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Pseudo AGP-injective rings

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Abstract. A ring R is called right Pseudo AGP-injective or right $PAGP$-injective for short, if for any $a \in R$ there exists a positive integer n and a left ideal X_{a^n} such that $lr(a^n) = Ra^n \oplus X_{a^n}$. In this Article, we investigate properties of right $PAGP$-injective rings satisfying some additional conditions.

1 Introduction

Throughout this paper, R denotes an associative ring with identity . The left and right annihilators of a subset X of R will be denoted as $\mathbf{l}(X)$ and $\mathbf{r}(X)$, respectively. We write $J = J(R)$ and Z_r, respectively for the Jacobson radical of R and the right singular ideal of R. For regular rings we mean von Neumann regular rings.

At first, we recall that a ring R is called *right P-injective* [1] if, for any $a \in R$, any right R-homomorphism from aR to R extends to an endomorphism of R, this is equivalent to say that $\mathbf{lr}(a) = Ra$ for every $a \in R$. We recall also that a ring R is called *right GP-injective* if, for any $0 \neq a \in R$ there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism from a^nR to R extends to an endomorphism of R, this is equivalent to say that for any $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and $\mathbf{lr}(a^n) = Ra^n$. P-injective rings and GP-injective rings have been studied by many authors such as [1, 2, 3, 4, 5]. In paper [6], Stanley S.Page and Yiqiang Zhou generalized the concepts of P-injective rings and GP-injective rings to *AP-injective rings* and *AGP-injective rings* respectively. Following Page and Zhou, a ring R is called right AP-injective, if for any $a \in R$ there exists a a left ideal X_a such that $\mathbf{lr}(a) = Ra \oplus X_a$, and a ring R is called right AGP-injective, if for any $0 \neq a \in R$ there exist a positive integer n and a left ideal X_{a^n} such that $a^n \neq 0$ and $\mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$. In this paper, we generalize the concept of right AGP-injective rings to right Pseudo AGP-injective rings (or right $PAGP$-injective rings for short), some interesting properties of $PAGP$-injective rings are obtained.

We start with the following Definition.

Definition 1 *Let R be a ring and M be a right R-module with $S = End(M_R)$. Then M_R is called Pseudo GP-injective or PGP-injective for short if, for any $a \in R$ there exists a positive integer n such that $\mathbf{l}_M\mathbf{r}_R(a^n) = Ma^n$. M_R is called Pseudo AGP-injective or PAGP-injective for short if, for any $a \in R$ there exist a positive integer n and an S-submodule X_{a^n} such that $\mathbf{l}_M\mathbf{r}_R(a^n) = Ma^n \oplus X_{a^n}$ as left S-modules. R is called right Pseudo GP-injective or right PGP-injective for short if R_R is PGP-injective. R is called right Pseudo AGP-injective or right PAGP-injective for short if R_R is PAGP-injective.*

The concepts of PGP-injective modules and $PAGP$-injective modules are explained by the following theorem.

Theorem 2. *Let M be a right R-module with $S = End(M_R)$. Then*

(1) *M is PGP-injective if and only if for any $a \in R$ there exists a positive integer n such that every homomorphism from a^nR to M extends to a homomorphism of R to M.*

(2) *M is PAGP-injective if and only if, for any $a \in R$ there exist a positive integer n such that $Hom_R(R, M)$ is a direct summand of $Hom_R(a^nR, M)$ as left S-modules.*

Proof (1) is obvious. (2) by [6, Lemma 1.2(3)].

Clearly, The following implications hold:

right P-injective $\Rightarrow$ right GP-injective $\Rightarrow$ right AGP-injective $\Rightarrow$ right $PAGP$-injective.

right GP-injective $\Rightarrow$ right PGP-injective $\Rightarrow$ right $PAGP$-injective.

right P-injective $\Rightarrow$ right AP-injective $\Rightarrow$ right AGP-injective $\Rightarrow$ right $PAGP$-injective.

But right AGP-injective (even if for right AP-injective) $\nRightarrow$ right GP-injective $\nRightarrow$ right P-injective by [6, Examples 1.5] and [4, Example 1].

right AGP-injective (even if for right GP-injective) $\nRightarrow$ right AP-injective $\nRightarrow$ right P-injective (even if right GP-injective) by [4, Proposition 2] and [6, Examples 1.5].

We don't know whether right $PAGP$-injective rings (even if for right AP-injective rings) are PGP-injective, but we have that right PGP-injective rings (and hence right $PAGP$-injective rings) need not be right AGP-injective by the following Example 3, and right PGP-injective right AP-injective rings need not be right GP-injective by the following Example 4.

Example 3. *A finite commutative ring which is PGP-injective but not AGP-injective.*

Let $R = Z_8 \propto 2Z_8$ be the trivial extension of Z_8 and the Z_8-module $2Z_8$. For $a = (\overline{n}, 2\overline{x}) \in R$. If $n = 1, 3, 5, 7$, then a is invertible in R, thus $\mathbf{lr}(a) = Ra$. If $n = 0, 2, 4, 6$, then $a^3 = 0$, and so $\mathbf{lr}(a^3) = Ra^3$. Therefore R is PGP-injective and hence $PAGP$-injective . For $b = (\overline{0}, \overline{2})$, we have $b^2 = 0$, $\mathbf{lr}(b) = 2Z_8 \propto 2Z_8$ and $Rb = (0) \propto 2Z_8$. Clearly, Rb is not a direct summand of $\mathbf{lr}(b)$. Hence R is not AGP-injective .

Example 4. *A finite commutative ring which is AP-injective and PGP-injective, but not GP-injective.*

Let $R = Z_4 \propto (Z_4 \oplus Z_4)$. For any $a = (\overline{n}, \overline{l}, \overline{m}) \in R$, if $n = 1, 3$, then a is invertible, thus $\mathbf{lr}(a) = Ra$. If $n = 0, 2$, then $a^2 = 0$, and so $\mathbf{lr}(a^2) = Ra^2$. Hence R is PGP-injective, moreover, by [6, Examples 1.5(2)], R is a finite commutative ring which is AP-injective. Let $b = (\overline{0}, \overline{1}, \overline{0})$. Then $b^2 = 0$ and $\mathbf{lr}(b) = (0) \propto (Z_4 \oplus Z_4) \neq (0) \propto (Z_4 \oplus (0)) = Rb$. Therefore, R is not GP-injective.

Following [6], let A, B are two left ideals of a ring R, then we write $A \mid B$ to indicate that A is a direct summand of B.

Lemma 5. *Let R be a ring, $a \in R$ with $\mathbf{r}(a) = 0$. Then for any positive integer i, $a^iR \mid a^{i-1}R$ if and only if $a^{i+1}R \mid a^iR$.*

Proof $\Rightarrow$. Suppose $a^{i-1}R = a^iR \oplus K$ for some right ideal K. Then for any $r \in R$, we have $a^ir = a(a^{i-1}r) = a(a^ir' + k) = a^{i+1}r' + ak$, where $r' \in R, k \in K$, and so $a^iR = a^{i+1}R + aK$. Now if $x \in a^{i+1}R \cap aK$, let $x = a^{i+1}r = ak, r \in R, k \in K$. Then $a(a^ir - k) = 0$, and so $a^ir - k = 0$ because $\mathbf{r}(a) = 0$. Hence $a^ir = k \in a^iR \cap K = 0$, this implies that $x = 0$, and whence $a^{i+1}R \cap aK = 0$. Therefore, $a^iR = a^{i+1}R \oplus aK$.

$\Leftarrow$. If $a^{i+1}R \mid a^iR$, let $a^iR = a^{i+1}R \oplus N$ and write $N' = \{a^{i-1}r : a^ir \in N\}$. Then for any $a^{i-1}r \in a^{i-1}R$, there exist a $r' \in R$ and an $n \in N$ such that $a^ir = a^{i+1}r' + n$. Hence $a^{i-1}r = a^ir' + a^{i-1}(r - ar')$. Since $a^i(r - ar') = a^ir - a^{i+1}r' = n \in N$, $a^{i-1}(r - ar') \in N'$, and then $a^{i-1}R = a^iR + N'$. If $x \in a^iR \cap N'$, let $x = a^ir_1 = a^{i-1}r_2$, where $r_1, r_2 \in R, a^ir_2 \in N$. Then $a^{i+1}r_1 = a^ir_2 \in a^{i+1}R \cap N = 0$, which shows that $x = a^ir_1 = 0$, and so $a^iR \cap N = 0$. Hence $a^{i-1}R = a^iR \oplus N$.

Recall that a module M_R is said to satisfy the generalized C_2-condition (or GC_2 for short) [7] if for any $N \leq M$ with $N \cong M$, N is a direct summand of M. The ring R is called right GC_2 if R_R is GC_2.

Theorem 6. *If R is a right PAGP-injective ring, then*

(1) *R is right GC_2.*

(2) *R is a classical quotient ring.*

Proof (1) Let I be a right ideal of R with $I \cong R_R$. Then $I = aR$ for some $a \in R$ with $\mathbf{r}(a) = 0$. Since R is right $PAGP$-injective, there exist a positive integer n and a left ideal X_{a^n} such that

$$\mathbf{lr}(a^n) = Ra^n \oplus X_{a^n} \qquad (*)$$

By Lemma 5, $aR \mid R \Leftrightarrow a^nR \mid a^{n-1}R$. Thus, to prove $aR \mid R$, we need only to prove that $a^nR \mid R$. Since $\mathbf{r}(a^n) = 0$, by (*), we have $R = Ra^n \oplus X_{a^n}$. Let $1 = ba^n + x$, where $b \in R, x \in X_{a^n}$. Then $a^n = a^nba^n + a^nx$, this follows that $a^n - a^nba^n = a^nx \in Ra^n \cap X_{a^n} = 0$, and thus $a^n = a^nba^n$. Let $e = a^nb$. Then $e^2 = e$ and $a^nR = eR$, as required.

(2) Let $\mathbf{l}(a) = \mathbf{r}(a) = 0$. Then $\mathbf{l}(a^k) = \mathbf{r}(a^k) = 0$ for every positive integer k. By the right $PAGP$-injectivity of R, $\mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$ holds for some positive integer n and some left ideal

X_{a^n}. Thus, $R = \mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$. Let $1 = ba^n + x$, where $b \in R$, $x \in X_{a^n}$. Then $a^n = a^n ba^n$. Hence $1 = a(a^{n-1}b) = (ba^{n-1})a$, and the result follows.

By Theorem 6(1) and [7, Corollary 2.5 and Proposition 2.6], we have immediately the following corollary.

Corollary 7. Let *R be a right PAGP-injective ring. Then*

(1) $Z_r \subseteq J(R)$.

(2) *If R is right finite dimensional, then it is semilocal.*

Recall that a ring R is called *π-regular* if for every $a \in R$, there exist a positive integer n and $b \in R$ such that $a^n = a^n ba^n$; R is called *right PP* if every right ideal of R is projective; R is called a *Baer ring* [8] if the right annihilator of every nonempty subset of R is generated by an idempotent; Clearly, every Baer ring is right PP. R is called a *right IN ring* [9] if $\mathbf{l}(A \cap B) = \mathbf{l}(A) + \mathbf{l}(B)$ for every pair of right ideals A and B of R.

Theorem 8 *Let R be a right PAGP-injective ring. Then*

(1) *If R is right PP (in particular, if R is a Baer ring), then R is π-regular.*

(2) *If R is a right nonsingular and right IN-ring, then R is π-regular.*

Proof First of all , for any $a \in R$, by the right $PAGP$-injectivity of R , there exist a positive integer n and a left ideal X_{a^n} such that $\mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$.

(1) Since R is right PP, a^nR is projective, and so there exists $e^2 = e \in R$ such that $\mathbf{r}(a^n) = eR$. Thus we have $R(1-e) = \mathbf{l}(eR) = \mathbf{lr}(a^n) = Ra^n \oplus X_{a^n}$. Let $1-e = ba^n + x$, where $b \in R$ and $x \in X_{a^n}$. Then $a^n = a^n(1-e) = a^n ba^n + a^n x$, this follows that $a^n = a^n ba^n$. Therefore R is π-regular.

(2) Let I be a right ideal of R such that $\mathbf{r}(a^n) \oplus I$ is essential in R. Then we have $\mathbf{l}(\mathbf{r}(a^n)) + \mathbf{l}(I) = \mathbf{l}(\mathbf{r}(a^n) \cap I) = R$ and $\mathbf{lr}(a^n) \cap \mathbf{l}(I) \subseteq \mathbf{l}(\mathbf{r}(a^n) + I) = 0$ because R is right IN and right nonsingular. Thus, $R = \mathbf{lr}(a^n) \oplus \mathbf{l}(I) = Ra^n \oplus X_{a^n} \oplus \mathbf{l}(I)$. Write $1 = ra^n + x$, where $r \in R$, $x \in X_{a^n} \oplus \mathbf{l}(I)$. Then $a^n = a^n ra^n$. This implies that R is π-regular.

Corollary 9 *Let R be a semiprime, right PAGP-injective right IN-ring. If each essential right ideal of R is an ideal, then R is a π-regular ring.*

Proof By Theorem 8(2), we need only to prove R is nonsingular. Indeed, if $a \in R$ such that $\mathbf{r}(a)$ is essential in R, then $\mathbf{r}(a)$ is an ideal of R by hypotheses, hence $\mathbf{lr}(a)$ is also an ideal. Since $(\mathbf{lr}(a) \cap \mathbf{r}(a))^2 \subseteq (\mathbf{lr}(a))\mathbf{r}(a) = 0$ and R is semiprime, $\mathbf{lr}(a) \cap \mathbf{r}(a) = 0$, and so $a \in \mathbf{lr}(a) = 0$ for $\mathbf{r}(a)$ is essential in R. As required.

We call an element $x \in R$ *right generalized π-regular* if there exists a positive integer n such that $x^n = xyx^n$ for some $y \in R$. R is called right generalized π-regular if every element in R is right generalized π-regular.

The results of Lemma 10 and Lemma 11 are similar to [3, Lemma 2.1] and [3 , Theorem 2.2] respectively, and they are included here for the completeness.

Lemma 10 *Let R be a ring and $a \in R$. If $a^n - ara^n$ is regular for some positive integer n and $r \in R$, then there exists $y \in R$ such that $a^n = aya^n$, whence R is right generalized π-regular.*

Proof Let $d = a^n - ara^n$. Since d is regular, $d = dud$ for some $u \in R$. Hence

$$a^n = d + ara^n = (a^n - ara^n)u(a^n - ara^n) + ara^n$$

$$= a(a^{n-1} - ra^n)u(1 - ar)a^n + ara^n = aya^n$$

,where $y = (a^{n-1} - ra^n)u(1 - ar) + r$.

Lemma 11 *The following are equivalent for a ring R.*

(1) *R is regular.*

(2) $N(R) = \{a \in R \mid a^2 = 0\}$ *is regular and R is right generalized π-regular.*

Proof (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let $a \in R$. Since R is right generalized π-regular, there exist a positive integer n and an element r in R such that $a^n = ara^n$. Next we shall show that a is regular. In fact, if $n = 1$, we are done. Let $n > 1$. Put $d = a^{n-1} - ara^{n-1}$. Then $da = 0$, and so $d^2 = d(a^{n-1} - ara^{n-1}) = 0$. Since $N(R) = \{a \in R \mid a^2 = 0\}$ is regular, d is regular. Hence $a^{n-1} = ay_1a^{n-1}$ for some $y_1 \in R$ by Lemma 10 . If $n - 1 > 1$, then there exists $y_2 \in R$ such that $a^{n-2} = ay_2a^{n-2}$ by the preceding proof. Continus in this way, we will get $b \in R$ such that $a = aba$, i.e., a is regular.

Next we give a new characterization of regular rings.

Theorem 12 *The following are equivalent for a ring R.*

(1) *R is regular.*

(2) *Every principally right ideal of R is PGP-injective and* $N(R) = \{a \in R \mid a^2 = 0\}$ *is regular.*

Proof (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1) Let $a \in R$. Write $M = aR$. Since M is PGP-injective, there exists a positive integer n such that $\mathbf{l}_M\mathbf{r}_R(a^n) = Ma^n$, so $a^n = aba^n$ for some $b \in R$. Hence, R is right generalized π-regular. Therefore, R is regular by Lemma 11.

Recall that a ring R is called *strongly regular* if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$; R is called *reduced* if it has no nonzero nilpotent elements. Clearly, a ring R is reduced if and only if $\mathbf{r}(a^k) = \mathbf{r}(a)$ for any $a \in R$ and any positive integer k; a reduced ring is AGP-injective if and only if it is $PAGP$-injective.

Theorem 13 *The following statements are equivalent for a ring R:*

(1) *R is a strongly regular ring.*

(2) *R is a reduced right AGP-injective ring.*

(3) *R is a reduced right PAGP-injective ring.*

(4) *R is a reduced and right generalized* π*-regular ring.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are Obvious.

(3) $\Rightarrow$ (4). Let $a \in R$. Since R is right $PAGP$-injective, there exist a positive integer n and a left ideal $X_{a^{2n}}$ such that $\mathbf{lr}(a^{2n}) = Ra^{2n} \oplus X_{a^{2n}}$. Since R is reduced, $\mathbf{r}(a^{2n}) = \mathbf{r}(a)$, and so $a \in \mathbf{lr}(a) = \mathbf{lr}(a^{2n}) = Ra^{2n} \oplus X_{a^{2n}}$. Let $a = ba^{2n} + x, b \in R, x \in X_{a^{2n}}$. Then $a^{2n} - a^{2n-1}ba^{2n} = a^{2n-1}x \in Ra^{2n} \cap X_{a^{2n}} = 0$, i.e., $a^{2n} = a(a^{2n-2}b)a^{2n}$. And (4) follows.

(4) $\Rightarrow$ (1). Assume (4), then by Lemma 11 , R is regular. Let $a \in R$. Then $a = aba$ for some $b \in R$. Since $(a - a^2b)^2 = a^2 - a^3b - a^2ba + a^2ba^2b = a^2 - a^3b - a^2 + a^2ab = 0$ and R is reduced, $a - a^2b = 0$, i.e., $a = a^2b$. Therefore, R is strongly regular.

Theorem 14 *If R is a semiprime right PAGP-injective ring, then the center of R is a strongly regular ring.*

Proof Let $a \in C(R)$. Since R is right $PAGP$-injective, there exist a positive integer n and a left ideal $X_{a^{2n}}$ such that $\mathbf{lr}(a^{2n}) = Ra^{2n} \oplus X_{a^{2n}}$. If $a^{2n}r = 0$, then $(Ra^{2n-1}r)^2 = 0$, and so $a^{2n-1}r = 0$ as R is semiprime. Continues in this way, we get $ar = 0$, this follows that $\mathbf{r}(a) = \mathbf{r}(a^{2n})$. Hence, $a \in \mathbf{lr}(a) = \mathbf{lr}(a^{2n}) = Ra^{2n} \oplus X_{a^{2n}}$. Let $a = ba^{2n} + x, b \in R, x \in X_{a^{2n}}$. Then $a^{2n} = a^{2n-1}ba^{2n} + a^{2n-1}x$, and thus $a^{2n} - a^{2n-1}ba^{2n} = a^{2n-1}x \in Ra^{2n} \cap X_{a^{2n}} = 0$, which shows that $a^{2n} = a^{2n-1}ba^{2n}$, i.e., $1 - a^{2n-1}b \in \mathbf{r}(a^{2n}) = \mathbf{r}(a)$. Hence, $a = a^{2n}b = a^2a^{2n-2}b$. Let $c = a^{2n-2}b$. Then $a = a^2c$. Now we claim that $c \in C(R)$. In fact, for any $x \in R$, we have $a^2(xc - cx) = a^2(xa^{2n-2}b - a^{2n-2}bx) = xa^{2n}b - a^{2n}bx = xa - ax = 0 \Rightarrow xc - cx \in \mathbf{r}(a^2) \Rightarrow xc - cx \in \mathbf{r}(a) \Rightarrow 0 = a(xc - cx) = a^{2n-1}(xb - bx) \Rightarrow xb - bx \in \mathbf{r}(a^{2n-1}) \Rightarrow xb - bx \in \mathbf{r}(a^{2n-2}) \Rightarrow 0 = a^{2n-2}(xb - bx) = xa^{2n-2}b - a^{2n-2}bx = xc - cx \Rightarrow xc = cx$, so $c \in C(R)$, and therefore $C(R)$ is strongly regular.

References

[1] Nicholson, W. K., Yousif, M. F., Principally injective rings, J. Algebra, **174** (1) (1995), 77-93.

[2] Chen, J. L., Ding, N. Q., On general principally injective rings, Comm.Algebra, **27** (5) (1999), 2097-2116.

[3] Chen, J. L., Ding, N. Q., On regularity of rings, Algebra colloq. **8** (3) (2001), 267-274.

[4] Chen, J. L., Zhou, Y. Q., Zhu, Z. M., GP-injective rings need not be P-injective, Comm. Algebra, **33** (7) (2005), 2395-2402.

[5] Nam, S. B., Kim, N. K., Kim, J. Y., On simple *GP*-injective modules, Comm. Algebra, **23** (14) (1995), 5437-5444.

[6] Page, S. S., Zhou, Y. Q., Generalizations of Principally injective rings, J. Algebra, **206** (2) (1998), 706-721.

[7] Yousif, M. F., Zhou, Y. Q., Rings for which certain elements have the principal extension property, Algebra Colloq, **10** (4) (2003), 501-512.

[8] Kaplansky, I., Rings of Operators, Benjamin, New York, 1968, 3.

[9] Camillo, V., Nicholson, W. K., Yousif, M. F., Ikeda-Nakayama rings, J. Algebra, **226** (2) (2000), 1001-1010.

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