (1989).

[6] F. Bruhat and J. Tits, "Groups reductifs sur un corps local", Institut des Hautes Etudes Scientifiques. *Publications Mathematiques* **41**, 5–251 (1972).

[7] L. B. Ciric, A generalization of Banach principle, *Proc. Amer. Math. Soc.* **45**, 727–730 (1974).

[8] P. Chaoha and A. Phon-on, A note on fixed point sets in CAT(0) spaces, *J. Math. Anal. Appl.* **320(2)**, 983–987 (2006).

[9] S. K. Chatterjee, Fixed point theorems compactes, *Rend. Acad. Bulgare Sci.* **25**, 727–730 (1972).

[10] C. E. Chidume and N. Shahzad, Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, *Nonlinear Anal.* **62**, 1149–1156 (2005).

[11] S. Dhompongsa, A. Kaewkho and B. Panyanak, Lim's theorems for multivalued mappings in CAT(0) spaces, *J. Math. Anal. Appl.* **312(2)**, 478–487 (2005).

[12] S. Dhompongsa and B. Panyanak, On $\triangle$-convergence theorem in CAT(0) spaces, *Comput. Math. Appl.* **56(10)**, 2572–2579 (2008).

[13] R. Espinola and A. Fernandez-Leon, CAT(k)-spaces, weak convergence and fixed point, *J. Math. Anal. Appl.* **353(1)**, 410–427 (2009).

[14] K. Goebel and S. Reich, Uniform convexity, hyperbolic geometry, and nonexpansive mappings, Vol. **83** of Monograph and Textbooks in Pure and Applied Mathematics, Marcel Dekker Inc., New York, NY, USA, (1984).

[15] F. Gu, Strong and weak convergence of implicit iterative process with errors for asymptotically nonexpansive mappings, *J. Appl. Anal.* **12(2)**, 267–282 (2006).

[16] N. Hussain and M. A. Khamsi, On asymptotic pointwise contractions in metric spaces, *Nonlinear Anal.: TMA* **71(10)**, 4423–4429 (2009).

[17] R. Kannan, Some results on fixed point theorems, *Bull. Calcutta Math. Soc.* **10**, 71–76 (1968).

[18] M. A. Khamsi and W. A. Kirk, An introduction to metric spaces and fixed point theory, *Pure Appl. Math*, Wiley-Interscience, New York, NY, USA, (2001).

[19] S. H. Khan and M. Abbas, Strong and $\triangle$-convergence of some iterative schemes in CAT(0) spaces, *Comput. Math. Appl.* **61(1)**, 109–116 (2011).

[20] A. R. Khan, M. A. Khamsi and H. Fukhar-ud-din, Strong convergence of a general iteration scheme in CAT(0) spaces, *Nonlinear Anal.: TMA* **74(3)**, 783–791 (2011).

[21] W. A. Kirk, Fixed point theory in CAT(0) spaces and $\mathbb{R}$-trees, *Fixed Point Theory and Applications* **2004:4**, 309–316 (2004).

[22] W. A. Kirk, Geodesic geometry and fixed point theory, in Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), Vol. **64** of Coleccion Abierta, 195-225, University of Seville Secretary of Publications, Seville, Spain, (2003).

[23] W. A. Kirk, Geodesic geometry and fixed point theory II, in International Conference on Fixed point Theory and Applications, 113-142, Yokohama Publishers, Yokohama, Japan, (2004).

[24] W. Laowang and B. Panyanak, Strong and $\triangle$ convergence theorems for multivalued mappings in CAT(0) spaces, *J. Inequal. Appl.* Vol. 2009, Article ID 730132, 16 pages.

[25] L. Leustean, A quadratic rate of asymptotic regularity for CAT(0)-spaces, *J. Math. Anal. Appl.* **325(1)**, 386–399 (2007).

[26] S. Moradi, Kannan fixed point theorem on complete metric spaces and on generalized metric spaces depended on another function, arXiv:0903.1577v1 [math.FA].

[27] J. Morales and E. Rojas, Some results on T-Zamfirescu operators, *Revista Notas de Matematics* **5(1)**, 64–71 (2009).

[28] J. Morales and E. Rojas, Cone metric spaces and fixed point theorems of T-Kannan contractive mappings, *Int. J. Math. Anal.* **4(4)**, 175–184 (2010).

[29] J. Morales and E. Rojas, T-Zamfirescu and T-weak contraction mappings on cone metric spaces, arXiv:0909.1255v1 [math.FA].

[30] Y. Niwongsa and B. Panyanak, *Noor iterations for asymptotically nonexpansive mappings in CAT(0) spaces*, Int. J. Math. Anal. Vol. **4(13)** (2010), 645-656.

[31] M. O. Osilike, Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps, *J. Math. Anal. Appl.* **294**, 73–81 (2004).

[32] S. Plubtieng, R. Wangkeeree and R. Punpaeng, On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.* **322(2)**, 1018–1029 (2006).

[33] S. Pulickakunnel and N. Singh, Approximation of common fixed points for a finite family of Zamfirescu operators, *General Math.* **18(4)**, 175–184 (2010).

[34] S. Pulickakunnel and N. Singh, A new three-step iterative process for common fixed points of a finite family of Z-operators, *J. Contem. App. Math.* **1(1)**, 13–19 (2011).

[35] A. Rafiq, Strong convergence of a modified implicit iteration process for a finite family of Z-operators, *Int. J. Math. Math. Sci.* (2006), Article ID 10328, 1–6.

[36] A. Rafiq, Fixed points of Ciric quasi-contractive operators in generalized convex metric spaces, *General Math.* **14(3)**, 79–90 (2006).

[37] P. Raphael and S. Pulickakunnel, Convergence theorems for a finite family of T-Zamfirescu operators, *Funct. Anal. Approx. Comput.* **5(2)**, 1–9 (2013).

[38] S. Saejung, Halpern's iteration in CAT(0) spaces, *Fixed Point Theory and Applications* Vol. 2010, Article ID 471781, 13 pages.

[39] G. S. Saluja and H. K. Nashine, Strong convergence of an implicit iteration process for a finite family of strictly asymptotically pseudocontractive mappings, *CUBO A Mathematical Journal* **13(1)**, 137–147 (2011).

[40] G. S. Saluja, Convergence theorems of two-step implicit iteration process with errors for a finite family of asymptotically nonexpansive mappings, *Mat. Vesnik* **65(3)**, 285–298 (2013).

[41] N. Shahzad, Fixed point results for multimaps in CAT(0) spaces, *Topology and its Applications* **156(5)**, 997–1001 (2009).

[42] Z. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, *J. Math. Anal. Appl.* **286(1)**, 351-358 (2003).

[43] R. Wittmann, Approximation of fixed points of nonexpansive mappings, *Arch. Math.* **58**, 486–491 (1992).

[44] B. L. Xu and M. A. Noor, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **267(2)**, 444–453 (2002).

[45] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, *Numer. Funct. Anal. Optim.* **22 (5-6)**, 767–773 (2001).

[46] T. Zamfirescu, Fixed point theorems in metric space, *Arch. Math. (Basel)*, **23**, 292–298 (1972).

Author information

G. S. Saluja, Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur - 492010 (C.G.), India.
E-mail: saluja1963@gmail.com

Received: November 8, 2013.

Accepted: May 22, 2014.