Pseudo AGP-injective rings

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Abstract. A ring \( R \) is called right Pseudo AGP-injective or right PAGP-injective for short, if for any \( a \in R \) there exists a positive integer \( n \) and a left ideal \( X_a \) such that \( \text{lr}(a^n) = Ra^n \oplus X_a \). In this Article, we investigate properties of right PAGP-injective rings satisfying some additional conditions.

1 Introduction

Throughout this paper, \( R \) denotes an associative ring with identity \( 1 \). The left and right annihilators of a subset \( X \) of \( R \) will be denoted as \( l(X) \) and \( r(X) \), respectively. We write \( J = J(R) \) and \( Z_r \), respectively for the Jacobson radical of \( R \) and the right singular ideal of \( R \). For regular rings we mean von Neumann regular rings.

At first, we recall that a ring \( R \) is called \textit{right P-injective} [1] if, for any \( a \in R \) , any right \( R \)-homomorphism from \( aR \) to \( R \) extends to an endomorphism of \( R \), this is equivalent to say that \( \text{lr}(a) = Ra \) for every \( a \in R \). We recall also that a ring \( R \) is called \textit{right GP-injective} if, for any \( 0 \neq a \in R \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and any right \( R \)-homomorphism from \( a^nR \) to \( R \) extends to an endomorphism of \( R \), this is equivalent to say that for any \( 0 \neq a \in R \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and \( \text{lr}(a^n) = Ra^n \). \( P \)-injective rings and \( GP \)-injective rings have been studied by many authors such as [1, 2, 3, 4, 5]. In paper [6], Stanley S. Page and Yiqiang Zhou generalized the concepts of \( P \)-injective rings and \( GP \)-injective rings to \( AP \)-injective rings and \( AGP \)-injective rings respectively. Following Page and Zhou, a ring \( R \) is called right \( AP \)-injective, if for any \( a \in R \) there exists a \( a \) left ideal \( X_a \) such that \( \text{lr}(a) = Ra \oplus X_a \), and a ring \( R \) is called right \( AGP \)-injective, if for any \( 0 \neq a \in R \) there exist a positive integer \( n \) and a left ideal \( X_a \) such that \( a^n \neq 0 \) and \( \text{lr}(a^n) = Ra^n \oplus X_a \). In this paper, we generalize the concept of right \( AGP \)-injective rings to right Pseudo \( AGP \)-injective rings (or right \( PAGP \)-injective rings for short), some interesting properties of \( PAGP \)-injective rings are obtained.

We start with the following Definition.

Definition 1. Let \( R \) be a ring and \( M \) be a right \( R \)-module with \( S = \text{End}(M_R) \). Then \( M_R \) is called \textit{Pseudo GP-injective} or \textit{PGP-injective} for short if, for any \( a \in R \) there exists a positive integer \( n \) such that \( \text{lr}_R(a^n) = Ma^n \). \( M_R \) is called \textit{Pseudo AGP-injective} or \textit{PAGP-injective} for short if, for any \( a \in R \) there exist a positive integer \( n \) and a \( S \)-submodule \( X_a \) such that \( \text{lr}_R(a^n) = Ma^n \oplus X_a \) as left \( S \)-modules. \( R \) is called right \textit{Pseudo GP-injective} or \textit{PGP-injective} for short if \( R_R \) is \( GP \)-injective. \( R \) is called right \textit{Pseudo AGP-injective} or \textit{PAGP-injective} for short if \( R_R \) is \( PAGP \)-injective.

The concepts of \( PGP \)-injective modules and \( PAGP \)-injective modules are explained by the following theorem.

Theorem 2. Let \( M \) be a right \( R \)-module with \( S = \text{End}(M_R) \). Then

(1) \( M \) is \textit{PGP-injective} if and only if for any \( a \in R \) there exists a positive integer \( n \) such that every homomorphism from \( a^nR \) to \( M \) extends to a homomorphism of \( R \) to \( M \).

(2) \( M \) is \textit{PAGP-injective} if and only if, for any \( a \in R \) there exists a positive integer \( n \) such that \( \text{Hom}_R(R, M) \) is a direct summand of \( \text{Hom}_R(a^nR, M) \) as \( S \)-modules.

Proof. (1) is obvious. (2) by [6, Lemma 1.2(3)].

Clearly, The following implications hold:

right \( P \)-injective \( \Rightarrow \) right \( GP \)-injective \( \Rightarrow \) right \( AGP \)-injective \( \Rightarrow \) right \( PAGP \)-injective.
right $P$-injective $\Rightarrow$ right $AP$-injective $\Rightarrow$ right $AGP$-injective $\Rightarrow$ right $PAGP$-injective.

But right $AGP$-injective (even if for right $AP$-injective) $\Rightarrow$ right $GP$-injective $\Rightarrow$ right $P$-injective by [6, Examples 1.5] and [4, Example 1].

right $AGP$-injective (even if for right $GP$-injective) $\Rightarrow$ right $AP$-injective $\Rightarrow$ right $P$-injective (even if right $GP$-injective) by [4, Proposition 2] and [6, Examples 1.5].

We don’t know whether right $PAGP$-injective rings (even if for right $AP$-injective rings) are $PGP$-injective, but we have that right $PGP$-injective rings (and hence right $PAGP$-injective rings) need not be right $AGP$-injective by the following Example 3, and right $PGP$-injective right $AP$-injective rings need not be right $GP$-injective by the following Example 4.

Example 3. A finite commutative ring which is $PGP$-injective but not $AGP$-injective.

Let $R = Z_2 \vartriangleleft 2Z_8$ be the trivial extension of $Z_2$ and the $Z_2$-module $2Z_8$. For $a = (\bar{n}, \bar{2}) \in R$. If $n = 1, 3, 5, 7$, then $a$ is invertible in $R$, thus $lr(a) = Ra$. If $n = 0, 2, 4, 6$, then $a^2 = 0$, and so $lr(a^2) = Ra^3$. Therefore $R$ is $PGP$-injective and hence $PAGP$-injective. For $b = (\bar{0}, \bar{2})$, we have $b^2 = 0$, $lr(b) = 2Z_8 \vartriangleleft 2Z_8$ and $Rb = (0) \vartriangleleft 2Z_8$. Clearly, $Rb$ is not a direct summand of $lr(b)$. Hence $R$ is not $AGP$-injective.

Example 4. A finite commutative ring which is $AP$-injective and $PGP$-injective, but not $GP$-injective.

Let $R = Z_4 \vartriangleleft (Z_4 \oplus Z_4)$. For any $a = (\bar{n}, \bar{r}, \bar{m}) \in R$, if $n = 1$, then $a$ is invertible, thus $lr(a) = Ra$. If $n = 0, 2$, then $a^2 = 0$, and so $lr(a^2) = Ra^2$. Hence $R$ is $PGP$-injective, moreover, by [6, Examples 1.5(2)], $R$ is a finite commutative ring which is $AP$-injective. Let $b = (\bar{1}, \bar{0}, \bar{0})$. Then $b^2 = 0$ and $lr(b) = (0) \vartriangleleft (Z_4 \oplus Z_4) \neq (0) \vartriangleleft (Z_4 \oplus (0)) = Rb$. Therefore, $R$ is not $GP$-injective.

Following [6], let $A, B$ are two left ideals of a ring $R$, then we write $A \mid B$ to indicate that $A$ is a direct summand of $B$.

Lemma 5. Let $R$ be a ring, $a \in R$ with $ra = 0$. Then for any positive integer $i$, $aR \mid a^{-1}R$ if and only if $a^{-1}R \mid aR$.

Proof $\Rightarrow$. Suppose $a^{-1}R = a'R \oplus K$ for some right ideal $K$. Then for any $r \in R$, we have $a^{-1}r = a^{-1}(ar') = a^{-1}(ar' + k) = a^{-1}r' + ak$, where $r' \in R, k \in K$, and so $a'R = a^{-1}R + aK$. Now if $x \in a^{-1}R \cap aK$, let $x^{-1}r = ak, r \in R, k \in K$. Then $a^{-1}r = 0$, and so $a^{-1}r - k = 0$ because $a(r-k) = 0$. Hence $a^{-1}r' + ak = 0$, this implies that $x = 0$, and whence $a^{-1}R \cap aK = 0$. Therefore, $a'R = a^{-1}R + aK$.

$\Leftarrow$. If $a^{-1}R \mid aR$, let $a'R = a^{-1}R \oplus N$ and write $N' = \{a^{-1}r : ar \in N\}$. Then for any $a^{-1}r \in a^{-1}R$, there exist a $r' \in R$ and an $n \in N$ such that $a^{-1}r = a^{-1}(r' + n)$. Hence $a^{-1}r = a^{-1}r' + a^{-1}(r - ar')$. Since $a^{-1}(r - ar') = a^{-1}r - a^{-1}r' = n \in N$, $a^{-1}r - a^{-1}r' \in N'$, and then $a^{-1}R = a^{-1}R + N'$. If $x \in a^{-1}R \cap N'$, let $x = a^{-1}r_1 + a^{-1}r_2$, where $r_1, r_2 \in R, dr_2 \in N$. Then $a^{-1}r_1 = a^{-1}r_2 \in a^{-1}R \cap N = 0$, which shows that $x = a^{-1}r_1 = 0$, and so $a^{-1}R \cap N = 0$. Hence $a^{-1}R = a^{-1}R \oplus N$.

Recall that a module $M_R$ is said to satisfy the generalized $C_2$-condition (or $GC_2$ for short) [7] if for any $N \leq M$ with $N \cong M$, $N$ is a direct summand of $M$. The ring $R$ is called right $GC_2$ if $R_R$ is $GC_2$.

Theorem 6. If $R$ is a right PAGP-injective ring, then

(1) $R$ is right $GC_2$.

(2) $R$ is a classical quotient ring.

Proof (1) Let $I$ be a right ideal of $R$ with $I \cong R_R$. Then $I = aR$ for some $a \in R$ with $ra = 0$. Since $R$ is right $PAGP$-injective, there exist a positive integer $n$ and a left ideal $X_{a^n}$ such that $lr(a^n) = Ra^n \oplus X_{a^n}$ (1)

By Lemma 5, $aR \mid R \Rightarrow a^{-1}R \mid a^{-1}R$. Thus, to prove $aR \mid R$, we need only to prove that $a^{-1}R \mid R$. Since $r(a^n) = 0$, by (1), we have $R = Ra^n \oplus X_{a^n}$. Let $1 = ba^n + x$, where $b \in R, x \in X_{a^n}$. Then $a^x = a^x ba^n + a^x x$, this follows that $a^x - a^x ba^n = a^x x \in Ra^n \cap X_{a^n} = 0$, and thus $a^x = a^x ba^n$. Let $e = a^x$. Then $e^2 = e$ and $a^{-1}R = eR$, as required.

(2) Let $I(a) = r(a) = 0$. Then $lr(a^k) = r(a^k) = 0$ for every positive integer $k$. By the right $PAGP$-injectivity of $R$, $lr(a^n) = Ra^n \oplus X_{a^n}$ holds for some positive integer $n$ and some left ideal
Let \( R \) be a ring and \( a \in R \).

Lemma 10. The following are equivalent for a ring \( R \).

(1) \( R \) is regular.
(2) \( N(R) = \{a \in R \mid a^2 = 0\} \) is regular and \( R \) is right generalized \( \pi \)-regular.

Proof

First of all, for any \( a \in R \), by the right \( PAGP \)-injectivity of \( R \), there exist a positive integer \( n \) and a left ideal \( X_{a^n} \) such that \( \text{Ir}(a^n) = Ra^n \oplus X_{a^n} \).

(1) Since \( R \) is right \( PP \), \( a^2 R \) is projective, and so there exists \( e^2 = e \in R \) such that \( \text{r}(a^n) = e R \).

Thus we have \( R(1 - e) = (1)(eR) = \text{Ir}(a^n) = Ra^n \oplus X_{a^n} \).

Let \( 1 - e = ba^n + x \), where \( b \in R \) and \( x \in X_{a^n} \).

Then \( a^n = a^n(1 - e) = a^n ba^n + a^n x \), this follows that \( a^n = a^n ba^n \). Therefore \( R \) is \( \pi \)-regular.

(2) Let \( I \) be a right ideal of \( R \) such that \( \text{r}(a^n) \oplus I \) is essential in \( R \). Then we have \( \text{Ir}(a^n) + I = \text{Ir}(a^n) \cap I = R \) and \( \text{Ir}(a^n) \cap I = \text{Ir}(a^n) + I = 0 \) because \( R \) is right \( IN \) and right nonsingular.

Therefore, \( R = \text{Ir}(a^n) \oplus I = Ra^n \oplus X_{a^n} \oplus I \).

Write \( 1 = ra^n + x \), where \( r \in R \), \( x \in X_{a^n} \oplus I \). Then \( a^n = a^n ra^n \). This implies that \( R \) is \( \pi \)-regular.

Corollary 7. Let \( R \) be a right \( PAGP \)-injective ring. Then

(1) \( Z_\pi \subseteq J(R) \).
(2) If \( R \) is right finite dimensional, then it is semilocal.

Theorem 8. Let \( R \) be a right \( PAGP \)-injective ring. Then

(1) If \( R \) is right \( PP \) (in particular, if \( R \) is a Baer ring), then \( R \) is \( \pi \)-regular.
(2) If \( R \) is a right nonsingular and right \( IN \)-ring, then \( R \) is \( \pi \)-regular.

Proof

First of all, for any \( a \in R \), by the right \( PAGP \)-injectivity of \( R \), there exist a positive integer \( n \) and a left ideal \( X_{a^n} \) such that \( \text{Ir}(a^n) = Ra^n \oplus X_{a^n} \).

(1) Since \( R \) is right \( PP \), \( a^2 R \) is projective, and so there exists \( e^2 = e \in R \) such that \( \text{r}(a^n) = e R \).

Thus we have \( R(1 - e) = (1)(eR) = \text{Ir}(a^n) = Ra^n \oplus X_{a^n} \).

Let \( 1 - e = ba^n + x \), where \( b \in R \) and \( x \in X_{a^n} \).

Then \( a^n = a^n(1 - e) = a^n ba^n + a^n x \), this follows that \( a^n = a^n ba^n \). Therefore \( R \) is \( \pi \)-regular.

(2) Let \( I \) be a right ideal of \( R \) such that \( \text{r}(a^n) \oplus I \) is essential in \( R \). Then we have \( \text{Ir}(a^n) + I = \text{Ir}(a^n) \cap I = R \) and \( \text{Ir}(a^n) \cap I = \text{Ir}(a^n) + I = 0 \) because \( R \) is right \( IN \) and right nonsingular.

Thus, \( R = \text{Ir}(a^n) \oplus I = Ra^n \oplus X_{a^n} \oplus I \).

Write \( 1 = ra^n + x \), where \( r \in R \), \( x \in X_{a^n} \oplus I \). Then \( a^n = a^n ra^n \). This implies that \( R \) is \( \pi \)-regular.

Corollary 9. Let \( R \) be a semiprime, right \( PAGP \)-injective right \( IN \)-ring. If each essential right ideal of \( R \) is an ideal, then \( R \) is a \( \pi \)-regular ring.

Proof

By Theorem 8(2), we need only to prove \( R \) is nonsingular. Indeed, if \( a \in R \) such that \( \text{r}(a) \) is essential in \( R \), then \( \text{r}(a) \) is an ideal of \( R \) by hypotheses, hence \( \text{Ir}(a) \) is also an ideal. Since \( (\text{Ir}(a) \cap \text{r}(a))^2 \subseteq (\text{Ir}(a) \cap \text{r}(a)) = 0 \) and \( R \) is semiprime, \( \text{Ir}(a) \cap \text{r}(a) = 0 \), and so \( a \in \text{Ir}(a) = 0 \) for \( \text{r}(a) \) is essential in \( R \). As required.

We call an element \( x \in R \) right generalized \( \pi \)-regular if there exists a positive integer \( n \) such that \( x^n = x y x^n \) for some \( y \in R \). \( R \) is called right generalized \( \pi \)-regular if every element in \( R \) is right generalized \( \pi \)-regular.

The results of Lemma 10 and Lemma 11 are similar to [3, Lemma 2.1] and [3, Theorem 2.2] respectively, and they are included here for the completeness.

Lemma 10. Let \( R \) be a ring and \( a \in R \). If \( a^n - ar a^n \) is regular for some positive integer \( n \) and \( r \in R \), then there exists \( y \in R \) such that \( a^n = ay a^n \), whence \( R \) is right generalized \( \pi \)-regular.

Proof

Let \( d = a^n - ar a^n \). Since \( d \) is regular, \( d = du d \) for some \( u \in R \). Hence

\[
a^n = d + ar a^n = (a^n - ar a^n)u(a^n - ar a^n) + ar a^n = a^n - ra^n u(1 - ar) a^n + ar a^n = ay a^n.
\]

where \( y = (a^{n-1} - ra^n u)(1 - ar) + r \).

Lemma 11. The following are equivalent for a ring \( R \).

(1) \( R \) is regular.
(2) \( N(R) = \{a \in R \mid a^2 = 0\} \) is regular and \( R \) is right generalized \( \pi \)-regular.
Proof. (1) ⇒ (2) is clear.

(2) ⇒ (1) Let \(a \in R\). Since \(R\) is right generalized \(\pi\)-regular, there exist a positive integer \(n\) and an element \(r\) in \(R\) such that \(a^n = ara^n\). Next we shall show that \(a\) is regular. In fact, if \(n = 1\), we are done. Let \(n > 1\). Put \(d = a^{n-1} - ara^{n-1}\). Then \(da = 0\), and so \(d^2 = d(ara^{n-1}) = 0\). Since \(N(R) = \{a \in R \mid a^2 = 0\}\) is regular, \(d\) is regular. Hence \(a^{n-1} = a\gamma yer_{a^{n-1}}\) for some \(y_1 \in R\) by Lemma 10. If \(n - 1 > 1\), then there exists \(y_2 \in R\) such that \(a^{n-2} = a\gamma yer_{a^{n-2}}\) by the preceding proof. Continues in this way, we will get \(b \in R\) such that \(a = aba\), i.e., \(a\) is regular.

Next we give a new characterization of regular rings.

Theorem 12 The following are equivalent for a ring \(R\):

(1) \(R\) is regular.

(2) Every principally right ideal of \(R\) is PGP-injective and \(N(R) = \{a \in R \mid a^2 = 0\}\) is regular.

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1) Let \(a \in R\). Write \(M = aR\). Since \(M\) is PGP-injective, there exists a positive integer \(n\) such that \(I_M(e^{(a)} = M^{(a)}\), so \(a^n = aba^n\) for some \(b \in R\). Hence, \(R\) is right generalized \(\pi\)-regular. Therefore, \(R\) is regular by Lemma 11.

Recall that a ring \(R\) is called strongly regular if for every \(a \in R\), there exists \(b \in R\) such that \(a = a^2b\); \(R\) is called reduced if it has no nonzero nilpotent elements. Clearly, a ring \(R\) is reduced if and only if \(r(a^k) = r(a)\) for any \(a \in R\) and any positive integer \(k\); a reduced ring is AGP-injective if and only if it is PAGP-injective.

Theorem 13 The following statements are equivalent for a ring \(R\):

(1) \(R\) is a strongly regular ring.

(2) \(R\) is a reduced right AGP-injective ring.

(3) \(R\) is a reduced right PAGP-injective ring.

(4) \(R\) is a reduced and right generalized \(\pi\)-regular ring.

Proof. (1) ⇒ (2) ⇒ (3) are obvious.

(3) ⇒ (4). Let \(a \in R\). Since \(R\) is right PAGP-injective, there exist a positive integer \(n\) and a left ideal \(X_{a^n}\) such that \(\mathfrak{I}(a^{2n}) = Ra^{2n} \oplus X_{a^n}^n\). Since \(R\) is reduced, \(r(a^{2n}) = r(a)\), and so \(a \in \mathfrak{I}(a) = \mathfrak{I}(a^{2n}) = Ra^{2n} \oplus X_{a^n}^n\). Let \(a = ba^n + x, b \in R, x \in X_{a^n}\). Then \(a^{2n} - a^{2n-1}ba^{2n} = a^{2n-1} \in Ra^{2n} \cap X_{a^n} = 0, i.e., a^{2n} = a(a^{2n-2}b)a^{2n}\). And (4) follows.

(4) ⇒ (1). Assume (4), then by Lemma 11, \(R\) is regular. Let \(a \in R\). Then \(a = aba\) for some \(b \in R\). Since \((a - a^2b)^2 = a^2 - a^3b - a^2ba + a^2ba^2 = a^2 - a^3b - a^2 + a^2ab = 0\) and \(R\) is reduced, \(a - a^2b = 0, i.e., a = b^2\). Therefore, \(R\) is strongly regular.

Theorem 14 If \(R\) is a semiprime right PAGP-injective ring, then the center of \(R\) is a regular ring.

Proof. Let \(a \in C(R)\). Since \(R\) is right PAGP-injective, there exist a positive integer \(n\) and a left ideal \(X_{a^n}\) such that \(\mathfrak{I}(a^{2n}) = Ra^{2n} \oplus X_{a^n}\). If \(a^{2n}r = 0\), then \((Ra^{2n-1}r)^2 = 0\), and so \(a^{2n-1}r = 0\) as \(R\) is semiprime. Continues in this way, we get \(ar = 0\), this follows that \(r(a) = r(a^n)\). Hence, \(a \in \mathfrak{I}(a) = \mathfrak{I}(a^{2n}) = Ra^{2n} \oplus X_{a^n}\). Let \(a = ba^n + x, b \in R, x \in X_{a^n}\). Then \(a^{2n} = a^{2n-1}ba^{2n} + a^{2n-1}x\), and thus \(a^{2n} - a^{2n-1}ba^{2n} = a^{2n-1} \in Ra^{2n} \cap X_{a^n} = 0, i.e., a^{2n} = a(a^{2n-2}b)a^{2n}\). Hence, \(a = a^{2n}b = a^{2n}a^{2n-2}b\). Let \(c = a^{2n-2}b\). Then \(a = ac\). Now we claim that \(c \in C(R)\). In fact, for any \(x \in R\), we have \(a^2(xc - cx) = x(a^{2n-2}b - a^{2n-2}bx) = xa^{2n} - b - x - ax - 0 = xc - cx \in R(a^2) = xc - cx \in r(a) \Rightarrow 0 = (a(xc - cx) = a^{2n-1}(xc - cx) = a^{2n-1}a^{2n-1}b) = x - b - x \in r(a^{2n-1}) = x - b - x \in r(a^{2n-1}) \Rightarrow 0 = a^{2n-2}(xc - cx) = xa^{2n-2} - a^{2n-2}b = xc - cx \Rightarrow xc = cx\), so \(c \in C(R)\), and therefore \(C(R)\) is strongly regular.

References


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