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On order statiscal limit points

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Abstract. Following the concept of order statistical convergence in a particular metric lattice we define the order statistical limit points and order statistical cluster points and obtain some related results.

1 INTRODUCTION

The concept of statistical convergence was formally introduced by H. Fast [1]. This was also studied by several authors J. S. Connor [2], J. A. Fridy [3], Miller [4], Freedman Sember [5], Erdös and Tenenbaum [6] and many others in different aspects. The notion of order statistical convergence on a linearly order lattice associated with a suitably chosen metric was given by Ganguly and Biswas [7]. In this paper the authors proved that the statistical convergence is a particular case of order statistical convergence. Also the concept of order statistically Cauchy sequence and order statistically bounded sequence have been introduced in this paper.
In the present paper following the concept of statistical limit points and cluster points [8] we introduce the notion of order statistical limit points and order statistical cluster points and give some basic properties of these limit points and cluster points. In the last part of this paper we have also proved a few results relating to the order statistically bounded sequence and it is shown that an order statistically bounded sequence has an order statistically convergent subsequence.

2 DEFINITIONS AND NOTATIONS

A partially ordered set or poset is set P in which a binary relation $x \leq y$ is defined, which satisfies for all $x, y, z \in P$ the following conditions.
(i) $x \leq x$ for all $x \in P$,
(ii) if $x \leq y$ and $y \leq x$, then $x = y$,
(iii) if $x \leq y$ and $y \leq z$, then $x \leq z$.
If $x \leq y$ and $x \neq y$, we write $x < y$. The relation $x \leq y$ is also written as $y \geq x$. Similarly, $x < y$ is also written as $y > x$.
L is called an additive system, if every two elements $a, b \in L$ possess a least upper bound(l.u.b.) $a \vee b \in L$ and L is said to be a multiplicative system, if every two elements $a, b \in L$ possess a

greatest lower bound(g.l.b.) $a \wedge b \in L$.
A poset L is a lattice if L is both additive and multiplicative.
A lattice L is said to be complete if each of its subset has a l.u.b. and a g.l.b. in L.
An element $\theta \in L$ is said to be the null element of L if $x \vee \theta = x$ and $x \wedge \theta = \theta$ for all $x \in L$.
If L is a lattice, we say that a sequence $\{a_i\} \in L$ is increasing (decreasing) if $a_i \leq a_j$ $(a_i \geq a_j)$ for $i < j$.

Definition 2.1. [9] A sequence $\{x_n\}$ of a lattice L is said to be Order convergent (O-convergent) to $x \in L$, if there exist sequences $\{y_n\}$ of elements of L with $y_n \downarrow 0$ such that

$$|x_n - x| < y_n \text{ for each } n \in \mathbb{N},$$

where in L, $|x| = x^+ + x^-$ and $x^+ = x \vee \theta$, $x^- = (-x) \vee \theta$.

Definition 2.2. [10] If K be a subset of the set of positive integers $\mathbb{N}$, then the natural density of K, denoted by $\delta(K)$ is defined by

$$\delta(K) = lim_{n\to\infty} \frac{|K_n|}{n}, \text{ where}$$

$K_n = \{k \in K : k \leq n\}$ and $|K_n|$ is the number of elements of K_n.

Definition 2.3. A sequence $\{x_n\}$ of real numbers is said to be statistically convergent to some number l, if for any $\epsilon > 0$,

$$\delta(\{k \in \mathbb{N} : |x_k - l| \geq \epsilon\}) = 0.$$

If $\{x_n\}$ is statistically convergent to l then we write $st - lim_n\, x_n = l$.

Definition 2.4. [11] (i) Let L be an additive system and D be a real valued function defined on L. Then define a function γ on L as

$$\gamma(a, b) = 2D(a \vee b) - D(a) - D(b), \forall a, b \in L.$$

$D(a)$ is said to be monotone increasing (decreasing) when

$$D(a) \leq D(b)(D(a) \geq D(b)), \text{ for } a < b \text{ and } a, b \in L.$$

The function $D(a)$ is a norm if $\gamma(a, b)$ is a metric for L.

(ii) Let L be an additive system and $\gamma(a, b)$ be a real valued function defined for every $a, b \in L$; then define,

$$\Delta(a, b, c) = \frac{1}{2}\{\gamma(a, b) + \gamma(b, c) - \gamma(a, c)\}, \text{ for } a, b, c \in L.$$

Lemma 2.5. [11] (A) If $D(a)$ is a real valued function defined on an additive system L, then for $a, b \in L$
(i) $D(a) - D(b) = \gamma(a, b)$ if $a \geq b$.

(ii) If $D(a)$ is a monotone increasing, then $|D(a) - D(b)| \leq \gamma(a, b)$.
(iii) $\gamma(a, b) = \gamma(b, a)$, $\gamma(a, a) = 0$.
(iv) $\Delta(a, a \vee b, b) = 0$.
(v) $D(a)$ is monotone increasing if and only if $\gamma(a, b) \geq 0$.
(vi) $D(a)$ is properly monotone increasing if and only if $\gamma(a, b) > 0$ for $a \neq b$.

(B) If $D(a)$ is a real valued function defined on an additive system L and $\Delta(a, b, c) \geq 0$ for every $a, b, c \in L$, then the following statements are equivalent.
(i) $\gamma(a \vee c, b \vee c) \leq \gamma(a, b)$ for all $a, b \in L$.
(ii) $\gamma(a \vee c, b \vee c) \leq \gamma(a, b)$ for all $b \leq a$.
(iii) $D(a \vee c) + D(b) \leq D(a) + D(c \vee b)$ for $b \leq a$.
(iv) $\gamma(a \vee c, b \vee d) \leq \gamma(a, b) + \gamma(c, d)$.
(C) If $D(a)$ is monotone increasing then $\Delta(a, b, c) \geq 0$ if and only if one of the equivalent statements of (B) holds.
Note : If $D(a)$ is monotone increasing and $\Delta(a, b, c) \geq 0$ for $a, b, c \in L$, then lemma 2.5(A) implies that γ is a metric on L.

The following definitions and results established in the paper [7] are mentioned here to pursue some result relating to order statistical limit points and order statistical cluster points.
Through out the paper we consider D to be monotone increasing on L with $D(\theta) = 0$.

Definition 2.6. [7] A sequence $\{x_n\}_n$ in a metric lattice (L, γ) is said to be order statistically convergent (i.e ost-convergent) to $x \in L$ if, there exists a sequence $\{y_n\} \in L$ with $y_n \downarrow \theta$ such that

$$\delta(\{k \in \mathbb{N} : \gamma(x_k, x) \geq D(y_k)\}) = 0,$$

where D is a real valued monotone increasing function on L with $D(\theta) = 0$ and $\Delta(a, b, c) \geq 0$ for all $a, b, c \in L$.

If a sequence $\{x_n\}_n$ is order statistically convergent to $x \in L$, then the order statistical limit (i.e. ost-limit) of $\{x_n\}_n$ is x and we denote it by $x_n \xrightarrow{ost} x$.

Theorem 2.7. A sequence in L can have at most one ost-limit.

Definition 2.8. A sequence $\{x_n\}$ in L is said to be order statistically bounded (i.e. ost-bounded) if there exists $B \in \mathbb{R}$ such that

$$\delta(\{n \in \mathbb{N} : D(x_n) \geq B\}) = 0.$$

Theorem 2.9. Any ost-convergent sequence in a metric lattice (L, γ) is ost-bounded.

Theorem 2.10. If $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric lattice (L, γ) such that $x_n \xrightarrow{ost} x$ and $y_n \xrightarrow{ost} y$ then $x_n \vee y_n \xrightarrow{ost} x \vee y$.

Theorem 2.11. If a sequence $\{x_n\}_n$ in L is order statistically convergent to x if and only if there is a set $K = \{k_1 < k_2 < k_3 < ....\} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that $lim_{n\to\infty} x_{k_n} = x$ with respect to the metric γ.

Definition 2.12. A sequence $\{x_n\}$ in a metric lattice (L, γ) is said to be order statistically Cauchy (i.e. ost-Cauchy) if there is a sequence $\{y_n\}$ in L with $y_n \downarrow \theta$ such that

$$\delta(\{n \in \mathbb{N} : \gamma(x_{n+p}, x_n) \geq D(y_n)\}, p = 1, 2, 3, ...) = 0.$$

Theorem 2.13. An ost-Cauchy sequence is ost-bounded.

Theorem 2.14. A sequence $\{x_n\}_n$ in L is ost-Cauchy if it is ost-convergent.

Theorem 2.15. If a sequence $\{x_n\}_n$ in L is ost-Cauchy then there exists a set $K = \{k_1 < k_2 < k_3 < ....\} \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that
$\gamma(x_{k_n}, x_{k_m}) \to 0$ as $m, n \to \infty$.

3 MAIN RESULTS

In this section we first define order statistical limit point (ost-limit point) and order statistical cluster point (ost-cluster point) analogous to the statistical limit point and statistical cluster point and investigate some properties relating to this points. First we give some basic definitions and examples.

Definition 3.1. [8] Let $\{x_n\}$ be a sequence in a metric lattice (L, γ) and $\{x_{k_n}\}$ be a subsequence of $\{x_n\}$. Then
(i) $\{x_{k_n}\}$ is said to be a subsequence of density zero or a thin subsequence if $\delta\{k_n : n \in \mathbb{N}\} = 0$.
(ii) $\{x_{k_n}\}$ is said to be a non-thin subsequence if either $\delta\{k_n : n \in \mathbb{N}\} > 0$ or density of the set $\{k_n : n \in \mathbb{N}\}$ does not exist.

Definition 3.2. An element $\lambda \in L$ is said to be an ost-limit point of a sequence $x = \{x_n\} \in L$ provided there is a non-thin subsequence $\{x_{k_n}\}$ of x and $\{y_n\} \in L$ with $y_n \downarrow \theta$ such that $\gamma(x_{k_n}, \lambda) < D(y_n)$ for all $n \in \mathbb{N}$.

Definition 3.3. An element $\xi \in L$ is said to be an ost-cluster point of a sequence $x = \{x_n\} \in L$ provided there is a sequence $\{y_n\} \in L$ with $y_n \downarrow \theta$ such that $\{k \in \mathbb{N} : \gamma(x_k, \xi) < D(y_k)\}$ does not have density zero.

$Notation$: We denote the set of all order statistical limit points of a sequence x by $OSL(x)$ and $OSC(x)$ denotes the set of all order statistical cluster points of x and $L(x)$ denotes the set of all ordinary limit points of the sequence x.

We now give an example of a sequence x for which $L(x) \subseteq OSL(x)$ and $L(x) \subseteq OSC(x)$.

Example1 : Let

$$\begin{aligned} x_k &= 1; \textit{when } k = n^2, n = 1, 2, ... \\ &= 0; \textit{otherwise}. \end{aligned}$$

In the lattice $\mathbb{R}$, the set of real numbers if D be the identity mapping, then γ be the usual metric on $\mathbb{R}$. Clearly $L(x) = \{0, 1\}$. Also if $M = \{k \in \mathbb{N} : k \neq n^2\}$, $n = 1, 2, ...$, then $\gamma(x_n, 0) < \frac{1}{n}$

for all $n \in M$. This shows that $OSL(x) = OSC(x) = \{0\}$

Now we determine some properties of the two sets $OSL(x)$ and $OSC(x)$.

Theorem 3.4. For any sequence $x = \{x_n\}$ in L, $OSL(x) \subseteq OSC(x)$.

Proof: Let $\lambda \in OSL(x)$. Then there exists a non-thin subsequence $\{x_{k_n}\}$ of x so that there is a sequence $\{y_n\} \in L$ with $y_n \downarrow \theta$ and

$$\gamma(x_{k_n}, \lambda) < D(y_{k_n}) \text{ for all } n \in \mathbb{N}.$$

Also since $\{x_{k_n}\}$ is non-thin, $\limsup \frac{1}{n}|\{k_n \leq n : n \in \mathbb{N}\}| > 0$.
Again $\{k_n : n \in \mathbb{N}\} \subseteq \{k \in \mathbb{N} : \gamma(x_k, \lambda) < D(y_k)\}$.
Therefore,

$$\limsup \frac{1}{n}|\{k \leq n : \gamma(x_k, \lambda) < D(y_k)\}| \geq \limsup \frac{1}{n}|\{k_n \leq n : n \in \mathbb{N}\}| > 0.$$

Thus $\{k \leq n : \gamma(x_k, \lambda) < D(y_k)\}$ does not have density zero. This implies that $\lambda \in OSC(x)$, hence $OSL(x) \subseteq OSC(x)$.

$Note$: The converse of the above theorem is not true and it will be clear from the following example.
Example2 : We consider the sequence $x = \{x_n\}$ as
$\{0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, ...\}$ and taking γ as the usual metric on $\mathbb{R}$.
Now for $\alpha \in [0, 1]$ let there exists a subsequence $\{x_{n_k}\}$ of x and $\{y_n\}$ be a sequence of real numbers in the lattice $\mathbb{R}$ with $y_n \downarrow 0$ such that $\gamma(x_{n_k}, \alpha) < y_k$ for all $k \in \mathbb{N}$. Let $K = \{n_k : k \in \mathbb{N}\}$.
Since $y_n \downarrow 0$ then for $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that $y_n < \varepsilon$ for all $n \geq m$.
Let $K_1 = \{n_k : k \in \mathbb{N}\} - \{1, 2, ..., m - 1\}$.
Now $|K_1(n)| \leq |n_k \in K_1(n) : \gamma(x_{n_k}, \alpha) < \varepsilon| + |n_k \in K_1(n) : \gamma(x_{n_k}, \alpha) \geq \varepsilon|$.
$\Rightarrow \delta(K_1) \leq 2\varepsilon$.
Since ε is arbitrary, then $\delta(K_1) = 0$ and consequently $\delta(K) = 0$. This shows that $\alpha \notin OSL(x)$.i.e. $OSL(x) = \phi$.
Again let $\beta \in [0, 1]$. Then for any $\varepsilon > 0$,
$\delta\{k \in \mathbb{N} : \gamma(x_k, \beta) < \frac{1}{k}\} = \delta\{k \in \mathbb{N} : \gamma(x_k, \beta) < \varepsilon\} = \delta\{k \in \mathbb{N} : x_k \in (\beta - \varepsilon, \beta + \varepsilon)\} > \varepsilon > 0$.
So, $OSC(x) = [0, 1]$ and thus $OSC(x) \neq OSL(x)$.

Lemma 3.5. If $x = \{x_n\} \in L$ be such that $\lim_{n\to\infty} x_n = \xi$ with respect to the metric γ then there is a sequence $\{\alpha_n\} \in L$ with $\alpha_n \downarrow \theta$ such that $\gamma(x_n, \xi) < D(\alpha_n)$, for all $n \in \mathbb{N}$.

Proof: Since $\lim_{n\to\infty} x_n = \xi$, then for $\epsilon > 0$ there is $m \in \mathbb{N}$ such that $\gamma(x_n, \xi) < \epsilon$ for all $n \geq m$.
Let $\{y_n\}$ be a sequence in L such that $y_n \downarrow \theta$. Then for each y_i there is a smallest positive integer m_i such that $\gamma(x_n, \xi) < D(y_i)$ for all $n \geq m_i$, $i = 1, 2, 3, ...$
Choose $z_1 \in L$ such that, $D(z_1) > max\{D(y_1), \gamma(x_1, \xi), \gamma(x_2, \xi), ..., \gamma(x_{m_1 - 1}, \xi)\}$,

Choose $z_2 \in L$ such that,
$\gamma(x_{m_1}, \xi) \geq D(z_2) > max\{D(y_2), \gamma(x_{m_1+1}, \xi), \gamma(x_{m_1+2}, \xi), ..., \gamma(x_{m_2-1}, \xi)\}$,

Choose $z_3 \in L$ such that,
$\gamma(x_{m_2}, \xi) \geq D(z_3) > max\{D(y_3), \gamma(x_{m_2+1}, \xi), \gamma(x_{m_2+2}, \xi), ..., \gamma(x_{m_3-1}, \xi)\}$,
and so on.
Now set,

$$\begin{aligned} \alpha_i &= z_1; i = 1, 2, ..., m_1 - 1 \\ &= y_1; i = m_1 \\ &= z_2; i = m_1 + 1, m_1 + 2, ..., m_2 - 1 \\ &= y_2; i = m_2 \\ &.. \quad .... \end{aligned}$$

Then $\gamma(x_n, \xi) < D(\alpha_n)$, for all $n \in \mathbb{N}$ and $\alpha_n \downarrow \theta$.

Theorem 3.6. $OSC(x)$ is a closed set for any sequence $x = \{x_n\}$ in L.

Proof: Let $x = \{x_n\}$ be a sequence in L and p be a limit point of $OSC(x)$. Consider a sequence $\{y_n\}$ in L with $y_n \downarrow \theta$.
Then for each $n \in \mathbb{N}$ there exists $\alpha_n \in OSC(x)$ such that $\alpha_n \in B(p, D(y_n))$, where $B(p, D(y_n))$ denotes the open ball with centre at p and radius $D(y_n)$.
Since $\alpha_n \in OSC(x)$ there exists $\{z_k^{(n)}\}_k$ with $z_k^{(n)} \downarrow \theta$ as $k \to \infty$ such that

$$\limsup \tfrac{1}{m}|\{k \leq m : \gamma(x_k, \alpha_n) < D(z_k^{(n)})\}| > 0 \text{ for all } n \in \mathbb{N}.$$

Consider $A_n = \{k \in \mathbb{N} : \gamma(x_k, \alpha_n) < D(z_k^{(n)})\}$, $n \in \mathbb{N}$. Now $\alpha_n \in B(p, D(y_n))$ implies that $\gamma(p, \alpha_n) < D(y_n)$. Thus for $k \in A_n$

$$\begin{aligned} \gamma(x_k, p) &\leq \gamma(x_k, \alpha_n) + \gamma(\alpha_n, p) \\ &< D(z_k^{(n)}) + D(y_n). \end{aligned}$$

Now $k, n \to \infty$ implies that $D(z_k^{(n)}) + D(y_n) \to 0$. i.e. $\gamma(x_k, p) \to 0$ when $k, n \to \infty$. Using Lemma 3.5 we can choose a sequence $\{w_k\}$ in L such that $\gamma(x_k, p) < D(w_k)$ for all $k \in \mathbb{N}$ and $w_k \downarrow \theta$.
i.e. $k \in A_n$ implies that $k \in \{k \in \mathbb{N} : \gamma(x_k, p) < D(w_k)\} = B$, say.
So, $A_n \subseteq B$ and consequently,

$$\begin{aligned} \limsup \tfrac{1}{m}|\{k \leq m : \gamma(x_k, p) < D(w_k)\}| &\geq \limsup \tfrac{1}{m}|\{k \leq m : \gamma(x_k, \alpha_n) < D(z_k^{(n)})\}| \\ &> 0. \end{aligned}$$

So, $p \in OSC(x)$ and hence $OSC(x)$ is closed.
We now show that the set $OSL(x)$ need not be closed and for this purpose we give the following example.
Example3 : Let $x = \{x_n\}$ be a sequence in $\mathbb{R}$, the set of real numbers the with usual metric

such that
$x_n = \frac{1}{p}$ where $n = 2^{p-1}(2q+1)$, p, q are positive integers.
Then for each p

$$\begin{aligned} \delta(\{n \in \mathbb{N} : \gamma(x_n, \tfrac{1}{p}) < \tfrac{1}{n}\}) &= \delta(\{n \in \mathbb{N} : |x_n - \tfrac{1}{p}| < \tfrac{1}{n}\}) \\ &\geq \delta(\{n \in \mathbb{N} : x_n = \tfrac{1}{p}\}) \\ &= 2^{-p} \\ &> 0 \end{aligned}$$

Thus $\frac{1}{p} \in OSL(x)$.
Clearly 0 is a limit point of $OSL(x)$. Let there is a subsequence $\{x_{n_k}\}$ of x and $\{y_k\}$ be a sequence of real number with $y_k \downarrow 0$ such that
$\gamma(x_{n_k}, 0) < y_k$ for all $k \in \mathbb{N}$ i.e. $x_{n_k} < y_k$ for all $k \in \mathbb{N}$.
Since $y_k \downarrow 0$ then there exists a least positive integer l such that $y_k < \frac{1}{l}$ for some k.
Let $K = \{n_k : k \in \mathbb{N}\}$. Then for each $p \geq l$,
$|K_n| \leq |\{k \in K_n : x_k \geq \frac{1}{p}\}| + |\{k \in K_n : x_k < \frac{1}{p}\}| \leq |\{k \in K_n : x_k \geq \frac{1}{p}\}| + |\{k \in \mathbb{N} : x_k < \frac{1}{p}\}| \leq |\{k \in K_n : x_k \geq \frac{1}{p}\}| + \frac{n}{2^p}$.
Thus $\delta(K) \leq \frac{1}{2^p}$. Since p is arbitrary then we have $\delta(K) = 0$. Therefore $0 \notin OSL(x)$

Theorem 3.7. Let $x = \{x_k\}$ and $y = \{y_k\}$ be two sequences in L such that $x_k = y_k$ for almost all k, then
(i) $OSL(x) = OSL(y)$ and
(ii) $OSC(x) = OSC(y)$.

Proof: (i) We have $\delta(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$ and $\lambda \in OSL(x)$. Then there exists a non-thin subsequence $\{x_{k_n}\}$ of x with the property that there exists $\{z_k\} \in L$ with $z_k \downarrow \theta$ so that

$$\gamma(x_{k_n}, \lambda) < D(z_{k_n}) \text{ for all } n \in \mathbb{N}.$$

Now $\{k_n \in \mathbb{N} : x_{k_n} \neq y_{k_n}\} \subseteq \{k \in \mathbb{N} : x_k \neq y_k\}$.
Then $\delta(\{k_n \in \mathbb{N} : x_{k_n} \neq y_{k_n}\}) = 0$. Thus $\{k_n \in \mathbb{N} : x_{k_n} = y_{k_n}\}$ does not have density zero. If $\{k_n \in \mathbb{N} : x_{k_n} = y_{k_n}\} = \{p_n : n \in \mathbb{N}\}$, then clearly $\{x_{p_n}\}$ is a non-thin subsequence of x and

$$\gamma(x_{p_n}, \lambda) < D(z_{p_n}) \text{ for all } n \in \mathbb{N}.$$

Consequently,

$$\gamma(y_{p_n}, \lambda) < D(z_{p_n}) \text{ for all } n \in \mathbb{N},$$

since $x_{p_n} = y_{p_n}$ for all $n \in \mathbb{N}$. Therefore $\lambda \in OSL(y)$ and hence $OSL(x) \subseteq OSL(y)$. By symmetry $OSL(y) \subseteq OSL(x)$ and thus $OSL(x) = OSL(y)$.
(ii) Let $\alpha \in OSC(x)$. Then there exists $\{z_k\} \in L$ with $z_k \downarrow \theta$ so that $\{k \in \mathbb{N} : \gamma(x_k, \alpha) < D(z_k)\}$ does not have density zero.
Let $A = \{k \in \mathbb{N} : \gamma(x_k, \alpha) < D(z_k)\}$ and consider $\{k \in \mathbb{N} : x_k = y_k\} = \{k_n : n \in \mathbb{N}\}$.
Then $\delta(\{k_n \in \mathbb{N}\}) = 1$. Since A does not have density zero then $\{k_n \in \mathbb{N} : \gamma(x_{k_n}, \alpha) < D(z_{k_n})\}$ does not have density zero. So, $\{k_n \in \mathbb{N} : \gamma(y_{k_n}, \alpha) < D(z_{k_n})\}$ does not have density zero as

$x_{k_n} = y_{k_n}$ for all $n \in \mathbb{N}$. Thus $\{k \in \mathbb{N} : \gamma(y_k, \alpha) < D(z_k)\}$ does not have density zero. i.e. $\alpha \in OSC(x)$. Therefore, $OSC(x) \subseteq OSC(y)$. By symmetry $OSC(y) \subseteq OSC(x)$ and thus $OSC(x) = OSC(y)$.

Theorem 3.8. Let $\{x_n\}$ be an ost-bounded sequence in L such that the density of the set $M = \{n \in \mathbb{N} : x_{n+1} \geq x_n\}$ is 1. Then $\{x_n\}$ is ost-convergent.

Proof: Let $M = \{n_k : k \in \mathbb{N}\}$. Since $\{x_n\}$ is ost-bounded then there exists $B \in \mathbb{R}$ such that $\delta(\{n \in \mathbb{N} : D(x_n) \geq B\}) = 0$ and so
$\delta(\{n_k \in \mathbb{M} : D(x_{k_n}) \geq B\}) = 0$.
Let $M_1 = \{p_k \in \mathbb{M} : D(x_{p_k}) < B\}$. Then $\delta(M_1) = 1$ and $x_{p_k} \leq x_{p_{k+1}}$ for all $k \in \mathbb{N}$.
Since D is monotone increasing then $D(x_{p_1}) \leq D(x_{p_2}) \leq D(x_{p_3}) \leq ..... < B$.
Here $sup_{k\in\mathbb{N}}\, x_{p_k} = \vee_{k\in\mathbb{N}}\, x_{p_k}$ exists.
Let $sup_{k\in\mathbb{N}}\, x_{p_k} = \alpha$. Since D is increasing then $sup_{k\in\mathbb{N}}\, D(x_{p_k}) = D(\alpha)$. Now,

$$\begin{aligned}\gamma(x_{p_k}, \alpha) &= 2D(x_{p_k} \vee \alpha) - D(x_{p_k}) - D(\alpha)\\ &= 2D(\alpha) - D(x_{p_k}) - D(\alpha);\, since\ x_{p_k} \vee \alpha = \alpha\\ &= D(\alpha) - D(x_{p_k}).\end{aligned}$$

$$\begin{aligned}So,\ x_{p_k} &\leq x_{p_{k+1}}\\ \Rightarrow D(x_{p_k}) &\leq D(x_{p_{k+1}})\\ \Rightarrow D(\alpha) - D(x_{p_k}) &\geq D(\alpha) - D(x_{p_{k+1}})\\ i.e.\ \gamma(x_{p_k}, \alpha) &\geq \gamma(x_{p_{k+1}}, \alpha).\end{aligned}$$

Thus $\{\gamma(x_{p_k}, \alpha)\}$ is a monotone decreasing sequence.
Also it is clear that $\gamma(x_{p_k}, \alpha) \to 0$ as $k \to \infty$.
Consider a sequence $\{y_k\} \in L$ with $y_k \downarrow \theta$. Then for x_{p_1} there exists some $y_{q_1} \in \{y_n\}$ such that

$$\begin{aligned}D(\alpha) - D(y_{q_1}) &< D(x_{p_1})\\ or,\, D(\alpha) - D(x_{p_1}) &< D(y_{q_1}).\\ i.e.,\, \gamma(x_{p_1}, \alpha) &< D(y_{q_1}).\end{aligned}$$

Again for x_{p_2} we can choose some $y_q \in \{y_n\}$ such that $\{\gamma(x_{p_2}, \alpha)\} < D(y_{q_2})$ since $\{\gamma(x_{p_k}, \alpha)\}$ is a monotone decreasing.
Thus for the sequence $\{x_{p_k}\}$ we can construct the sequence $\{y_{p_k}\}$ such that $\{\gamma(x_{p_k}, \alpha)\} < D(y_{q_k})$ for all $k \in \mathbb{N}$ and $y_{q_1} \geq y_{q_2} \geq y_{q_3}$........
Construct a sequence $\{z_n\} \in L$ as follows:

$$\begin{aligned}z_i &= y_{q_1};\, 1 \leq i \leq p_1\\ &= y_{q_2};\, p_1 < i \leq p_2\\ &= y_{q_3};\, p_2 < i \leq p_3\\ ... \quad &.... \quad ....\end{aligned}$$

Then $z_{p_i} = y_{q_i}$ for all $i = 1, 2, 3, ...$
So

$$\gamma(x_{p_k}, \alpha) < D(z_{p_k})$$

for all $k \in \mathbb{N}$ and $\{z_{p_k}\}$ is a sequence in L and $z_{p_k} \downarrow \theta$. Thus $\delta(\{k \in \mathbb{N} : \gamma(x_k, \alpha) \geq D(z_k)\}) = 0$ and hence $\{x_k\}$ is ost-convergent to α.

Corollory 3.9. A monotone increasing ost-bounded sequence is ost-convergent.

Theorem 3.10. Let $\{x_n\}$ be an ost-bounded sequence in L such that the density of the set $M = \{n \in \mathbb{N} : x_{n+1} \leq x_n\}$ is 1. Then $\{x_n\}$ is ost-convergent.

Proof: The proof is similar to the previous theorem.

Corollory 3.11. A monotone decreasing ost-bounded sequence is ost-convergent.

Theorem 3.12. A ost-bounded sequence has an ost-convergent subsequence.

Proof: Let $\{x_n\}$ be an ost-bounded sequence in L. Since L is an ordered lattice then we can choose a monotone subsequence $\{x_{n_k}\}$ of $\{x_n\}$. $\{x_{n_k}\}$ is monotone and ost-bounded and hence it is ost-convergence.

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