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Quasi-Conformal Curvature Tensor for the Spacetime of General Relativity

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Abstract. A detailed study of quasi-conformal curvature tensor for the spacetime of general relativity has been made. The spacetimes satisfying Einstein field equations with vanishing quasi-conformal curvature tensor have been considered and the existence of Killing and conformal Killing vector fields has been established. Perfect fluid spacetimes with vanishing quasi-conformal curvature tensor have also been considered. The divergence of quasi-conformal curvature tensor is studied in the setting of perfect fluid with the derivation of many physical results.

1. INTRODUCTION

In 1968, Yano and Sewaki [8] have given the concept of quasi-conformal curvature tensor, which is given by the expression

$$\begin{aligned}\widetilde{C}(X,Y)Z &= \mathrm{A}R(X,Y)Z \\ &+\mathrm{B}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &-\frac{R}{n}\left(\frac{\mathrm{A}}{n-1}+2\mathrm{B}\right)[g(Y,Z)X - g(X,Z)Y]\end{aligned} \tag{1}$$

Or, for covariant form, one can write

$$\begin{aligned}\widetilde{C}(X,Y,Z,T) &= g(\widetilde{C}(X,Y)Z,T) \\ &= \mathrm{A}R(X,Y,Z,T) + \mathrm{B}[\mathrm{Ric}(Y,Z)g(X,T) - \mathrm{Ric}(X,Z)g(Y,T) \\ &+g(Y,Z)g(QX,T) - g(X,Z)g(QY,T)] - \frac{R}{n}\left(\frac{\mathrm{A}}{n-1}+2\mathrm{B}\right) \\ &[g(Y,Z)g(X,T) - g(X,Z)g(Y,T)]\end{aligned} \tag{2}$$

where, $R(X,Y,Z,T) = g(R(X,Y,Z),T)$ and $R(X,Y,Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$ is the Riemann curvature tensor, $\mathrm{Ric}(X,Y) = g(R(X),Y)$ is the (0,2) type Ricci tensor, R is the scalar curvature, D is the Riemannian connection and A & B are the constants. Also Ricci operator Q is defined as $\mathrm{Ric}(X,Y) = g(QX,Y)$. Now equation (2) can be written in local coordinates as following

$$\begin{aligned}\widetilde{C}_{ijkl} &= \mathrm{A}R_{ijkl} + \mathrm{B}[R_{jl}g_{ik} + R_{ik}g_{jl} - R_{il}g_{jk} - R_{jk}g_{il}] \\ &-\frac{R}{n}\left(\frac{\mathrm{A}}{n-1}+2\mathrm{B}\right)[g_{ik}g_{jl} - g_{il}g_{jk}]\end{aligned} \tag{3}$$

and satisfies the following properties:

$$\widetilde{C}_{ijkl} = -\widetilde{C}_{jikl},\ \widetilde{C}_{ijkl} = -\widetilde{C}_{ijlk} \tag{4}$$

$$\widetilde{C}_{ijkl} = \widetilde{C}_{klij} \tag{5}$$

$$\widetilde{C}_{ijkl} + \widetilde{C}_{iklj} + \widetilde{C}_{iljk} = 0 \tag{6}$$

which in index-free notation can be expressed as

$$\widetilde{C}(X,Y,Z,T) = -\widetilde{C}(Y,X,Z,T),\ \widetilde{C}(X,Y,Z,T) = -\widetilde{C}(X,Y,T,Z) \tag{4a}$$

$$\widetilde{C}(X,Y,Z,T) = \widetilde{C}(Z,T,X,Y) \tag{5a}$$

$$\widetilde{C}(X,Y,Z,T) + \widetilde{C}(X,Z,T,Y) + \widetilde{C}(X,T,Y,Z) = 0 \tag{6a}$$

For A $= 1$ and B $= -\dfrac{1}{n-2}$, equation (3) gives

$$\begin{aligned}\widetilde{C}_{ijkl} &= R_{ijkl} - \frac{1}{n-2}[R_{jl}g_{ik} + R_{ik}g_{jl} - R_{il}g_{jk} - R_{jk}g_{il}] \\ &+ \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &\equiv C_{ijkl}\end{aligned} \tag{7}$$

where, C_{ijkl} is Weyl conformal tensor.

Motivation for the study of quasi-conformal curvature tensor comes from the work of Chaki and Ghosh, 1997 ([2]) for Riemannian manifold. The study of quasi-conformal curvature tensor for perfect fluid spacetime has been done by Sarbari Guha in 2003 (c.f., [5]). In this paper we have made a detailed study of this tensor on the spacetime of general relativity. Section-wise we have given algebraic properties of quasi-conformal curvature tensor. A detailed study of divergence of quasi-conformal curvature tensor and perfect fluid spacetimes is also given.

2. PRELIMINARIES

In general theory of relativity, the curvature tensor describing the gravitational field consists of two parts viz., the matter part and the free gravitational part. The interaction between these parts is described through Bianchi identities. For a given distribution of matter, the construction of gravitational potential satisfying Einstein field equations is the principal aim of all investigations in gravitational physics; and this has often been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of spacetime are expressed through the equation

$$£_\xi A - 2\Omega A = 0 \tag{8}$$

where A represents a geometrical/physical quantity, $£_\xi$ denotes the Lie derivative with respect to a vector field ξ and Ω is a scalar [7]. The symmetry assumptions on the spacetime manifold are also known as collineations. The literature on collineations is very large and still expanding with results of elegance. As an example, we define a symmetry (conformal motion) of a spacetime as following

Definition 1 : Consider an n-dimensional Riemannian space V_n and referred to co-ordinate system (x) in V_n we consider the point transformation

$$T:\ '\xi^x = f^x(\xi^\nu);\ Det\left(\frac{\partial f^x}{\partial \xi^\lambda}\right) \neq 0 \tag{9}$$

When point transformation (9) does not change the angle between two directions at a point it is said to define a conformal motion in V_n. The necessary and sufficient condition for this is that the infinitesimal Lie difference of g_{ij} is proportional to g_{ij}

$$£_\xi g_{ij} = 2\sigma g_{ij} \tag{10}$$

Similarly, we define

Definition 2 : A symmetry property of a spacetime is said to be “the symmetry inheritance" if and only if the Lie derivative of the energy momentum tensor T_{ij} is proportional to T_{ij} i.e.,

$$£_\xi T_{ij} = 2\Omega T_{ij} \tag{11}$$

where Ω and σ are constants.

The Einstein field equations, with a cosmological term are given by (c.f., [6])

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = -kT_{ij} \tag{12}$$

The Riemann curvature tensor satisfies ([1])

$$R_{ijkl;m} + R_{ijlm;k} + R_{ijmk;l} = 0 \tag{13}$$

where, a semi-colon denotes the covariant differentiation. Equation (13) called as Bianchi identity can be written in index-free notation as

$$(\nabla_U R)(X, Y, Z, T) + (\nabla_Z R)(X, Y, T, U) + (\nabla_T R)(X, Y, U, Z) = 0 \tag{14}$$

Now for spacetime of general relativity (4-dimensional), equation (3) changes to

$$\begin{aligned} \widetilde{C}(X, Y, Z, T) &= \mathrm{A}R(X, Y, Z, T) \\ &+\mathrm{B}[\mathrm{Ric}(Y, Z)g(X, T) - \mathrm{Ric}(X, Z)g(Y, T) + g(Y, Z)\mathrm{Ric}(X, T) \\ &-g(X, Z)\mathrm{Ric}(Y, T)] - \frac{R}{4}\left(\frac{\mathrm{A}}{3} + 2\mathrm{B}\right) \\ &.[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] \end{aligned} \tag{15}$$

In local coordinates, we write above equation as

$$\begin{aligned} \widetilde{C}_{ijkl} &= \mathrm{A}R_{ijkl} + \mathrm{B}[R_{jl}g_{ik} + R_{ik}g_{jl} - R_{il}g_{jk} - R_{jk}g_{il}] \\ &-\frac{R}{4}\left(\frac{\mathrm{A}}{3} + 2\mathrm{B}\right)[g_{ik}g_{jl} - g_{il}g_{jk}] \end{aligned} \tag{15a}$$

Using equations (13) and (15a), we get

$$\begin{aligned} &\widetilde{C}_{ijkl;m} + \widetilde{C}_{ijlm;k} + \widetilde{C}_{ijmk;l} \\ &= \mathrm{B}[g_{ik}(R_{jl;m} - R_{jm;l}) + g_{jl}(R_{ik;m} - R_{im;k}) + g_{il}(R_{jm;k} - R_{jk;m}) \\ &+g_{jm}(R_{il;k} - R_{ik;l}) + g_{im}(R_{jk;l} - R_{jl;k}) + g_{jk}(R_{im;l} - R_{il;m})] \end{aligned} \tag{16}$$

Definition 3 : If the Ricci tensor R_{ij} is of Codazzi type, then

$$(\nabla_X \mathrm{Ric})(Y, Z) = (\nabla_Y \mathrm{Ric})(X, Z) = (\nabla_Z \mathrm{Ric})(X, Y) \tag{17}$$

or, in local coordinates

$$R_{ij;k} = R_{ik;j} = R_{jk;i} \tag{18}$$

Note 1 : The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian or Pseudo Riemannian manifold have been given by Derdzinski and Shen [3].

Note 2 : The simplest Codazzi tensors are parallel one.

3. MAIN RESULTS

From equations (16) and (18), we have

$$\widetilde{C}_{ijkl;m} + \widetilde{C}_{ijlm;k} + \widetilde{C}_{ijmk;l} = 0 \tag{19}$$

equation (19) can be called as Bianchi-like identity for quasi-conformal curvature tensor. Now, Conversely if quasi-conformal curvature tensor satisfies the Bianchi-like identity (19), then equation (16) reduces to

$$\begin{aligned} &g_{ik}(R_{jl;m} - R_{jm;l}) + g_{jl}(R_{ik;m} - R_{im;k}) + g_{il}(R_{jm;k} - R_{jk;m}) \\ &+g_{jm}(R_{il;k} - R_{ik;l}) + g_{im}(R_{jk;l} - R_{jl;k}) + g_{jk}(R_{im;l} - R_{il;m}) = 0 \end{aligned} \tag{20}$$

On contraction, we get

$$R_{jl;k} = R_{jk;l}$$

which shows that the Ricci tensor is Codazzi.

Thus, we have

Theorem 1 : In a 4-dimensional spacetime of general relativity, the Ricci tensor is Codazzi if and only if quasi-conformal curvature tensor satisfies the Bianchi-like identity (19).

From equation (3), we have

$$\begin{aligned} \widetilde{C}^h_{ijk} &= \mathrm{A}R^h_{ijk} + \mathrm{B}[g_{ik}R^h_j + \delta^h_j R_{ik} - g_{jk}R^h_i - \delta^h_i R_{jk}] \\ &- \frac{R}{n}\left(\frac{\mathrm{A}}{n-1} + 2\mathrm{B}\right)[\delta^h_j g_{ik} - \delta^h_i g_{jk}] \end{aligned} \tag{21}$$

Contraction over h and k leads to

$$\widetilde{C}_{ij} = AR_{ij} \tag{22}$$

For non-null electromagnetic field, the energy momentum tensor T_{ij} is expressed as

$$T_{ij} = F_{im}F^m_j - \frac{1}{4}g_{ij}F_{pq}F^{pq} \tag{23}$$

and Einstein's equation (12), without cosmological term, reduces to

$$R_{ij} = kT_{ij} \tag{24}$$

Thus by using equations (22) and (24), we have

Theorem 2 : In a non-null electromagnetic field, the quasi-conformal curvature tensor and energy momentum tensor are related through

$$\widetilde{C}_{ij} = AkT_{ij} \tag{25}$$

The vanishing of Lie derivative of quasi-conformal curvature tensor gives rise to a new symmetry of the spacetime, one can termed as "quasi-conformal collineation". From equation (22), we write

$$\pounds_\xi \widetilde{C}_{ij} = A\pounds_\xi R_{ij} \tag{26}$$

and we have following

Theorem 3 : A four dimensional spacetime admits quasi-conformal collineation if it admits the Ricci collineation and conversely.

Thus, by using equation (25), we have the following

Corollary 1 : A four dimensional spacetime admits quasi-conformal collineation if it admits the symmetry inheritance property and conversely.

The Bianchi identities in contravariant form are given by

$$R^h_{ijk;l} + R^h_{ikl;j} + R^h_{ilj;k} = 0 \tag{27}$$

Contracting equation (27) over h and l, using the symmetry properties of Riemann curvature tensor, we get

$$R^h_{ijk;h} = R_{ij;k} - R_{ik;j} \tag{28}$$

We know that Riemannian manifolds for which the divergence of curvature tensor vanish identically are identified as manifolds with harmonic curvature. The curvature of such manifolds arise as a special case of Young-Mills fields. These manifolds also form a natural generalization of Einstein spaces and of conformally flat manifolds with constant scalar curvature.

Now from equation (15a), we have

$$\begin{aligned}\widetilde{C}^h_{ijk} &= \mathrm{A}R^h_{ijk} + \mathrm{B}[g_{ik}R^h_j + \delta^h_j R_{ik} - g_{jk}R^h_i - \delta^h_i R_{jk}] \\ &-\frac{R}{4}\left(\frac{\mathrm{A}}{3} + 2\mathrm{B}\right)[\delta^h_j g_{ik} - \delta^h_i g_{jk}]\end{aligned} \tag{29}$$

so that the divergence of quasi-conformal curvature tensor is given by

$$\widetilde{C}^h_{ijk;h} = \mathrm{A}R^h_{ijk;h} + \mathrm{B}[(R_{ik;j} - R_{jk;i}) + (g_{ik}R_{,j} - g_{jk}R_{,i})] \tag{30}$$

From equation (28), we thus have

Theorem 4 : A spacetime of constant curvature possess harmonic curvature if and only if spacetime has divergence-free quasi-conformal curvature tensor.

The Einstein field equations in the presence of matter are given by

$$R_{ij} - \frac{1}{2}Rg_{ij} = -kT_{ij} \tag{31}$$

On multiplication with g^{ij}, equation (31) leads to

$$R = -kT \tag{32}$$

From equations (32), equation (31) becomes

$$R_{ij} = -k(T_{ij} - Tg_{ij}) \tag{33}$$

Thus using equations (28) and (33), equation (30) leads to

$$\widetilde{C}^h_{ijk;h} = (A+B)k(T_{ij;k} - T_{ik;j} + \frac{1}{2}g_{ik}T_{,j} - \frac{1}{2}g_{ij}T_{,k}) + Bk(g_{ik}T_{,j} - g_{jk}T_{,i}) \tag{34}$$

which for purely electromagnetic distribution, reduces to

$$\widetilde{C}^h_{ijk;h} = (A+B)k(T_{ij;k} - T_{ik;j}) \tag{35}$$

Thus, we have

Theorem 5 : For a spacetime satisfying the Einstein field equations for a purely electromagnetic distribution, the quasi-conformal curvature tensor is conserved if the energy momentum tensor is Coddazi type and conversely.

The energy-momentum tensor for a perfect fluid is

$$T_{ij} = (\mu + p)u_i u_j + pg_{ij} \tag{36}$$

where p is isotropic pressure and u^i is velocity vector of the fluid.

Equation (36) leads to

$$T = -\mu + 3p \tag{37}$$

Now if T_{ij} is Coddazi and $\widetilde{C}^h_{ijk;h} = 0$, equation (34) then reduces to

$$-\left(\frac{1}{2}A + \frac{3}{2}B\right)kg_{ij}(\mu - 3p);j - \frac{(A+B)k}{2}g_{ij}(\mu - 3p);k + Bkg_{jk}(\mu - 3p);i = 0$$

which leads to

$$(\mu - 3p);i = 0$$

and this leads to

$$(\mu - 3p) = \text{Constant} \tag{38}$$

we thus have

Theorem 6 : If for a perfect fluid spacetime, the divergence of quasi-conformal curvature tensor vanishes and the energy-momentum tensor is Codazzi type then $(\mu - 3p)$ is constant.

It is known that for a radiative perfect fluid spacetime ($\mu = 3p$) the resulting universe is isotropic and homogeneous ([4]). Thus by choosing the constant in equation (38) as zero, we have

Corollary 2 : If the energy-momentum tensor for a divergence-free quasi-conformal curvature tensor is Codazzi type in perfect fluid spacetime then the resulting spacetime is radiative and consequently isotropic and homogeneous.

Now for spacetimes which have divergence-free quasi-conformal curvature tensor, the equation (34) leads to

$$(A+B)\left(T_{ij;k} - T_{ik;j} - \frac{1}{2}g_{ij}T_{,k}\right) + \left(\frac{A}{2} + \frac{3}{2}B\right)g_{ik}T_{,j} - Bg_{jk}T_{,i} = 0 \tag{39}$$

Using equations (36) and (37), we have

$$\begin{aligned}&(A+B)[(\mu+p)_{;k}u_iu_j + (\mu+p)u_{i;k}u_j + (\mu+p)u_iu_{j;k} + p_{;k}g_{ij}\\ &-(\mu+p)_{;j}u_iu_k - (\mu+p)u_{i;j}u_k - (\mu+p)u_iu_{k;j} + p_{;j}g_{ik} + \frac{1}{2}g_{ij}(\mu-3p)_{;k}]\\ &-\left(\frac{A}{2} + \frac{3B}{2}\right)g_{ik}(\mu-3p)_{;j} + Bg_{jk}(\mu-3p)_{;i} = 0\end{aligned} \tag{40}$$

Contracting the equation (40)with u^k, we get

$$\begin{aligned}&(A+B)[(\dot{\mu+p})u_iu_j + (\mu+p)\dot{u}_iu_j + (\mu+p)u_i\dot{u}_j + \dot{p}g_{ij}\\ &-(\mu+p)_{;j}u_iu_k - (\mu+p)u_{i;j}u_k - (\mu+p)u_iu_{k;j} + p_{;j}g_{ik} + \frac{1}{2}g_{ij}(\dot{\mu-3p})]\\ &-(\frac{A}{2} + \frac{3B}{2})(\mu-3p)_{;j}u_i + B(\mu-3p)_{;i}u_j = 0\end{aligned} \tag{41}$$

where an over head dot denotes the covariant derivative along the fluid flow vector u_a (that is, $(\dot{\mu+p}) = (\mu+p)_{;k}u^k$, $\dot{u}_j = u_{bj;k}u^k$, $\dot{p} = p_{;k}u^k$, $u_{i;j}u^i = 0$, etc.).

Also, the conservation of energy-momentum tensor ($T^{ij}_{;j} = 0$) leads to

$$(\mu+p)\dot{u}_i = -p_{;i} + \dot{p}u_i \qquad \text{(force equation)} \tag{42}$$

$$\dot{\mu} = -(\mu+p)u^i_{;i} = -(\mu+p)\theta \qquad \text{(energy equation)} \tag{43}$$

Moreover, the covariant derivative of the velocity vector can be splitted into kinematical quantities as ([4])

$$u_{i;j} = \frac{1}{3}\theta(g_{ij} + u_iu_j) - \dot{u}_iu_j + \sigma_{ij} + \omega_{ij} \tag{44}$$

where $\theta = u^i_{;i}$, is the expansion scalar, $\dot{u}_i = u_{i;j}u^j$, the acceleration vector $\sigma_{ij} = h^k_ih^l_ju_{(k;l)} - \frac{1}{3}\theta h_{ij}$, the symmetric shear tensor $(h_{ij} = g_{ij} - u_iu_j)$ and $\omega_{ij} = h^k_ih^d_ju_{[k;l]}$ is the skew symmetric virticity or rotation tensor.

Using force equation (42) in equation (41), we get

$$\begin{aligned}&(A+B)[(\dot{\mu+p})u_iu_j - p_{;i}u_j + \dot{p}u_ju_i - p_{;j}u_i + \dot{p}u_iu_j + \dot{p}g_{ij}\\ &-(\mu+p)_{;j}u_i - (\mu+p)u_{i;j} + p_{;j}u_i + \frac{1}{2}g_{ij}(\dot{\mu-3p})]\\ &-(\frac{A}{2} + \frac{3B}{2})(\mu-3p)_{;j}u_i + B(\mu-3p)_{;i}u_j = 0\end{aligned} \tag{45}$$

which can be expressed as

$$
\begin{aligned}
&(A+B)[(\mu+3p)\dot{}u_iu_j - p_{;i}u_j - p_{;j}u_i + \dot{p}g_{ij} - p_{;j}u_i \\
&-(\mu+p)u_{i;j} - p_{;j}u_i + \frac{1}{2}\dot{\mu}g_{ij} - \frac{3}{2}\dot{p}g_{ij}] - (\frac{A}{2}+\frac{3B}{2})\mu_{;j}u_i \\
&+(\frac{3A}{2}+\frac{9B}{2})p_{;j}u_i + B\mu_{;i}u_j + 3Bp_{;i}u_j = 0
\end{aligned}
\tag{46}
$$

Now, contracting the equation (46) with u^i, we get

$$
\begin{aligned}
&(A+B)[(\mu+3p)\dot{}u_j - \frac{1}{2}(\dot{p}+\dot{\mu})u_j] + (\frac{-3A}{2}+\frac{3B}{2})p_{;j} \\
&-(A+4B)\dot{p}u_j - \left(\frac{3A}{2}+\frac{5B}{2}\right)\mu_{;j} + B\dot{\mu}u_j = 0
\end{aligned}
\tag{47}
$$

which on simplification

$$
\begin{aligned}
&\left(\frac{3A}{2}+\frac{5B}{2}\right)\dot{\mu}u_j + \left(\frac{3A}{2}-\frac{5B}{2}\right)\dot{p}u_j \\
&-\frac{3}{2}(A-B)p_{;j} - \frac{1}{2}(3A-5B)\mu_{;j} = 0
\end{aligned}
\tag{48}
$$

we thus have

Theorem 7 : For a perfect fluid spacetime with the divergence free quasi-conformal curvature tensor, the pressure and density are constant.

Moreover, contracting equation (46) with u^j, we get

$$(A+4B)(-p_{;i}+\dot{p}u_i) + B(\dot{\mu}u_i+\mu_{;i}) - (A+B)(\mu+p)\dot{u}_i \tag{49}$$

Using the force equation (42), we get

$$3B(-p_{;i}+\dot{p}u_i) + B(\dot{\mu}u_i+\mu_{;i}) = 0 \tag{50}$$

or,

$$(\mu-3p)_{;i} + (\mu+p)\dot{\theta}u_i = 0 \tag{51}$$

While using the energy equation (43), we have

$$(\mu-3p)_{;i} + p_{;j}u^ju_i - (\mu+p)\theta u_i = 0 \tag{52}$$

Thus, we have

Theorem 8 : For a perfect fluid spacetime having conserved quasi-conformal curvature tensor, either pressure and density are constant over the space-like hypersurface orthogonal to the fluid four velocity or the fluid is expansion-free.

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