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Coefficient inequality for certain subclass of *p*-valent functions

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Abstract. The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for certain subclass of p-valent functions, using Toeplitz determinants.

1 Introduction

Let A_p (p is a fixed integer ≥ 1) denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$ with $p \in N = \{1, 2, 3, ...\}$. Let S be the subclass of $A_1 = A$, consisting of univalent functions.
The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [19, 20] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by many authors in the literature [14]. For example, Noor [15] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with a bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [10]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [2] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order $\alpha (0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{1.3}$$

Janteng, Halim and Darus [9] have considered the functional$|a_2 a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [11]. In their work, they have shown that if $f \in$ RT then $|a_2 a_4 - a_3^2| \leq \frac{4}{9}$.
The same authors [8] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and shown that $|a_2 a_4 - a_3^2| \leq 1$ and $|a_2 a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Mishra and Gochhayat [12] have obtained the sharp bound to the non- linear functional $|a_2 a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2})$, defined as $Re \left\{ e^{i\alpha} \frac{\Omega_z^\lambda f(z)}{z} \right\} > \rho \cos \alpha$, using the fractional differential operator denoted by Ω_z^λ, defined by Owa and Srivastava [17]. These authors have shown that, if $f \in R_\lambda(\alpha, \rho)$ then $|a_2 a_4 - a_3^2| \leq \left\{ \frac{(1-\rho)^2 (2-\lambda)^2 (3-\lambda)^2 cos^2 \alpha}{9} \right\}$. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([1], [3], [13]).
Motivated by the above mentioned results obtained by different authors in this direction, in this

paper, we obtain an upper bound to the functional $|a_{p+1}a_{p+3}-a_{p+2}^2|$ for the function f belonging to certain subclass of p- valent functions, defined as follows.

Definition 1.1. A function $f(z) \in A_p$ is said to be in the class $I_p(\beta)$ (β is real) [16], if it satisfies the condition

$$Re\left\{(1-\beta)\frac{f(z)}{z^p}+\beta\frac{f'(z)}{pz^{p-1}}\right\}>0, \qquad \forall z \in E. \tag{1.4}$$

For the choice of $\beta = 1$ and $p = 1$, we obtain $I_1(1) = RT$. In the next section, we state the necessary Lemmas while, in Section 3, we present our main result.

2 Preliminary Results

Let *P* denote the class of functions

$$p(z) = (1 + c_1z + c_2z^2 + c_3z^3 + ...) = \left[1+\sum_{n=1}^{\infty}c_nz^n\right], \tag{2.1}$$

which are regular in E and satisfy $\text{Re}\{p(z)\} > 0$ for any $z \in E$. To prove our main result in the next section, we shall require the following two Lemmas:

Lemma 2.1. ([18, 21]) If $p \in P$, then $|c_k| \le 2$, for each $k \ge 1$.

Lemma 2.2. ([6]) The power series for p given in (2.1) converges in the unit disc E to a function in *P* if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m}\rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \ge m$. This necessary and sufficient condition is due to Caratheodory and Toeplitz, can be found in [6].

We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \ge 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some} \quad x, \quad |x| \le 1. \tag{2.2}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \ge 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \le 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}$$
$$\text{for some real value of} \quad z, \quad \text{with} \quad |z| \le 1. \tag{2.4}$$

3 Main Result

Theorem 3.1. If $f(z) \in I_p(\beta) \quad (\beta > 0 \quad \text{and} \quad p \in N)$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{4p^2}{(p+2\beta)^2}\right].$$

Proof. Since $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in I_p(\beta)$, from the Definition 1.1, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{(1-\beta)\frac{f(z)}{z^p} + \beta\frac{f'(z)}{pz^{p-1}}\right\} = p(z) \Rightarrow \{(1-\beta)pf(z) + \beta f'(z)\} = \{pz^p p(z)\}. \quad (3.1)$$

Replacing $f(z)$, $f'(z)$ with their equivalent p- valent series expressions and series expression for $p(z)$ in (3.1), we have

$$\left[(1-\beta)p\left\{z^p + \sum_{n=p+1}^{\infty} a_n z^n\right\} + \beta\left\{pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right\}\right] = pz^p\left[1 + \sum_{n=1}^{\infty} c_n z^n\right].$$

Upon simplification, we obtain

$$[(p+\beta)a_{p+1}z^{p+1} + (p+2\beta)a_{p+2}z^{p+2} + (p+3\beta)a_{p+3}z^{p+3} + ...] = [pc_1z^{p+1} + pc_2z^{p+2} + pc_3z^{p+3} + ...]. \quad (3.2)$$

Equating the coefficients of like powers of z^{p+1}, z^{p+2} and z^{p+3} respectively in (3.2), we have

$$[a_{p+1} = \frac{pc_1}{(p+\beta)}; a_{p+2} = \frac{pc_2}{(p+2\beta)}; a_{p+3} = \frac{pc_3}{(p+3\beta)}]. \quad (3.3)$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} from the relation (3.3) in the second Hankel functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in I_p(\beta)$, after simplifying, we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2}{(p+\beta)(p+2\beta)^2(p+3\beta)} \times \left|(p+2\beta)^2c_1c_3 - (p+\beta)(p+3\beta)c_2^2\right|.$$

The above expression is equivalent to

$$|a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2}{(p+\beta)(p+2\beta)^2(p+3\beta)} \times \left|d_1c_1c_3 + d_2c_2^2\right|. \quad (3.4)$$

Where

$$\{d_1 = (p+2\beta)^2; d_2 = -(p+\beta)(p+3\beta)\}. \quad (3.5)$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 in the right hand side of (3.4), we have

$$\left|d_1c_1c_3 + d_2c_2^2\right| = |d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 +$$
$$2(4-c_1^2)(1-|x|^2)z\} + d_2 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2|.$$

Using the facts $|z| < 1$ and $|xa + yb| \leq |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4\left|d_1c_1c_3 + d_2c_2^2\right| \leq |(d_1+d_2)c_1^4 + 2d_1c_1(4-c_1^2) + 2(d_1+d_2)c_1^2(4-c_1^2)|x| - \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|x|^2|. \quad (3.6)$$

Using the values of d_1, d_2 given in (3.5), upon simplification, we obtain

$$\{(d_1 + d_2) = \beta^2; d_1 = (p+2\beta)^2\} \quad (3.7)$$

$$\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} = \left\{\beta^2c_1^2 + 2(p+2\beta)^2c_1 + 4(p+\beta)(p+3\beta)\right\}. \quad (3.8)$$

Consider

$$\left\{\beta^2c_1^2+2(p+2\beta)^2c_1+4(p+\beta)(p+3\beta)\right\} = \beta^2\times\left[c_1^2+\frac{2(p+2\beta)^2}{\beta^2}c_1+\frac{4(p+\beta)(p+3\beta)}{\beta^2}\right]$$

$$=\beta^2\times\left[\left\{c_1+\frac{(p+2\beta)^2}{\beta^2}\right\}^2-\left\{\frac{(p+2\beta)^4}{\beta^4}+\frac{4(p+\beta)(p+3\beta)}{\beta^2}\right\}\right]$$

$$=\beta^2\times\left[\left\{c_1+\frac{(p+2\beta)^2}{\beta^2}\right\}^2-\left\{\frac{\sqrt{p^4+8p^3\beta^3+20p^2\beta^2+16p\beta^3+4\beta^4}}{\beta^4}\right\}^2\right]$$

$$=\beta^2\times\left[c_1+\left\{\frac{(p+2\beta)^2}{\beta^2}+\frac{\sqrt{p^4+8p^3\beta^3+20p^2\beta^2+16p\beta^3+4\beta^4}}{\beta^4}\right\}\right]$$

$$\times\left[c_1+\left\{\frac{(p+2\beta)^2}{\beta^2}-\frac{\sqrt{p^4+8p^3\beta^3+20p^2\beta^2+16p\beta^3+4\beta^4}}{\beta^4}\right\}\right]. \quad (3.9)$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b\geq 0$ in the right hand side of (3.9), upon simplification, we obtain

$$\left\{\beta^2c_1^2+2(p+2\beta)^2c_1+4(p+\beta)(p+3\beta)\right\}\geq\left\{\beta^2c_1^2-2(p+2\beta)^2c_1+4(p+\beta)(p+3\beta)\right\}. \quad (3.10)$$

From the relations (3.8) and (3.10), we get

$$-\ \left\{(d_1+d_2)c_1^2+2d_1c_1-4d_2\right\}\ \leq\ \left\{\beta^2c_1^2-2(p+2\beta)^2c_1+4(p+\beta)(p+3\beta)\right\}. \quad (3.11)$$

Substituting the calculated values from the expressions (3.7) and (3.11) in the right hand side of (3.6), we have

$$4\left|d_1c_1c_3+d_2c_2^2\right|\leq|\beta^2c_1^4+2(p+2\beta)^2c_1(4-c_1^2)+2\beta^2c_1^2(4-c_1^2)|x|-$$
$$\left\{\beta^2c_1^2-2(p+2\beta)^2c_1+4(p+\beta)(p+3\beta)\right\}(4-c_1^2)|x|^2|. \quad (3.12)$$

Choosing $c_1=c\in[0,2]$, applying Triangle inequality and replacing $\mid x\mid$ by μ in the right hand side of (3.12), we get

$$4\left|d_1c_1c_3+d_2c_2^2\right|\leq[\beta^2c^4+2(p+2\beta)^2c(4-c^2)+2\beta^2c^2(4-c^2)\mu+\left\{\beta^2c^2-2(p+2\beta)^2c+4(p+\beta)(p+3\beta)\right\}(4-c^2)\mu^2]$$
$$=F(c,\mu)(say),\quad\text{with}\quad 0\leq\mu=|x|\leq 1\quad\text{and}\quad 0\leq c\leq 2. \quad (3.13)$$

Where

$$F(c,\mu)=[\beta^2c^4+2(p+2\beta)^2c(4-c^2)+2\beta^2c^2(4-c^2)\mu$$
$$+\left\{\beta^2c^2-2(p+2\beta)^2c+4(p+\beta)(p+3\beta)\right\}(4-c^2)\mu^2]. \quad (3.14)$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (3.14) partially with respect to μ, we get

$$\frac{\partial F}{\partial\mu}\ =\ 2\left[\beta^2c^2+\left\{\beta^2c^2-2(p+2\beta)^2c+4(p+\beta)(p+3\beta)\right\}\mu\right]\ \times\ (4\ -\ c^2). \quad (3.15)$$

For $0<\mu<1$, for fixed c with $0<c<2$ with $p\in N$ and $\beta>0$, from (3.15), we observe that $\frac{\partial F}{\partial\mu}>0$. Therefore, $F(c,\mu)$ cannot have a maximum value in the interior of the closed square $[0,2]\times[0,1]$.
Moreover, for a fixed $c\in[0,2]$, we have

$$\max_{0\leq\mu\leq 1}F(c,\mu)=F(c,1)=G(c). \quad (3.16)$$

From the relations (3.14) and (3.16), upon simplification, we obtain

$$G(c)=\left\{-2\beta^2c^4-4p(p+4\beta)c^2+16(p+\beta)(p+3\beta)\right\}. \quad (3.17)$$

$$G'(c) = \left\{-8\beta^2 c^3 - 8p(p+4\beta)c\right\}. \tag{3.18}$$

From the expression (3.18), we observe that $G'(c) \leq 0$ for all values of $c \in [0, 2]$ with $p \in N$ and $\beta > 0$. Therefore, G(c) is a monotonically decreasing function of c in $0 \leq c \leq 2$. Also, we have $G(c) > G(2)$. Hence, the maximum value of G(c) occurs at $c = 0$. From (3.17), we obtain

$$\max_{0 \leq c \leq 2} G(c) = 16(p+\beta)(p+3\beta). \tag{3.19}$$

From the expressions (3.13) and (3.19), after simplifying, we get

$$\left|d_1 c_1 c_3 + d_2 c_2^2\right| \leq 4(p+\beta)(p+3\beta). \tag{3.20}$$

From the expressions (3.4) and (3.20), upon simplification, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{4p^2}{(p+2\beta)^2}\right]. \tag{3.21}$$

By setting $c_1 = c = 0$ and selecting $x = -1$ in (2.2) and (2.4), we find that $c_2 = -2$ and $c_3 = 0$. Using these values in (3.4), we observe that equality is attained, which shows that our result is sharp. This completes the proof of our Theorem 3.1.

Remarks.

1) For the choice of $\beta = 1$, we get $I_p(1) = RT_p$, class of p- valent functions, whose derivative has a positive real part, from (3.21), we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \left[\frac{4p^2}{(p+2)^2}\right].$$

2) Choosing $p = 1$ and $\beta = \alpha$ with $\alpha > 0$, we get $I_p(\beta) = I_1(\alpha)$, for which, from (3.21), we obtain $|a_2a_4 - a_3^2| \leq \left[\frac{4}{(1+2\alpha)^2}\right]$. This result coincides with that of Murugusundaramoorthy and Magesh [13].

3) Choosing $p = 1$ and $\beta = 1$, we have $I_p(\beta) = RT$, from (3.21), we obtain $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. This inequality is sharp and it coincides with the result obtained by Janteng, Halim and Darus [9].

References

[1] Afaf Abubaker and M. Darus, Hankel Determinant for a class of analytic functions involving a generalized linear differential operator. Int. J. Pure Appl.Math., 69(4)(2011), 429 - 435.

[2] R.M Ali, Coefficients of the inverse of strongly starlike functions, Bull. Malays. Math. Sci. Soc. (second series) 26(1)(2003), 63 - 71.

[3] Oqlah. Al- Refai and M. Darus, Second Hankel determinant for a class of analytic functions defined by a fractional operator, European J. Sci. Res., 28(2)(2009), 234 - 241.

[4] R. Ehrenborg, The Hankel determinant of exponential polynomials, Amer. Math. Monthly, 107(6)(2000), 557 - 560.

[5] A.W. Goodman, Univalent functions Vol.I and Vol.II, Mariner publishing Comp. Inc., Tampa, Florida, 1983.

[6] U. Grenander and G. Szegö, Toeplitz forms and their applications, Second edition. Chelsea Publishing Co., New York, 1984.

[7] A. Janteng, S.A.Halim and M. Darus, Estimate on the Second Hankel Functional for Functions whose derivative has a positive real part, J. Qual. Meas. Anal.(JQMA), 4(1)(2008), 189 - 195.

[8] A. Janteng, S. A. Halim and M. Darus, Hankel Determinant for starlike and convex functions, Int. J. Math. Anal. 1(13),(2007), 619 - 625.

[9] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, J. Inequal. Pure Appl. Math, 7(2)(2006), 1 - 5.

[10] J. W. Layman, The Hankel transform and some of its properties, J. Integer Seq., 4(1)(2001), 1 - 11.

[11] T. H. Mac Gregor, Functions whose derivative have a positive real part, Trans. Amer.Math.Soc. 104(3)(1962), 532 - 537.

[12] A. K. Mishra and P. Gochhayat, Second Hankel Determinant for a class of Analytic Functions Defined by Fractional Derivative, Int. J. Math. Math. Sci. Vol.2008, Article ID 153280, 2008, 1-10.

[13] Gangadharan. Murugusundaramoorthy and N. Magesh, Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant, Bull Math Anal. Appl. 1(3) (2009), 85 - 89.

[14] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of a really mean p - Valent functions, Trans. Amer. Math. Soc., 223(2) (1976), 337 - 346.

[15] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roum. Math. Pures Et Appl., 28(8) (1983), 731 - 739.

[16] Öznur Özkan Kilic, Sufficient conditions for subordination of multivalent functions, J. Inequal. Pure Appl. Math,(2008), Article ID 374756, 8 pages

[17] S. Owa and H. M. Srivastava, Univalent and starlike generalised hypergeometric functions, Canad. J. Math., 39(5), (1987), 1057 - 1077.

[18] CH. Pommerenke, Univalent functions, Vandenhoeck and Ruprecht, Gottingen, (1975).

[19] CH. Pommerenke, On the Hankel determinants of univalent functions, Mathematica, (1967), 108-112.

[20] CH. Pommerenke, On the coefficients and Hankel determinants of univalent functions, J. London Math. Soc., (1966), 111-122.

[21] B. Simon, Orthogonal polynomials on the unit circle, Part 1. Classical theory. American Mathematical Society Colloquium Publications, 54, Part 1. American Mathematical Society, Providence, RI, 2005.

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