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Total Mean Cordiality of $K_n^c + 2K_2$

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Abstract. A Total Mean Cordial labeling of a graph $G = (V, E)$ is a mapping $f : V(G) \to \{0, 1, 2\}$ such that $f(xy) = \left\lceil \frac{f(x)+f(y)}{2} \right\rceil$ where $x, y \in V(G)$, $xy \in G$, and the total number of 0, 1 and 2 are balanced. That is $|ev_f(i) - ev_f(j)| \leq 1$, $i, j \in \{0, 1, 2\}$ where $ev_f(x)$ denotes the total number of vertices and edges labeled with x $(x = 0, 1, 2)$. If there exists a total mean cordial labeling on a graph G, we will call G is Total Mean Cordial. In this paper, it is shown that $K_n^c + 2K_2$ is Total Mean Cordial iff $n = 1$ or 2 or 4 or 6 or 8.

1 Introduction

By a graph we mean a finite unoriented graph without loops and multiple edges. A general reference for graph theoretic ideas can be seen in [2]. A vertex labeling of a graph G is an assignment f of labels to the vertices of G that induces for each $uv \in E(G)$ a label depending on the vertex labels $f(u)$ and $f(v)$. The vertex and edge set of a graph G are denoted by $V(G)$ and $E(G)$ so that the order and size of G are respectively $|V(G)|$ and $|E(G)|$. Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their join $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2 \cup \{uv : u \in V_1 \text{ and } v \in V_2\}$. The notion of Total Mean cordial labeling was introduced and studied by Ponraj, Ramasamy and Sathish Narayanan [3]. Let f be a function from $V(G) \to \{0, 1, 2\}$. For each edge uv, assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a Total Mean Cordial labeling if $|ev_f(i) - ev_f(j)| \leq 1$ where $ev_f(x)$ denote the total number of vertices and edges labeled with x $(x = 0, 1, 2)$. A graph with a Total Mean Cordial labeling is called Total Mean Cordial graph. In this paper, we investigate the Total Mean Cordial labeling behaviour of $K_n^c + 2K_2$. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x.

2 Main result

Theorem 2.1. $K_n^c + 2K_2$ is Total Mean Cordial if and only if $n = 1$ or 2 or 4 or 6 or 8.

Proof. Let $V(K_n^c + 2K_2) = \{u, v, x, y, u_i : 1 \leq i \leq n\}$ and $E(K_n^c + 2K_2) = \{uv, xy, \} \cup \{uu_i, vu_i, xu_i, yu_i : 1 \leq i \leq n\}$. It is clear that $|V(G)| + |E(G)| = 5n + 6$. Let l denotes the number of zeros to be used in u_i $(1 \leq i \leq n)$ and that for the label 2, we use r. Suppose f is a Total Mean Cordial labeling of $K_n^c + 2K_2$.
Case 1. $n \equiv 0 (\text{mod } 3)$.
Let $n = 3t$. Then $|V(G)| + |E(G)| = 15t + 6$. Here $ev_f(0) = ev_f(1) = ev_f(2) = 5t + 2$. Consider the set $S = \{u, v, x, y\}$ and the label 0. Here there are five possible cases.

* All the four vertices of S are labeled by 0.
* Any three of them are labeled by 0.
* Any two of them are labeled with 0. [This may be adjacent vertices or two non adjacent vertices.]
* Only one vertex is labeled by 0.
* None of them received the label 0.

Now we discuss all the cases given above.
Subcase 1. $f(u) = f(v) = f(x) = f(y) = 0$.

Here, $ev_f(2) \leq 3t$, a contradiction.
Subcase 2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t$, a contradiction.
Subcase 3. Any three of them are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. So the vertices in S contributes only 4 zeros. We should utilize the remaining $5t - 2$ zeros for both vertices and edges. In this case, if u_i is labeled with 0, apart from this label, each vertex u_i contributes 3 zeros. So we have $l+3l = 5t-2$. Therefore $l = \frac{5t-2}{4}$. This is possible only when $t \equiv 2 \pmod 4$. Now consider the label 2. Suppose $f(y) = 2$ then $r + r + 1 = 5t + 2$. This implies $r = \frac{5t+1}{2}$, a contradiction since $t \equiv 2 \pmod 4$. Suppose $f(y) = 1$. Then $r = \frac{5t+2}{2}$. But $l + r > 3t$, a contradiction.
Subcase 4. Any two vertices from S are labeled by zero.
First we assume that any two adjacent vertices in u, v, x, y are labeled with zero. Without loss of generality we assume that $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. At present we have used 3 zeros. In this case each u_i contributes two edges with label zero. So, $l + 2l + 3 = 5t + 2$. That is $l = \frac{5t-1}{3}$. Such a positive integer l exists only if $t \equiv 2 \pmod 3$. Suppose $t \equiv 2 \pmod 3$. Consider the label 2. If $f(x) = f(y) = 2$. Then $r = \frac{5t-1}{3}$. It is clear that $l + r \leq 3t$. This is true only when $t \leq 2$. Since $t \equiv 2$ (mod 3), $t \neq 1$. If $t = 2$, the following figure 1 shows that $K_6^c + 2K_2$ is Total Mean Cordial.

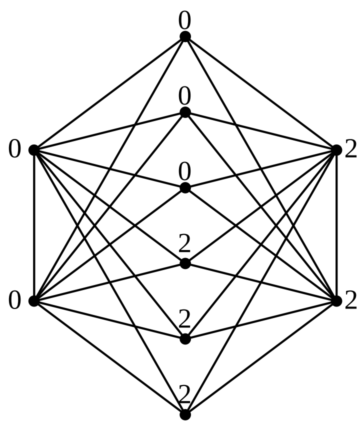

Figure 1.

Suppose $f(x) = 2$, $f(y) = 1$. In this case $r + 2r + 2 = 5t + 2$. That is $r = \frac{5t}{3}$. Since $t \equiv 2 \pmod 3$, such a postive integer r does not exists. If $f(x) = f(y) = 1$ then $r+2r = 5t+2$. Hence $r = \frac{5t+2}{3}$. Then $l + r > 3t$, a contradiction. Now consider the case, zero is labeled with any two non adjcent vertices from the set S. Without loss of generality assume that $f(u) = 0$ and $f(y) = 0$. Here, $l + 2l + 2 = 5t + 2$. Therefore $l = \frac{5t}{3}$. This is true only if $t \equiv 0 \pmod 3$. If $f(v) = f(y) = 2$ then in this case $r + 2r + 2 = 5t + 2$. This implies $r = \frac{5t}{3}$. Here $l + r > 3t$, a contradiction. Suppose $f(v) = 1$, $f(y) = 2$. Here $r + 2r + 1 = 5t + 2$. Then $r = \frac{5t+1}{3}$. Since $t \equiv 0$ (mod 3), r can not be a positive integer. Assume that $f(v) = f(y) = 1$. In this case $r + 2r = 5t + 2$. This implies $r = \frac{5t+2}{3}$. Since $t \equiv 0$ (mod 3), r is not a positive integer.
Subcase 5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. In this case each vertex u_i contributes one edge with label zero. Hence $l + l + 1 = 5t + 2$. That is $l = \frac{5t+1}{2}$. Since l is an positive integer, $t \equiv 1 \pmod 2$. Now assume $f(x) = f(y) = f(v) = 2$. Then $r + 3r + 4 = 5t + 2$. Therefore $r = \frac{5t-2}{4}$. Since $t \equiv 1 \pmod 2$, r can not be a positive integer. Suppose $f(x) = f(y) = f(v) = 1$. In this case, $r + 3r = 5t + 2$. Hence $r = \frac{5t+2}{4}$. Since $t \equiv 1 \pmod 2$, r can not be a positive integer. If $f(x) = f(y) = 1$ and $f(v) = 2$. Then $r + 3r + 1 = 5t + 2$. That is $r = \frac{5t+1}{4}$. Again a contradiction since $l+r > 3t$. For $f(x) = 1$ and $f(y) = f(v) = 2$, we have $r+3r+2 = 5t+2$. Hence $r = \frac{5t}{4}$, a contradiction since $t \equiv 1 \pmod 2$. If $f(v) = 1$ and $f(x) = f(y) = 2$ then $r + 3r + 3 = 5t + 2$. That is $r = \frac{5t-1}{4}$. It follows that $l + r > 3t$, a contradiction. Suppose $f(v) = f(x) = 1$ and $f(y) = 2$. In this case $r + 3r + 2 = 5t + 2$. Hence $r = \frac{5t}{4}$ a contradiction since $t \equiv 1 \pmod 2$.
Case 2. $n \equiv 1 \pmod 3$.

Let $n = 3t+1$. Then $|V(G)| + |E(G)| = 15t + 11$. Here we have three possibilities.

a. $ev_f(0) = ev_f(2) = 5t+4$, $ev_f(1) = 5t+3$ or

b. $ev_f(0) = ev_f(1) = 5t+4$, $ev_f(2) = 5t+3$ or

c. $ev_f(1) = ev_f(2) = 5t+4$, $ev_f(0) = 5t+3$.

Suppose $ev_f(0) = ev_f(2) = 5t+4$, $ev_f(1) = 5t+3$.
Subcase a1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \le 3t+1$, a contradiction.
Subcase a2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \le 3t+1$, a contradiction.
Subcase a3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. Here, if a vertex u_i is labeled with zero then it contributes three edges with label zero. So we have $l + 3l + 4 = 5t+4$. Therefore $l = \frac{5t}{4}$. Such a positive integer l exisits only when $t \equiv 0 \pmod 4$. Now suppose $f(y) = 2$. A vertex u_i with label 2 contributes one edge with label 2. So $r + r + 1 = 5t + 4$. That is $r = \frac{5t+3}{2}$. Since $t \equiv 0 \pmod 4$, r can not be a positive integer. If $f(y) = 1$ then in this case also a vertex u_i with label 2 contributes one edge with label 2. Therefore $r + r = 5t + 4$. That is $r = \frac{5t+4}{2}$. But $l + r > 3t + 1$, a contradiction.
Subcase a4. Any two vertices of S are labeled with zero.
First we assume that $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. In this case, if a vertex u_i is labeled by zero then it gives two edges with label zero. Therefore $l + 2l + 3 = 5t + 4$. That is $l = \frac{5t+1}{3}$. Since l is a positive integer, $t \equiv 1 \pmod 3$. Consider the vertices x and y. Suppose these two vertices are labeled with 1. If a vertex u_i is labeled by 2 then each u_i contributes two edges with labele 2. It follows that $r + 2r = 5t + 4$. Hence $r = \frac{5t+4}{3}$. But $l + r > 3t + 1$, a contradiction. Suppose the vertices x and y are labeled by 2. Here a vertex u_i with label 2 contributes two edges with label 2. Then $r + 2r + 3 = 5t + 4$. Therefore, $r = \frac{5t+1}{3}$. We know that $l + r \le 3t + 1$. This is true only when $t = 1$. The Total Mean Cordial labeling of $K_4^c + 2K_2$ is given in figure 2. Suppose $f(x) = 1$ and $f(y) = 2$. Then each vertex u_i with a

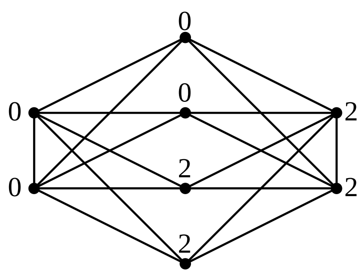

Figure 2.

label 2 contributes two edges with label 2. This implies $r + 2r + 2 = 5t + 4$ and hence $r = \frac{5t+2}{3}$. Since $t \equiv 1 \pmod 3$, such a positive integer r does not exists. Suppose $f(u) = f(x) = 0$. Here $l + 2l + 2 = 5t + 4$. Therefore $l = \frac{5t+2}{3}$. It follows that $t \equiv 2 \pmod 3$. Now assume $f(y) = f(v) = 2$. Then $r + 2r + 2 = 5t + 4$. That is $r = \frac{5t+2}{3}$. But $l + r > 3t + 1$, a contradiction. If $f(y) = f(v) = 1$ then $r + 2r = 5t + 4$. Hence $r = \frac{5t+4}{3}$, a contradiction since $t \equiv 2 \pmod 3$. For $f(y) = 1$ and $f(v) = 2$, we have $r + 2r + 1 = 5t + 4$. That is $r = \frac{5t+3}{3}$, again a contradiction since $t \equiv 2 \pmod 3$.
Subcase a5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. In this case $l + l + 1 = 5t + 4$ and hence $l = \frac{5t+3}{2}$. Since l is a positive integer, $t \equiv 1 \pmod 2$. If $f(v) = f(x) = f(y) = 1$ then $r + 3r = 5t + 4$. This implies $r = \frac{5t+4}{4}$. For the values of t, r could not be an integer. If $f(v) = 1$ and $f(x) = f(y) = 2$ then $r + 3r + 3 = 5t + 4$. Therefore $r = \frac{5t+1}{4}$. Here, $l + r > 3t + 1$, a contradiction. If $f(v) = 2$ and $f(x) = f(y) = 1$ then $r + 3r + 1 = 5t + 4$. This implies $r = \frac{5t+3}{4}$. Here also $l + r > 3t + 1$, a contradiction. Suppose $f(v) = f(x) = 1$ and $f(y) = 2$. Here $r + 3r + 2 = 5t + 4$ and hence $r = \frac{5t+2}{4}$. This is impossible since $t \equiv 1 \pmod 2$. For $f(v) = f(x) = f(y) = 2$, we have $3r + r + 4 = 5t + 4$. Therefore $r = \frac{5t}{4}$. But $t \equiv 1 \pmod 2$, a contradiction.

Consider the case $ev_f(0) = ev_f(1) = 5t + 4$ and $ev_f(2) = 5t + 3$.
Subcase b1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \leq 3t + 1$, a contradiction.
Subcase b2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t + 1$, a contradiction.
Subcase b3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. In this case $l+3l+4 = 5t+4$. That is $l = \frac{5t}{4}$. Since l is a positive integer, $t \equiv 0 \pmod 4$. Suppose $f(y) = 2$ then $r + r + 1 = 5t + 3$. Therefore $r = \frac{5t+2}{2}$. But $l + r \leq 3t + 1$. This is true only if $t = 0$. A Total Mean Cordial labeling of $K_1^c + 2K_2$ is given in figure 3. For $f(y) = 1$, $r + r = 5t + 3$.

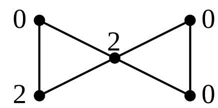

Figure 3.

Therefore $r = \frac{5t+3}{2}$. This is a contradiction to $t \equiv 0 \pmod 4$.
Subcase b4. Any two vertices of S are labeled with zero.
First we assume that $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. Then $l + 2l + 3 = 5t + 4$. This implies $l = \frac{5t+1}{3}$. It follows that $t \equiv 1 \pmod 3$. Now assume that $f(x) = f(y) = 2$. In this case $r + 2r + 3 = 5t + 3$. Then $r = \frac{5t}{3}$, a contradiction to $t \equiv 1 \pmod 3$. If $f(x) = f(y) = 2$ then $r + 2r = 5t + 3$ and hence $r = \frac{5t+3}{3}$. This is impossible since $t \equiv 1 \pmod 3$. For $f(x) = 1$ and $f(y) = 2$, we have $r+2r+2 = 5t+3$. Then $r = \frac{5t+1}{3}$. But $l+r \leq 3t+1$. This is true only when $t \leq 1$. Since $t \equiv 1 \pmod 3$, $t \neq 0$. For $t = 1$, the Total Mean Cordial labeling of $K_4^c + 2K_2$ is given in figure 4. Suppose $f(u) = f(x) = 0$ and $f(v) \neq 0$, $f(y) \neq 0$. Then $l + 2l + 2 = 5t + 4$

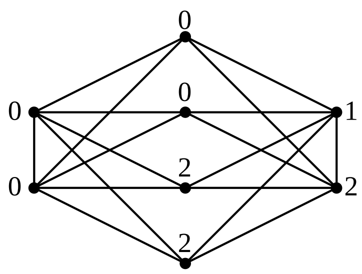

Figure 4.

and therefore $l = \frac{5t+2}{3}$. It follows that $t \equiv 2 \pmod 3$. Consider the vertices v and y. Suppose these two vertices are labeled by 2 then $r + 2r + 2 = 5t + 3$. So we have $r = \frac{5t+1}{3}$. Since $t \equiv 2 \pmod 3$, such a positive integer does not exists. Suppose v and y are labeled by 1. In this case $r + 2r = 5t + 3$ and hence $r = \frac{5t+3}{3}$. This is impossible since $t \equiv 2 \pmod 3$. If $f(v) = 1$ and $f(y) = 2$ then $r + 2r + 1 = 5t + 3$. That is $r = \frac{5t+2}{3}$. But $l + r > 3t + 1$ a contradiction.
Subcase b5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. In this case $l + l + 1 = 5t + 4$. Thus $l = \frac{5t+3}{2}$. This is true only if $t \equiv 1 \pmod 2$. Suppose $f(v) = f(x) = f(y) = 2$. Here $r + 3r + 4 = 5t + 3$. Then $r = \frac{5t-1}{4}$. But $l + r > 3t + 1$, a contradiction. If $f(v) = f(x) = f(y) = 1$ then $r + 3r = 5t + 3$ and hence $r = \frac{5t+3}{4}$. In this case $l + r > 3t + 1$, a contradiction. Suppose $f(v) = 2$ and $f(x) = f(y) = 1$. Here $r + 3r + 1 = 5t + 3$. Thus $r = \frac{5t+2}{4}$. This is impossible since $t \equiv 1 \pmod 2$. For $f(v) = f(y) = 2$ and $f(x) = 1$, we have $r + 3r + 3 = 5t + 3$. This implies $r = \frac{5t}{4}$, a contradiction to $t \equiv 1 \pmod 2$. If $f(v) = 1$ and $f(x) = f(y) = 2$ then $r+3r+3 = 5t+3$. Hence $r = \frac{5t}{3}$. Again a contradiction to $t \equiv 1 \pmod 2$. For $f(v) = f(x) = 1$ and $f(y) = 2$, $r + 3r + 2 = 5t + 3$. Therefore $r = \frac{5t+1}{4}$. But $l + r > 3t + 1$, a contradiction.
Suppose $ev_f(1) = ev_f(2) = 5t + 4$ and $ev_f(0) = 5t + 3$.

Subcase c1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \leq 3t + 1$, a contradiction.
Subcase c2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t + 1$, a contradiction.
Subcase c3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. Here $l + 3l + 4 = 5t + 3$. Thus $l = \frac{5t-1}{4}$. This implies $t \equiv 1 \pmod 4$. Now assume $f(y) = 2$. Then $r + r + 1 = 5t + 4$. Hence $r = \frac{5t+3}{2}$. But $l + r > 3t + 1$, a contradiction. For $f(y) = 2$, we have $r + r = 5t + 4$. This implies $\frac{5t+4}{2}$. This is a contradiction to $t \equiv 1 \pmod 4$.
Subcase c4. Any two vertices of S are labeled with zero.
Assume $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. In this case $l + 2l + 3 = 5t + 3$. Therefore $l = \frac{5t}{3}$. Such a positive integer l exists only if t is a multiple of 3. Suppose $f(x) = f(y) = 2$ then $r + 2r + 3 = 5t + 4$. Thus $r = \frac{5t+1}{3}$. This shows that t is not a multiple of 3, a contradiction. If $f(x) = f(y) = 1$ then $r + 2r = 5t + 4$. Therefore $r = \frac{5t+4}{3}$. Since $t \equiv 0 \pmod 3$, such a positive integer r does not exists. Suppose $f(x) = 1$ and $f(y) = 2$ then $r + 2r + 2 = 5t + 4$. Therefore $r = \frac{5t+2}{3}$, a contradiction since $t \equiv 0 \pmod 3$. Consider the case $f(u) = f(x) = 0$ and $f(v) \neq f(y) \neq 0$. Here $l + 2l + 2 = 5t + 3$. Thus $l = \frac{5t+1}{3}$. It follows that $t \equiv 1 \pmod 3$. Suppose $f(v) = f(y) = 2$ then $r + 2r + 2 = 5t + 4$. Hence $r = \frac{5t+2}{3}$. Since $t \equiv 1 \pmod 3$, r could not be a positive integer. If $f(v) = f(y) = 1$ then $r + 2r = 5t + 4$. Therefore $r = \frac{5t+4}{3}$. But $l + r > 3t + 1$, a contradiction. For $f(v) = 1$ and $f(y) = 2$, we have $r + 2r + 1 = 5t + 4$ and hence $r = \frac{5t+3}{3}$, a contradiction since $t \equiv 1 \pmod 3$.
Subcase c5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. Here, $l + l + 1 = 5t + 3$ and hence $l = \frac{5t+2}{2}$. It follows that $t \equiv 0 \pmod 2$. Now assume $f(v) = f(x) = f(y) = 2$. Then $r + 3r + 4 = 5t + 4$. This implies $r = \frac{5t}{4}$. But $l + r \leq 3t + 1$. This is true only when $t = 0$. $K_1^c + 2K_2$ with a Total Mean Cordial labeling is given in figure 5. Suppose $f(v) = f(x) = f(y) = 1$. In this case

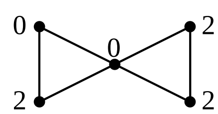

Figure 5.

$r + 3r = 5t + 4$. Therefore $r = \frac{5t+4}{4}$. But $l + r > 3t + 1$, a contradiction. If $f(v) = 2$ and $f(x) = f(y) = 1$ then $r + 3r + 1 = 5t + 4$. That is $r = \frac{5t+3}{4}$. Such a positive integer r does not exists since $t \equiv 0 \pmod 2$. Assume $f(v) = f(y) = 2$ and $f(x) = 1$. Here $r + 3r + 3 = 5t + 4$. Then $r = \frac{5t+1}{4}$. This is a contradiction to $t \equiv 0 \pmod 2$. If $f(x) = f(y) = 2$ and $f(v) = 1$ then $r + 3r + 3 = 5t + 4$ and hence $r = \frac{5t+1}{4}$. Here also a contradiction arises since $t \equiv 0 \pmod 2$. Further if $f(v) = f(x) = 1$ and $f(y) = 2$ then $r + 3r + 2 = 5t + 4$. Therefore $r = \frac{5t+2}{4}$. But $l + r > 3t + 1$, a contradiction.
Case 3. $n \equiv 2 \pmod 3$.
Let $n = 3t + 2$. Then $|V(G)| + |E(G)| = 15t + 16$. In this case we have three possibilities.

a. $ev_f(0) = ev_f(1) = 5t + 5$, $ev_f(2) = 5t + 6$ or

b. $ev_f(0) = ev_f(2) = 5t + 5$, $ev_f(1) = 5t + 6$ or

c. $ev_f(1) = ev_f(2) = 5t + 5$, $ev_f(0) = 5t + 6$.

Suppose $ev_f(0) = ev_f(1) = 5t + 5$, $ev_f(2) = 5t + 6$.
Subcase a1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \leq 3t + 2$, a contradiction.
Subcase a2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t + 2$, a contradiction.
Subcase a3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. Then $l + 3l + 4 = 5t + 5$. That is $l = \frac{5t+1}{4}$. It follows that $t \equiv 3 \pmod 4$. Suppose $f(y) = 2$. In this case $r + r + 1 = 5t + 6$. This implies $r = \frac{5t+5}{2}$. Here, $l + r > 3t + 2$, a contradiction. For $f(y) = 1$, we have $r + r = 5t + 6$ and hence $r = \frac{5t+6}{2}$. This is impossible since $t \equiv 3 \pmod 4$.

Subcase a4. Any two vertices of S are labeled with zero.
Assume $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. Here, $l + 2l + 3 = 5t + 5$. This implies $l = \frac{5t+2}{3}$. It follows that $t \equiv 2 \pmod 3$. Now consider the vertices x and y. Suppose these two vertices are labeled by 2 then $r + 2r + 3 = 5t + 6$. Hence $r = \frac{5t+3}{3}$, a contradiction to the nature of t. If $f(x) = f(y) = 1$ then $r + 2r = 5t + 6$. Therefore $r = \frac{5t+6}{3}$. Here also a contradiction arises to the values of t. Now we consider the case that $f(x) = 1$ and $f(y) = 2$. In this case $r + 2r + 2 = 5t + 6$. That is $r = \frac{5t+4}{3}$, a contradiction to $t \equiv 2 \pmod 3$. Now we consider the case $f(u) = f(x) = 0$ and $f(v) \neq 0$, $f(y) \neq 0$. In this case $l + 2l + 2 = 5t + 5$ and hence $l = \frac{5t+3}{3}$. This shows that t should be a multiple of 3. If $f(v) = f(y) = 2$. Then $r+2r+2 = 5t+6$. Hence $r = \frac{5t+4}{3}$, a contradiction to the values of t. Assume $f(v) = f(y) = 1$. Here $r + 2r = 5t + 6$ and then $r = \frac{5t+6}{3}$. But $l + r > 3t + 2$, a contradiction. Suppose $f(v) = 1$, $f(y) = 2$ then $r + 2r + 1 = 5t + 6$. That is $r = \frac{5t+5}{3}$. Such a positive integer r does not exists since $t \equiv 0 \pmod 3$.
Subcase a5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. Then $l + l + 1 = 5t + 5$ and therefore $l = \frac{5t+4}{2}$. This is possible only when t is a multiple of 2. If $f(v) = f(x) = f(y) = 2$ then $r + 3r + 4 = 5t + 6$. Therefore $r = \frac{5t+2}{4}$. But $l + r > 3t + 2$, a contradiction. Suppose $f(v) = f(x) = f(y) = 1$ then $r + 3r = 5t + 6$ and hence $r = \frac{5t+6}{4}$. Here also $l + r > 3t + 2$, a contradiction. For the case $f(v) = 2$ and $f(x) = f(y) = 1$, we have $r + 3r + 1 = 5t + 6$ and therefore $r = \frac{5t+5}{4}$. This is impossible since $t \equiv 0 \pmod 2$. Assume $f(v) = f(y) = 2$ and $f(x) = 1$. Here $r + 3r + 3 = 5t + 6$. Therefore $r = \frac{5t+3}{4}$. Here also a contradiction to the nature of t. If $f(v) = 1$ and $f(x) = f(y) = 2$ then $r + 3r + 3 = 5t + 6$. Hence $r = \frac{5t+3}{4}$, a contradiction to $t \equiv 0 \pmod 2$. Assume $f(v) = f(x) = 1$ and $f(y) = 2$. Here $r + 3r + 2 = 5t + 6$. That is $r = \frac{5t+4}{4}$. But $l + r > 3t + 2$, a contradiction.
Assume $ev_f(0) = ev_f(2) = 5t + 5$, $ev_f(1) = 5t + 6$.
Subcase b1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \leq 3t + 2$, a contradiction.
Subcase b2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t + 2$, a contradiction.
Subcase b3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. Then $l+3l+4 = 5t + 5$. This implies $l = \frac{5t+1}{4}$. This is true only if $t \equiv 3 \pmod 4$. Suppose $f(y) = 2$ then $r + r + 1 = 5t + 5$ and therefore $r = \frac{5t+4}{2}$, a contradiction to the values of t. For $f(y) = 1$, we have, $r + r = 5t + 5$ and hence $r = \frac{5t+5}{2}$. But $l + r > 3t + 2$, a contradiction.
Subcase b4. Any two vertices of S are labeled with zero.
Assume $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. Then $l + 2l + 3 = 5t + 5$ and therefore $l = \frac{5t+2}{3}$. It follows that $t \equiv 2 \pmod 3$. Now consider the vertices x and y. Suppose both of this two vertices are Simultaneously labeled by 2. Here $r + 2r + 3 = 5t + 5$. Therefore $r = \frac{5t+2}{3}$. Clearly the value of $l + r$ should be less than or equal to $3t + 2$. This should be true only if $t \leq 2$. But we discussed earlier that $t - 2$ is a multiple of 3 and t is a positive integer. Hence $t \neq 0$ and $t \neq 1$. For $t = 2$, we display a Total Mean Cordial labeling of $K_8^c + 2K_2$ in figure 6. Assume $f(x) = f(y) = 1$. Then $r + 2r = 5t + 5$ and therefore $r = \frac{5t+5}{3}$. But $l + r > 3t + 2$, a contradiction. Suppose $f(u) = f(x) = 0$ and $f(v) \neq 0$, $f(y) \neq 0$, then $l + 2l + 2 = \frac{5}{t} + 5$. This implies $l = \frac{5t+3}{3}$. It follows that t is a multiple of 3. If $f(v) = f(v) = 2$ then $r+2r+2 = 5t+5$ and hnece $r = \frac{5t+3}{3}$. Now $l+r$ should not exceed $3t+2$. This is possible only if $t \leq 0$. This implies $t = 0$. A Total Mean Cordial labeling of $K_2^c + 2K_2$ is given in figure 7. For $f(v) = f(y) = 1$ we have $r + 2r = 5t + 5$. Hence $r = \frac{5t+5}{3}$. This is a contradiction to the values of t. Assume $f(v) = 1$ and $f(y) = 2$. In this case $r + 2r + 1 = 5t + 5$ and therefore $r = \frac{5t+4}{3}$. This is also a contradiction to the nature of t.
Subcase b5. Only one vertex from the set S is labeled by zero.
Without loss of generality assume that $f(u) = 0$. Here $l + l + 1 = 5t + 5$. That is $l = \frac{5t+4}{2}$. This implies t is a multiple of 2. Now assume $f(v) = f(x) = f(y) = 2$. Then $r + 3r + 4 = 5t + 5$ and hence $r = \frac{5t+1}{4}$. Since t is a multiple of 2, r is not an integer, a contradiction. If $f(v) = f(x) = f(y) = 1$ then $r + 3r = 5t + 5$. This implies $r = \frac{5t+5}{4}$. For the same reason as discussed above, we have a contradiction. For $f(v) = 2$ and $f(x) = f(y) = 1$, we have $r + 3r + 1 = 5t + 5$. Therefore $\frac{5t+4}{4}$. Now $l + r > 3t + 2$, a contradiction. If $f(x) = 1$ and $f(v) = f(y) = 2$ then $r + 3r + 3 = 5t + 5$. That is $r = \frac{5t+2}{4}$. Here also the value of $l + r$ exceeds $3t + 2$, a contradiction. Consider the case when $f(v) = 1$ and $f(x) = f(y) = 2$. In this case $r + 3r + 3 = 5t + 5$. Therefore $r = \frac{5t+2}{4}$. Again $l + r > 3t + 2$, a contradiction. For $f(v) = f(x) = 1$ and $f(y) = 2$ we have $r + 3r + 2 = 5t + 5$. Hence $r = \frac{5t+3}{4}$. Such a positive

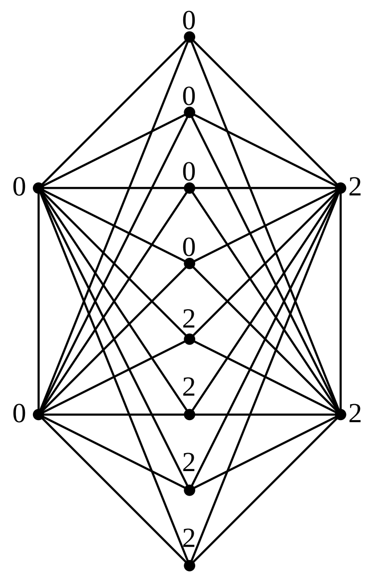

Figure 6.

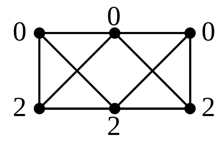

Figure 7.

integer r does not exists since t is a multiple of 2.
Consider the case $ev_f(0) = 5t + 6$ and $ev_f(2) = ev_f(1) = 5t + 5$.
Subcase c1. $f(u) = f(v) = f(x) = f(y) = 0$.
Here, $ev_f(2) \leq 3t + 2$, a contradiction.
Subcase c2. $f(u) \neq 0$, $f(v) \neq 0$, $f(x) \neq 0$, $f(y) \neq 0$.
In this case $ev_f(0) \leq 3t + 2$, a contradiction.
Subcase c3. Any three vertices of S are labeled with zero.
With out loss of generality assume that $f(u) = f(v) = f(x) = 0$ and $f(y) \neq 0$. Here $l+3l+4 = 5t + 6$. That is $l = \frac{5t+2}{4}$. It follows that $t \equiv 2 \pmod 4$. Now consider the vertex y. Suppose $f(y) = 2$ then $r + r + 1 = 5t + 5$ and hence $r = \frac{5t+4}{2}$. But $l + r > 3t + 2$, a contradiction. For $f(v) = 1$ we have $r + r = 5t + 5$. therefore $r = \frac{5t+5}{2}$, a contradiction to the values of t.
Subcase c4. Any two vertices of S are labeled with zero.
Assume $f(u) = f(v) = 0$ and $f(x) \neq 0$, $f(y) \neq 0$. In this case $l + 2l + 3 = 5t + 6$. This implies $l = \frac{5t+3}{3}$. It follows that t is a multiple of 3. Assume $f(x) = f(y) = 2$. Then $r+2r+3 = 5t+5$. That is $r = \frac{5t+2}{3}$. Such a positive integer r does not exists since $t \equiv 0 \pmod 3$. If $f(x) = f(y) = 1$ then $r + 2r = 5t + 5$. Hence $r = \frac{5t+5}{3}$. Suppose $f(x) = 1$, $f(y) = 2$. In this case $r + 2r + 2 = 5t + 5$ and hence $r = \frac{5t+3}{3}$. Now $l + r \leq 3t + 2$. This is possible only when $t = 0$. A Total Mean Cordial labeling of $K_2^c + 2K_2$ is given in figure 8. Consider the case

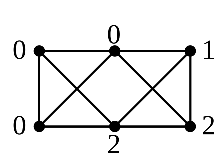

Figure 8.

$f(u) = f(x) = 0$ and $f(v) \neq 0$, $f(y) \neq 0$. In this case $l+2l+2 = 5t+6$. This implies $l = \frac{5t+4}{3}$. It follows that $t \equiv 1 \pmod 3$. Next consider the vertices v and y. If possible $f(v) = f(y) = 2$ then $r + 2r + 2 = 5t + 5$ and hence $r = \frac{5t+3}{3}$. This is impossible since $t \equiv 1 \pmod 3$. Suppose $f(v) = f(y) = 1$. Here $r + 2r = 5t + 5$. Therefore $r = \frac{5t+5}{3}$, a contradiction to the values of t. For $f(v) = 1$ and $f(y) = 2$ we have $r + 2r + 1 = 5t + 5$. Then $r = \frac{5t+4}{3}$. But $l + r > 3t + 2$, a contradiction.

Subcase c5. Only one vertex from the set S is labeled by zero.

Without loss of generality assume that $f(u) = 0$. Then $l + l + 1 = 5t + 6$. Hence $l = \frac{5t+5}{2}$. This implies $t \equiv 1 \pmod 2$. Suppose $f(v) = f(x) = f(y) = 2$ then $r + 3r + 4 = 5t + 5$ and hence $r = \frac{5t+1}{4}$. But $l + r > 3t + 2$, a contradiction. If $f(v) = f(x) = f(y) = 1$ then $r+3r = 5t+5$. Therefore $r = \frac{5t+5}{4}$. Here also $l+r > 3t+2$, a contradiction. For $f(v) = 2$ and $f(x) = f(y) = 1$ we have $r+3r+1 = 5t+5$. Then $r = \frac{5t+4}{4}$, a contradiction to $t \equiv 1 \pmod 2$. Assume $f(v) = f(y) = 2$ and $f(x) = 1$. In this case $r + 3r + 3 = 5t + 5$. Hence $r = \frac{5t+2}{4}$. This is impossible since $t \equiv 1 \pmod 2$. Consider the case $f(v) = 1$ and $f(x) = f(y) = 2$. Here $r + 3r + 3 = 5t + 3$. Then $r = \frac{5t+2}{4}$. Here also a contradiction to the values of t. When $f(v) = f(x) = 1$ and $f(y) = 2$, we have $r + 3r + 2 = 5t + 5$ and hence $r = \frac{5t+3}{4}$. But $l + r > 3t + 2$, a contradiction. □

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