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Reconstruction Property and Frames in Banach Spaces

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Abstract. Casazza and Christensen in [5], introduced and studied the reconstruction property in Banach spaces. In this paper sufficient conditions for the existence of the reconstruction property in Banach spaces are obtained. Casazza and Christensen gave Paley- Wiener type perturbation of the reconstruction property which does not force equivalence of the sequences. Some Paley-Wiener type perturbations concerning the reconstruction property are discussed. Motivated by a paper by Holub [18], the notion of Besselian type reconstruction property in Banach spaces is introduced and its application to Banach frames is obtained.

1 Introduction

The Fourier transform has been widely used in analysis for more than a century. However, it only provides frequency information, and hides (in its phases) information concerning the moment of emission and duration of a signal. D. Gabor in 1946, introduced a fundamental approach to signal decomposition in terms of elementary signals and resolve this problem [14]. Duffin and Schaeffer [11] in 1952, while addressing some deep problems in non-harmonic Fourier series, abstracted Gabor's method to define frames for Hilbert spaces. Later, in 1986, Daubechies, Grossmann and Meyer [10] found new applications to wavelet and Gabor transforms in which frames played an important role.

Let $\mathcal{H}$ be a separable Hilbert space. A countable system $\{f_n\} \subset \mathcal{H}$ is called *frame* (Hilbert) for $\mathcal{H}$ if there exists positive constants A and B such that

$$A\|f\|^2 \leq \|\{\langle f, f_n\rangle\}\|^2_{\ell^2} \leq B\|f\|^2, \quad \text{for all } f \in \mathcal{H}.$$

The positive constants A and B, respectively, are called *lower* and *upper frame bounds* of the frame $\{f_n\}$. They are not unique. The operator $T : \ell^2 \to \mathcal{H}$ defined as $T(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \{c_k\} \in \ell^2$, is called the *pre-frame operator* or the *synthesis operator* and its adjoint $T^* : \mathcal{H} \to \ell^2$ given by $T^*(f) = \{\langle f, f_k\rangle\},$ for all $f \in \mathcal{H}$, is called the *analysis operator*. Composing T and T^* we obtain the *frame operator* $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k, \text{ for all } f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint invertible operator on $\mathcal{H}$. For all $f \in \mathcal{H}$, we have

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k\rangle f_k = \sum_{k=1}^{\infty} \langle f, S^{-1}f_k\rangle f_k.$$

The series converges unconditionally and is called the *reconstruction formula* for the frame. The representation of f in the reconstruction formula need not be unique. Today, frames play important roles in many applications in mathematics, science and engineering. In the theoretical direction, powerful tools from operator theory and Banach spaces are being employed to study frames. For a nice introduction to theory of frames an interested reader may refer to $[1, 2, 8, 17, 34]$ and references therein.

During the development of redundant building blocks (elementary signals), in the later half of twentieth century, Coifman and Weiss in [9] introduced the notion of atomic decomposition for function spaces. Later, Feichtinger and Gröchenig [12, 13] extended this idea to Banach spaces. This concept was further generalized by Gröchenig [15] who introduced the notion of Banach frames for Banach spaces. Casazza, Han and Larson [4] also carried out a study of atomic decompositions and Banach frames. Recently, various generalization of frames in Banach spaces have been introduced and studied. Han and Larson [16] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (i.e. a compression) of a Schauder basis of $\mathcal{X}$. Schauder frames were further studied in [22, 25, 26, 31]. The reconstruction property in Banach spaces was introduced and studied by Casazza and Christensen in [5] and further studied in [32]. The reconstruction property is an important tool in several areas of mathematics and engineering. As the perturbation result of Paley and Wiener preserves reconstruction property, it becomes more important from an application point of view. Further, the reconstruction property is used as a tool to recover certain Banach spaces. The reconstruction property is also used to study the geometry of Banach spaces. In fact, it is related to bounded the approximated property as observed in [1, 3, 4]. In [5], Casazza and Christensen gave some perturbation results. In fact, they develop a more general perturbation theory that does not force equivalence of the frames.

This paper is organized as follows: In Section 2 we give basic definitions and results which will be used throughout the paper. Sufficient conditions for the existence of the reconstruction property are discussed in Section 3. Section 4 is devoted to perturbation of reconstruction property. Casazza and Christensen give Paley- Wiener type perturbation of reconstruction property which does not force equivalence of the sequences. A perturbation result concerning the reconstruction property in which equivalence of sequences is one of the sufficient conditions is given. Uniform approximation of a compact operator on a Banach space which admits a reconstruction property is discussed. By inspiration from a paper by Holub [18], we introduce the notion of Besselian type reconstruction property in Banach spaces in Section 5. An application of the Besselian type reconstruction property to Banach frames have also been obtained. Banach frames for operator spaces associated with the reconstruction property are discussed in Section 6.

2 Preliminaries

Throughout this paper $\mathcal{X}$ will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ (which will be $\mathbb{R}$ or $\mathbb{C}$), $\mathcal{X}^*$ the conjugate space (topological) of $\mathcal{X}$. The map $\pi : \mathcal{X} \to \mathcal{X}^{**}$ denotes the canonical mapping from $\mathcal{X}$ into $\mathcal{X}^{**}$. The closure of the linear hull of a system $\{f_n\} \subset \mathcal{X}$ in the norm topology of $\mathcal{X}$ is denoted by $[f_n]$. The space of all bounded linear operators from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. For a pair $(\{f_k\}, \{f_k^*\}) \subset \mathcal{X} \times \mathcal{X}^*$, $\{P_n\}$ is the sequence of finite rank operators defined by $P_n(f) = \sum_{k=1}^{n} f_k^*(f) f_k,\ f \in \mathcal{X}$. The sequence of canonical unit vectors in ℓ^2 is denoted by $\{e_k\}$.

Definition 2.1. [15] Let $\mathcal{X}$ be a Banach space and let $\mathcal{X}_d$ be an associated Banach space of scalar valued sequences indexed by $\mathbb{N}$. Let $\{f_k^*\} \subset \mathcal{X}^*$ and $\mathcal{S} : \mathcal{X}_d \to \mathcal{X}$ be given. The pair $(\{f_k^*\}, \mathcal{S})$ is called a *Banach frame* for $\mathcal{X}$ with respect to $\mathcal{X}_d$ if

(i) $\{f_k^*(f)\} \in \mathcal{X}_d$, for all $f \in \mathcal{X}$.

(ii) There exist positive constants A and B with $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathcal{X}} \leq \|\{f_k^*(f)\}\|_{\mathcal{X}_d} \leq B\|f\|_{\mathcal{X}}, \quad \text{for all } f \in \mathcal{X}. \tag{2.1}$$

(iii) $\mathcal{S}$ is a bounded linear operator such that

$$\mathcal{S}(\{f_k^*(f)\}) = f, \quad \text{for all } f \in \mathcal{X}.$$

Definition 2.2. [5] Let $\mathcal{X}$ be a separable Banach space . A sequence $\{f_k^*\} \subset \mathcal{X}^*$ has the *reconstruction property* for $\mathcal{X}$ with respect to a sequence $\{f_k\} \subset \mathcal{X}$ if

$$f = \sum_{n=1}^{\infty} f_n^*(f) f_n, \quad \text{for all } f \in \mathcal{X}. \tag{2.2}$$

In short, we will say that the pair $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. More precisely, we say that $(\{f_k\}, \{f_k^*\})$ is a *reconstruction system* or *reconstruction property* for $\mathcal{X}$.

Remark 2.3. An interesting example for the reconstruction property is given in [5]: Let $\{f_k^*\} \subset \ell^\infty$ and $\{f_k^*\}$ is unitarily equivalent to the unit vector basis of ℓ^2. Then, $\{f_k^*\}$ has the reconstruction property with respect to its own pre-dual (that is, expansions with respect to the orthonormal basis).

Regarding the existence of Banach spaces which have a reconstruction system, Casazza and Christensen proved the following result.

Proposition 2.4. [5] *There exists a Banach space $\mathcal{X}$ with the following properties:*

(i) *There is a sequence $\{f_k\}$ such that each $f \in \mathcal{X}$ has a expansion $f = \sum_{k=1}^{\infty} f_k^*(f) f_k$.*

(ii) *$\mathcal{X}$ does not have the reconstruction property with respect to any pair $(\{h_k\}, \{h_k^*\})$.*

Definition 2.5. [4] A separable Banach space $\mathcal{X}$ has the λ-*bounded approximation property* (i.e. λ-BAP) if there is a sequence of finite rank operators $\{T_i\}$ defined on $\mathcal{X}$ so that for every $f \in \mathcal{X}$, $T_i f \to f$ in norm. We say that $\mathcal{X}$ has the *Bounded approximation property* (denoted by BAP) if $\mathcal{X}$ has the λ-BAP, for some λ.

The notion of reconstruction property is related to the Bounded Approximation Property(BAP). If $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$, then $\mathcal{X}$ has the bounded approximation property. Conversely, if $\mathcal{X}$ has the bounded approximation property then there exists a Banach space $\mathcal{A} \supset \mathcal{X}$ with a basis and by using a projection $P : \mathcal{A} \to \mathcal{X}$ we can find a sequence $\{g_k^*\} \subset \mathcal{X}^*$ such that $\{g_k^*\}$ has reconstruction property for $\mathcal{X}$ with respect to $\{P(\bullet)\}_k$. So, $\mathcal{X}$ is isomorphic to a complemented subspace of a Banach space with a basis. The reconstruction property is also used to study geometry of Banach spaces [3]. For more results and basics on the reconstruction property and bounded approximation property one may refer to [4] and references therein.

3 Reconstruction Property in Banach Spaces

Suppose that each vector of a Banach space $\mathcal{X}$ is expressed as an infinite linear combination of a given system say $\{f_k\} \subset \mathcal{X}$. Then, a natural question arises to find further condition(s) which guarantee the existence of $\{f_k^*\} \subset \mathcal{X}^*$ such that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. This problem is very deep and we do not know the answer even for Hilbert spaces. In this direction the following proposition gives sufficient conditions for the existence of a sequence $\{f_k^*\} \subset \mathcal{X}^*$ such that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$.

Proposition 3.1. *Let $\{f_k\} \backslash \{0\} \subset \mathcal{X}$ be a sequence of vectors such that for each $f \in \mathcal{X}$, there exists a sequence $\{\gamma_k\} \subset \mathbb{K}$ such that $f = \sum\limits_{k=1}^{\infty} \gamma_k f_k$. Let $\mathcal{Y} = \{\{\gamma_k\} \subset \mathbb{K} : \sum_{k=1}^{\infty} \gamma_k f_k$ converges in the norm in $\mathcal{X}\}$ be a Banach space with norm given by $\|\{\gamma_k\}\|_{\mathcal{Y}} = \sup\limits_{1 \le n < \infty} \left\| \sum\limits_{k=1}^{n} \gamma_k f_k \right\|_{\mathcal{X}}$. If $\mathcal{Z} = \{\{\gamma_k\} \subset \mathbb{K} : \sum_{k=1}^{\infty} \gamma_k f_k = 0\}$*

is a complemented subspace of $\mathcal{Y}$, then there exists a sequence $\{f_k^*\} \subset \mathcal{X}^*$ such that $\{f_k^*\}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_k\}$.

Proof. Let us write $\mathcal{Y} = \mathcal{M} \oplus \mathcal{Z}$. Define $\mathfrak{S} : \mathcal{Y} \to \mathcal{X}$ by $\mathfrak{S}(\{\gamma_k\}) = \sum_{k=1}^{\infty} \gamma_k f_k, \{\gamma_k\} \in \mathcal{Y}$. Then, $\mathfrak{S}|_{\mathcal{M}}$ is an isomorphism of $\mathcal{M}$ onto $\mathcal{X}$.
Fix $f \in \mathcal{X}$. Choose $\{f_k^*(f)\} = (\mathfrak{S}|_{\mathcal{M}})^{-1}(f) \in \mathcal{M}$. Then, each f_k^* is linear and $f = \mathfrak{S}\{f_k^*(f)\} = \sum\limits_{k=1}^{\infty} f_k^*(f) f_k$. It can be verified that each f_k^* is bounded. Indeed, for each $k \in \mathbb{N}$, $\|f_k^*\| \leq \alpha \|(\mathfrak{S}|_{\mathcal{M}})^{-1}\|$, where $\alpha \leq \dfrac{2}{\|f_k\|}$. Thus, each f_k^* is bounded. Hence $\{f_n^*\} \subset \mathcal{X}^*$ is such that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. □

Remark 3.2. Note that in Proposition 3.1, corresponding to $f_k = 0$, we can choose arbitrary $g = f_k^* \in \mathcal{X}^*$.

Let $\mathfrak{X}_o$ be a finite dimensional Banach space. Then, we can find a pair $(\{f_j^{(n)}\}, \{f_j^{*(n)}\})_{j=1}^{m_n} \subset \mathfrak{X}_o \times \mathfrak{X}_o^*$ such that $(\{f_j^{(n)}\}, \{f_j^{*(n)}\})_{j=1}^{m_n}$ has the reconstruction property for $\mathfrak{X}_o$. Using this and of certain ideas developed in [21, 23, 24, 28, 29, 30], the following theorem give sufficient conditions for the existence of a reconstruction property in Banach spaces.

Theorem 3.3. *Let $\{p_n\}$ $(p_n : \mathcal{X} \to \mathcal{X})$ be a sequence of bounded linear operators of finite rank such that $f = \sum\limits_{i=1}^{\infty} p_i(f)$, $f \in \mathcal{X}$. For each n, suppose that $(\{f_j^{(n)}\}, \{f_j^{*(n)}\})_{j=1}^{m_n}$ has the reconstruction property for $p_n(\mathcal{X})$ such that*

$$\sup_{\substack{1 \leq n < \infty \\ 1 \leq j \leq m_n}} \|u_j^{(n)}\| \leq K < \infty, \quad \textit{where } u_j^{(n)}(\bullet) = \sum_{i=1}^{j} f_i^{*(n)}(\bullet) f_i^{(n)}.$$

Then, there exists a sequence $\{f_n^*\} \subset \mathcal{X}^*$ such that $\{f_n^*\}$ has the reconstruction property for $\mathcal{X}$ with respect to some $\{f_n\} \subset \mathcal{X}$.

Proof. Since $(\{f_j^{(n)}\}, \{f_j^{*(n)}\})_{j=1}^{m_n}$ has the reconstruction property for $p_n(\mathcal{X})$, we have

$$g = \sum_{j=1}^{m_n} f_j^{*(n)}(g) f_j^{(n)}, \quad \text{for all } g \in p_n(\mathcal{X}).$$

Define sequences $\{f_n\} \subset \mathcal{X}$ and $\{f_n^*\} \subset \mathcal{X}^*$ by

$$\left.\begin{aligned} f_{m_0+m_1+m_n+i} &= f_i^{(n+1)} \\ f^*_{m_0+m_1+m_n+i} &= f_i^{*(n+1)} \circ p_{n+1} \end{aligned}\right\} \; i = 1, \ldots, m_{n+1};\; n = 0, 1, 2, \ldots;\; m_0 = 0.$$

Put $\omega_n = m_0 + m_1 + \ldots + m_n$ $(n = 0, 1, 2, \ldots)$.

Then, for all $f \in \mathcal{X}$, we have

$$\begin{aligned} u_{\omega_{n+i_n}}(f) &= \sum_{j=1}^{\omega_n + i_n} f_j^*(f) f_j \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m_k} f_j^{*(k)}(p_k(f)) f_j^{(k)} + \sum_{j=1}^{i_n} f_j^{*(n+1)}(p_{n+1}(f)) f_j^{(n+1)} \\ &= \sum_{k=1}^{n} p_k(f) + u_{i_n}^{(n+1)}(p_{n+1}(f)), \quad i_n = 1, \ldots, m_{n+1}. \end{aligned}$$

Now $\sum\limits_{k=1}^{n} p_k(f) \to f$ and $\|u_{i_n}^{(n+1)}(p_{n+1}(f))\| \leq K\|p_{n+1}(f)\| \to 0$ as $n \to \infty$. Therefore, $\|u_{\omega_n+i_n}(f) - f\| \to 0$ as $n \to \infty$. Thus, $f = \sum\limits_{j=1}^{\infty} f_j^*(f)f_j$, for all $f \in \mathcal{X}$. Hence $\{f_n^*\} \subset \mathcal{X}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_n\} \subset \mathcal{X}$. □

4 Perturbation of Reconstruction Property in Banach Spaces

Perturbation theory is a very important tool in various areas of applied mathematics [5, 6, 7]. It began with the fundamental perturbation result of Paley and Wiener [27, 34]. The basic idea of Paley and Wiener was that a system that is sufficiently close to an orthonormal system (basis) in a Hilbert space is also form an orthonormal system (basis). Since then, a number of variations and generalization of this perturbation to the setting of Banach space and then to perturbation of atomic decompositions, frames (Hilbert) and Banach frames have been observed in [7]. Casazza and Christensen proved the following theorem in [5], which is a Banach space version of the Paley-Wiener theorem for frames in Hilbert spaces.

Theorem 4.1. [5] *Suppose that* $(\{f_i\}, \{f_i^*\})$ *has the reconstruction property for* $\mathcal{X}$. *Let* $\mathcal{X}_d$ *be a sequence space which has the unit vectors* $\{e_i\}$ *as a basis.*
Assume that

$$T\{c_i\} = \sum_{i=1}^{\infty} c_i f_i$$

defines a bounded linear operator from $\mathcal{X}_d$ *into* $\mathcal{X}$. *Assume further that the operator* $R : X \to \mathcal{X}_d$ *given by*

$$Rf = \{f_i^*(f)\}$$

is a bounded operator. Let $\{g_i\}$ *be a sequence in* $\mathcal{X}$ *for which there exist constants* λ, μ *such that* $\lambda + \mu\|R\| < 1$ *and*

$$\|\sum_{i=1}^{\infty} c_i(f_i - g_i)\| \leq \lambda\|\sum_{i=1}^{\infty} c_i f_i\| + \mu\|\{c_i\}\|,$$

for all finitely non-zero scalar sequences $\{c_i\}$. *Then, there are functionals* $\{g_i^*\} \subset \mathcal{X}^*$ *so that* $(\{g_i\}, \{g_i^*\})$ *has the reconstruction property for* $\mathcal{X}$.

Moreover, $U : \mathcal{X}_d \to \mathcal{X}$ *given by* $U\{c_i\} = \sum_{i=1}^{\infty} c_i g_i$ *is a bounded linear and surjective operator, and*

$$\frac{1}{\|R\|}(1 - (\lambda + \mu\|R\|))\|f\| \leq \|U^* f\| \leq \|T\|(1 + \lambda + \frac{\mu}{\|T\|})\|f\|,$$

for all $f \in \mathcal{X}$. *Finally, if the unit vectors form an unconditional basis for* $\mathcal{X}_d$, *then the series* $\sum_{i=1}^{\infty} c_i g_i$ *converges unconditionally for all* $\{c_i\} \in \mathcal{X}_d$.

Remark 4.2. An important aspect of Theorem 4.1 is that it does not require the perturbed family $\{g_i\}$ to be equivalent to the original reconstruction sequence $\{f_i\}$. Recall that two sequences $\{f_i\}$ and $\{g_i\}$ in a Banach space are said to be *equivalent* if the mapping $f_i \to g_i$ can be extended to a well defined bounded linear map of $[f_i]$ onto $[g_i]$.

The following proposition provides a Paley-Wiener type perturbation, where the equivalence of two sequences is one of the sufficient conditions.

Proposition 4.3. *Suppose that* $(\{f_k\}, \{f_k^*\})$ *has the reconstruction property for* $\mathcal{X}$. *Let* $\lambda \in (0, 1)$ *and let* $\{g_k\} \subset \mathcal{X}$ *be such that*

$$\|\sum_{k=1}^{\infty} f_k^*(f)(f_k - g_k)\| \leq \lambda\|f\|, \textit{ for all } f \in \mathcal{X}.$$

If there exists $U \in B(\mathcal{X}, \mathcal{X})$ such that $U(f_k) = g_k$, for all $k \in \mathbb{N}$, then we can find $\{g_k^*\} \subset \mathcal{X}^*$ such that $(\{g_k\}, \{g_k^*\})$ is a reconstruction system for $\mathcal{X}$.

Proof. Define $L : \mathcal{X} \to \mathcal{X}$ as $L(f) = \sum_{k=1}^{\infty} f_k^*(f) g_k$. Then, L is well defined bounded linear operator. Indeed, for all $n \geq m$, we have

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} f_k^*(f) g_k - \sum_{k=1}^{m} f_k^*(f) g_k \right\| \\
&= \left\| U\Big(\sum_{k=1}^{n} f_k^*(f) f_k - \sum_{k=1}^{m} f_k^*(f) f_k\Big) \right\| \\
&\leq \|U\| \left\| \sum_{k=1}^{n} f_k^*(f) f_k - \sum_{k=1}^{m} f_k^*(f) f_k \right\| \longrightarrow 0 \quad \text{as } n, m \to \infty.
\end{aligned}$$

Fix $f \in \mathcal{X}$. Then,

$$\begin{aligned}
&\|(I - L)f\| \\
&= \|f - L(f)\| \\
&= \left\| \sum_{k=1}^{\infty} f_k^*(f) f_k - \sum_{k=1}^{\infty} f_k^*(f) g_k \right\| \\
&\leq \lambda \|f\|.
\end{aligned}$$

Therefore, $\|I - L\| < 1$. Thus, L is an invertible operator. Choose $g_k^* = (L^{-1^*}) f_k^*$, for all $k \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} g_k^*(f)(g_k) &= \sum_{k=1}^{\infty} ((L^{-1})^* f_k^*)(f) g_k \\
&= \sum_{k=1}^{\infty} f_k^*(L^{-1}(f)) g_k \\
&= L(L^{-1} f) \\
&= f, \quad \text{for all } f \in \mathcal{X}.
\end{aligned}$$

Hence $(\{g_k\}, \{g_k^*\})$ is a reconstruction system for $\mathcal{X}$.

□

Recall that if a Banach space $\mathcal{X}$ has the reconstruction property $(\{f_k\}, \{f_k^*\})$, then each $f \in \mathcal{X}$ can be written as a linear combination (infinite) of $\{f_k\}$ over $\{f_k^*(f)\}$ (coefficients in the sense of Fourier). Consider a situation where a given system $\{g_k\} \subset \mathcal{X}$ is close to $\{f_k\}$, but it is not possible to find $\{g_k^*\} \subset \mathcal{X}$ such that $(\{g_k\}, \{g_k^*\})$ is a reconstruction property for $\mathcal{X}$. In such a situation one direction is to reconstruct the space $\mathcal{X}$ via its image under a continuous invertible operator over the same coefficients $\{f_k^*(f)\}$, which is discussed in Proposition 4.5. This leads us to define the *support* of the reconstruction property for the underlying space.

Definition 4.4. Let $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ be a reconstruction system for $\mathcal{X}$ and let $\{g_k\} \subset \mathcal{X}$. We say that $\mathcal{G} \equiv (\{g_k\}, \{f_k^*\})$ *support* $\mathcal{F}$, if there exists an invertible operator $T \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ such that $T(f) = \sum_{k=1}^{\infty} f_k^*(f) g_k$, for all $f \in \mathcal{X}$.

The following proposition provides sufficient conditions for a given reconstruction system $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ for $\mathcal{X}$ such that $\mathcal{G} \equiv (\{g_k\}, \{f_k^*\})$, $(\{g_k\} \subset \mathcal{X})$ support $\mathcal{F}$.

Proposition 4.5. *Let $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ be a reconstruction system for $\mathcal{X}$. Choose $\delta > 0$ such that*

$$\|\sum_{k=1}^{N} f_k^*(f)(f_k)\| \leq \delta\|f\|, \text{ for all } f \in \mathcal{X} \text{ and for all } N \geq 1. \tag{4.1}$$

Let $\epsilon \in (0,1)$ and let $\{g_n\} \subset \mathcal{X}$ be a sequence such that for all $n \leq m \leq l$

$$\|\sum_{k=n}^{m} f_k^*(f)(f_k - g_k)\| \leq \frac{\epsilon}{\delta} \sup_{n \leq l \leq m} \|\sum_{k=n}^{l} f_k^*(f)(f_k)\|, \ n \in \mathbb{N} \quad \text{for all } f \in \mathcal{X}. \tag{4.2}$$

Then, $\mathcal{G} \equiv (\{g_k\}, \{f_k^*\})$ support $\mathcal{F}$.

Proof. Define $\Theta : \mathcal{X} \longrightarrow \mathcal{X}$ by $\Theta(f) = \sum_{k=1}^{\infty} f_k^*(f)(f_k - g_k)$. Then, Θ is a well defined bounded linear operator. By using inequality (4.2), we have

$$\|\Theta f\| \leq \lim_{n\to\infty} \frac{\epsilon}{\delta} \sup_{1 \leq l \leq n} \|\sum_{k=1}^{l} f_k^*(f) f_k\|, \text{ for all } f \in \mathcal{X}.$$

Therefore, by using (4.1), we obtain $\|\Theta f\| \leq \epsilon\|f\|$, for all $f \in \mathcal{X}$. Thus, $\|\Theta\| < 1$. Hence $T = I - \Theta$ is a continuously invertible operator .
Now for all $f \in \mathcal{X}$, we have

$$\begin{aligned} T(f) &= T\left(\sum_{k=1}^{\infty} f_k^*(f) f_k\right) \\ &= f - \sum_{k=1}^{\infty} f_k^*(f)(f_k - g_k) \\ &= \sum_{k=1}^{\infty} f_k^*(f) g_k. \end{aligned}$$

Hence $\mathcal{G} \equiv (\{g_k\}, \{f_k^*\})$ support $\mathcal{F}$. □

The following proposition shows that if $\{f_k^*\}$ has the reconstruction property for $\mathcal{X}$, then we can determine a compact operator on $\mathcal{X}$ under certain closeness of $\{f_k^*\}$ and its image under a bounded linear operator on the conjugate space of the underlying space.

Proposition 4.6. *Suppose that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. Let $\Theta \in \mathcal{B}(\mathcal{X}^*, \mathcal{X}^*)$ be such that*

$$\sum_{k=1}^{\infty} \|\Theta(f_k^*) - f_k^*\| \|f_k\| < \infty.$$

Then, $f \mapsto \sum_{k=1}^{\infty} [(\Theta(f_k^*) - f_k^*)(f)] f_k$ defines a compact operator on $\mathcal{X}$.

Proof. Let $T : \mathcal{X} \to \mathcal{X}$ be an operator defined as

$$T(f) = \sum_{k=1}^{\infty} [(\Theta(f_k^*) - f_k^*)(f)] f_k, \ f \in \mathcal{X}.$$

Then, for $n \geq m$, we have

$$\begin{aligned}
&\left\| \sum_{k=m}^{n} [(\Theta(f_k^*) - f_k^*)(f)] f_k \right\| \\
&\leq \left| \sum_{k=m}^{n} [(\Theta(f_k^*) - f_k^*)(f)] \right| \|f_k\| \\
&\leq \left(\sum_{k=m}^{n} \|\Theta(f_k^*) - f_k^*\| \|f_k\| \right) \|f\|.
\end{aligned}$$

Therefore, T is a well defined bounded linear operator on $\mathcal{X}$. For each $n \in \mathbb{N}$, define $T_n : \mathcal{X} \to \mathcal{X}$ by

$$T_n(\bullet) = \sum_{k=1}^{n} [(\Theta(f_k^*) - f_k^*)(\bullet)] f_k.$$

Then, $\|T - T_n\| \leq \sum_{k \geq n+1} \|\Theta(f_k^*) - f_k^*\| \|f_k\| \longrightarrow 0$ as $n \to \infty$. Since each T_n is a finite dimensional and continuous operator, T is compact. □

Suppose that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. Then, the sequence of finite rank operators $P_n : \mathcal{X} \to \mathcal{X}, P_n(f) = \sum_{k=1}^{n} f_k^*(f) f_k$ has the property that $P_n(f) \to f$ in the norm, for all $f \in \mathcal{X}$. To conclude the section we show that if a Banach space admits a reconstruction property, then every compact operator can be approximated uniformly by a system of operators of finite rank defined on the underlying space. In Banach space theory, this is called the "Compact Approximation Property".

Proposition 4.7. *Suppose that $\{f_k^*\} \subset \mathcal{X}^*$ has the reconstruction property for $\mathcal{X}$ with respect to a sequence $\{f_k\} \subset \mathcal{X}$. Let $\Theta : \mathcal{X} \to \mathcal{X}$ be a compact operator. Then, there exists a sequence $\{\widehat{\Theta}_n\}$ of bounded linear operators of finite rank on $\mathcal{X}$ such that $\lim_{n\to\infty} \widehat{\Theta}_n = \Theta$ uniformly.*

Proof. Assume that $\mathcal{W} \subset \mathcal{X}$ is a compact set. Let $\mu = \sup_{1 \leq n < \infty} \|P_n\|$ and let $\epsilon > 0$ be arbitrary. Choose a finite $\frac{\epsilon}{2(1+\mu)} (= \epsilon_0)$-net, $\{g_l\}_{l=1}^{r}$ for $\mathcal{W}$. Let $f \in \mathcal{W}$ be arbitrary. Then, there exist an g_j such that $\|f - g_j\| < \epsilon_0$. Also, we can find a positive integer $N \equiv N(\epsilon)$ such that

$$\left\| g_k - \sum_{i=1}^{n} f_i^*(g_k) f_i \right\| < \frac{\epsilon}{2}, \quad \text{for all } n \geq N, \ (k = 1, 2, ..., r).$$

Now

$$\begin{aligned}
\left\| f - \sum_{i=1}^{n} f_i^*(f) f_i \right\| &\leq \|f - g_j\| + \left\| g_j - \sum_{i=1}^{n} f_i^*(g_j) f_i \right\| + \left\| \sum_{i=1}^{n} f_i^*(g_j) f_i - \sum_{i=1}^{n} f_i^*(f) f_i \right\| \\
&\leq (1+\mu)\|f - g_j\| + \left\| g_j - \sum_{i=1}^{n} f_i^*(g_j) f_i \right\| \\
&\leq (1+\mu)\epsilon_0 + \frac{\epsilon}{2} \\
&= \epsilon, \text{ for all } n \geq N.
\end{aligned}$$

Thus, since $f \in \mathcal{W}$ was arbitrary, we have

$$\lim \sup_{f \in \mathcal{W}} \left\| f - \sum_{i=1}^{n} f_i^*(f) f_i \right\| = 0. \tag{4.3}$$

Let $D = \{f \in \mathcal{X} : \|f\| \leq 1\}$. Then, $\mathrm{T} = \overline{\Theta(D)}$ is compact. Choose $\widehat{\Theta}_n = P_n\Theta$, for each $n \in \mathbb{N}$. Then, each $\widehat{\Theta}_n$ is of finite rank and by using (4.3), we have

$$\begin{aligned} \|\Theta - \widehat{\Theta}_n\| &= \sup_{\substack{f \in \mathcal{X} \\ \|f\| \leq 1}} \|\Theta(f) - \widehat{\Theta}_n(f)\| \\ &= \sup_{f_0 \in \mathrm{T}} \|f_0 - P_n(f_0)\| \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence $\{\widehat{\Theta}_n\}$ is the required system of bounded linear operators of finite rank such that $\lim_{n\to\infty} \widehat{\Theta}_n = \Theta$ uniformly. □

5 Reconstruction Property of Besselian Type

On the problem concerned with diagnosis of frames, which are actually a Riesz basis for a Hilbert space $\mathcal{H}$ or more generally, when a frame is a "near-Riesz basis" (in the sense that the deletion of a finitely many terms from a frame leaves a Riesz basis), Holub in [18] introduced and studied Besselian frames.

Definition 5.1. [18] A frame $\{f_k\}$ for $\mathcal{H}$ is said to be

(i) *Besselian* if whenever $\sum_{k=1}^{\infty} a_k f_k$ converges, then $\{a_k\} \in \ell^2$.

(ii) a *near Riesz basis* if there is a finite set σ for which $\{f_k\}_{k \notin \sigma}$ is a Riesz basis for $\mathcal{H}$.

One of the fundamental results in [18] says "A frame for Hilbert space is Besselian if and only if it a near-Riesz basis" . Further by motivation from a result which gives a characterization of a given frame in a Hilbert space as images of a complete orthonormal system under a quotient map, we introduce Besselian type reconstruction property in Banach spaces.

Definition 5.2. A reconstruction system $(\{f_k\}, \{f_k^*\})$ for $\mathcal{X}$ is said to be

(i) $\mathcal{X}$-Besselian if $\sum_{k=1}^{\infty} |f_k^*(f)|^2 < \infty$, for all $f \in \mathcal{X}$.

(ii) $\mathcal{X}^*$-Besselian if $\sum_{k=1}^{\infty} |f^*(f_k)|^2 < \infty$, for all $f^* \in \mathcal{X}^*$.

(iii) $\mathcal{X}^{**}$-Besselian if $\sum_{k=1}^{\infty} |\Phi(f_k^*)|^2 < \infty$, for all $\Phi \in \mathcal{X}^{**}$.

Example 5.3. Let $\mathcal{X} = L^1(\Omega)$, where Ω is the set of positive integers with counting measure. Consider a system $\{f_k\} \subset \mathcal{X}$ given by $f_1 = \chi_1, f_k = \chi_{k-1}$, where $\chi_k = \{0, 0, 0, \ldots\ldots, \underbrace{1}_{k^{th}-place}, 0, 0, \ldots\}$. Define $\{f_k^*\} \subset \mathcal{X}^*$ by $f_1^*(f) = 0$, $f_k^*(f) = \xi_{k-1}$, $f = \{\xi_j\} \in \mathcal{X}$. Then, it can be verified that $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}$-Besselian, but not $\mathcal{X}^*$-Besselian. Thus, we conclude from this that

$$\mathcal{X}\text{-Besselian} \not\Rightarrow \mathcal{X}^*\text{-Besselian} \not\Rightarrow \mathcal{X}^{**}\text{-Besselian}.$$

Remark 5.4. One may observe that if $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}^{**}$-Besselian, then $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}$-Besselian. If $\mathcal{X}$ is reflexive, then $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}$-Besselian if and only if $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}^{**}$-Besselian.

The following proposition provides a sufficient condition for a reconstruction property for $\mathcal{X}$ to be $\mathcal{X}$-Besselian.

Proposition 5.5. *If $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$ and $\widehat{\Theta} \in B(\mathcal{X}, \ell^2)$ is such that $\widehat{\Theta}(f_k) = e_k$, for all $k \in \mathbb{N}$, then $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}$-Besselian, where $\{e_k\}$ is the sequence of unit vectors in ℓ^2.*

Proof. Since $f = \sum_{k=1}^{\infty} f_k^*(f) f_k$, for all $f \in \mathcal{X}$,

$$\begin{aligned}
\infty &> \|\widehat{\Theta}(f)\|^2 \\
&= \left\| \sum_{k=1}^{\infty} f_k^*(f) \widehat{\Theta}(f_k) \right\|^2 \\
&= \left\| \sum_{k=1}^{\infty} f_k^*(f) e_k \right\|^2 \\
&= \sum_{k=1}^{\infty} |f_k^*(f)|^2, \text{ for all } f \in \mathcal{X}.
\end{aligned}$$

Hence $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}$-Besselian. □

Holub in [18] characterizes frames which are images of an orthonormal basis.

Theorem 5.6. [18] *A sequence of vectors $\{f_k\} \subset \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if there exists a bounded linear operator $\mathcal{Q}$ from ℓ^2 onto $\mathcal{H}$ for which $\mathcal{Q}e_k = f_k$, for all k.*

By motivation from this result, the following proposition gives sufficient conditions for a reconstruction property to be $\mathcal{X}^*$-Besselian.

Proposition 5.7. *Suppose that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. If there exists an operator $\widehat{\Theta} \in B(\ell^2, \mathcal{X})$ such that $\widehat{\Theta}(e_k) = f_k$, for all $k \in \mathbb{N}$, then $(\{f_k^*\}, \{f_k\})$ is $\mathcal{X}^*$-Besselian*

Proof. We compute

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} f^*(f_k) f_k \right\| \\
&= \left\| \sum_{k=1}^{n} f^*(f_k) \widehat{\Theta}(e_k) \right\| \\
&\le \|\widehat{\Theta}\| \left\| \sum_{k=1}^{n} f^*(f_k) e_k \right\| \\
&= \|\widehat{\Theta}\| \sqrt{\sum_{k=1}^{n} |f^*(f_k)|^2}, \text{ for all } n \in \mathbb{N} \text{ and for all } f^* \in \mathcal{X}^*.
\end{aligned} \tag{5.1}$$

Now for all $f^* \in \mathcal{X}^*$, by using (5.1), we have

$$\begin{aligned}
&\sum_{k=1}^{n} |f^*(f_k)|^2 \\
&= f^*\left(\sum_{k=1}^{n} \overline{f^*(f_k)} f_k\right) \\
&\leq \|f^*\| \left\|\sum_{k=1}^{n} \overline{f^*(f_k)} f_k\right\| \\
&\leq \|f^*\| \|\widehat{\Theta}\| \sqrt{\sum_{k=1}^{n} |f^*(f_k)|^2}, \text{ for all } n \in \mathbb{N}.
\end{aligned}$$

Therefore, $\sqrt{\sum_{k=1}^{n} |f^*(f_k)|^2} \leq \|f^*\| \|\widehat{\Theta}\|$, for all $n \in \mathbb{N}$ and for all $f^* \in \mathcal{X}^*$. Thus, $\sum_{k=1}^{\infty} |f^*(f_k)|^2 < \infty$, for all $f^* \in \mathcal{X}^*$. Hence the system $(\{f_k\}, \{f_k^*\})$ is $\mathcal{X}^*$-Besselian. □

Suppose that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$ and let $\{g_k\} \subset \mathcal{X}$ be close to $\{f_k\}$ in some sense. Then, in general, $(\{g_k^*\}, \{\pi(g_k)\})$ does not serve as reconstruction property for $\mathcal{X}^*$. Note that instead of $\{\pi(g_k)\}$ the choice of arbitrary $\{\psi_k\} \subset \mathcal{X}^{**}$ is also valid in the said argument. In such situations we may recover each element of $\mathcal{X}^*$ via a bounded linear operator (associated with $(\{\pi(g_k)\})$. The following theorem provides sufficient conditions for the existence of a reconstruction operator $\mathcal{S}$ such that $(\{\pi(g_k)\}, \mathcal{S})$ is a Banach frame for $\mathcal{X}^*$.

Theorem 5.8. *Suppose that $(\{f_k\}, \{f_k^*\})$ has reconstruction property for $\mathcal{X}$ which is $\mathcal{X}$-Besselian and let $\{g_k\} \subset \mathcal{X}$ be such that for every $f \in \mathcal{X}$*

$$\Delta \times \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2} < \delta \|f\| \quad (0 < \delta < 1),$$

where

$$\Delta = \sup_{\substack{\phi^* \in \mathcal{X}^* \\ \|\phi^*\| \leq 1}} \sqrt{\sum_{k=1}^{n} |\phi^*(f_k - g_k)|^2}, \; n \in \mathbb{N}.$$

Then, there exists a reconstruction operator $\mathcal{S}$ such that $(\{\pi(g_k)\}, \mathcal{S})$ is a Banach frame for $\mathcal{X}^*$.

Proof. For each n choose $\psi_n^* \in \mathcal{X}^*$ with $\|\psi_n^*\| = 1$ such that

$$\psi_n^*\left(\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right) = \left\|\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right\|.$$

Therefore, for all $f \in \mathcal{X}$, we have

$$\begin{aligned}
&\left\|\sum_{k=1}^{n} f_k^*(f)(f_k - g_k)\right\| \\
&= \sum_{k=1}^{n} f_k^*(f)\psi_n^*(f_k - g_k) \\
&\leq \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2} \times \sqrt{\sum_{k=1}^{n} |\psi_n^*(f_k - g_k)|^2} \\
&\leq \sqrt{\sum_{k=1}^{\infty} |f_k^*(f)|^2 \times \Delta} \\
&\leq \delta \times \|f\|. \qquad (5.2)
\end{aligned}$$

Assume that there is no reconstruction operator $\mathcal{S}$ such that $(\{\pi(g_k)\}, \mathcal{S})$ is a Banach frame for $\mathcal{X}^*$ with respect to an associated sequence space $\mathcal{Z}_d$ (say). Then, by Riesz Lemma [33], there exists $f_0 \in \mathcal{X}$ such that $\|f_0\| = 1$ and $\mathrm{dist}(f_0, [g_k]) > \epsilon$ $(0 \leq \epsilon < 1)$. Therefore, the Hahn-Banach Theorem gives a non-zero functional $\psi^* \equiv \psi_{f_0}^* \in \mathcal{X}^*$ such that $\psi^*(g_k) = 0,\ k \in \mathbb{N}; \psi^*(f_0) = 1$ and $\|\psi^*\| = \frac{1}{\mathrm{dist}(f_0, [g_k])}$.
By using (5.2), we have

$$\begin{aligned}
1 &= |\psi^*(f_0)| \\
&= \lim_{n\to\infty} \left|\psi^*\left(\sum_{k=1}^{n} f_k^*(f_0) f_k\right)\right| \\
&= \lim_{n\to\infty} \left|\psi^*\left(\sum_{k=1}^{n} f_k^*(f_0)(f_k - g_k)\right)\right| \\
&\leq \lim_{n\to\infty} \|\psi^*\| \left\|\sum_{k=1}^{n} f_k^*(f_0)(f_k - g_k)\right\| \\
&\leq \|\psi^*\| \times \delta \times \|f_0\|. \qquad (5.3)
\end{aligned}$$

In particular for $\delta = \epsilon$, (5.3) gives $\mathrm{dist}(f_0, [g_k]) \leq \epsilon$, a contradiction. □

6 Banach Frames Associated with Reconstruction Property

Suppose that $(\{f_k\}, \{f_k^*\})$ has the reconstruction property for $\mathcal{X}$. Then, we can find a reconstruction operator $\mathcal{S}$ such that $(\{f_k^*\}, \mathcal{S})$ is a Banach frame for $\mathcal{X}$, associated with the system $(\{f_k\}, \{f_k^*\})$. Similarly we can find a reconstruction operator associated with the system $\{f_k\}$. It is natural to ask whether we can find Banach frames for a large class of spaces associated with a given reconstruction system. In this direction we introduce Banach Λ-frames for the operator spaces.

Definition 6.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{Y}_d$ be a sequence space associated with $\mathcal{Y}$. A system $\{f_k\} \subset \mathcal{X}$ is called a *Banach* Λ-*frame* for $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ if there exist positive constants

$0 < \mathrm{a}_0 \leq \mathrm{b}_0 < \infty$ such that

$$\mathrm{a}_0\|\Lambda\| \leq \|\{\Lambda(f_k)\}\|_{\mathcal{Y}_d} \leq \mathrm{b}_0\|\Lambda\|, \text{ for all } \Lambda \in \mathcal{B}(\mathcal{X},\mathcal{Y}). \tag{6.1}$$

Remark 6.2. If $\mathcal{Y} = \mathbb{K}$, then $\mathcal{B}(\mathcal{X},\mathcal{Y}) \equiv \mathcal{X}^*$. Therefore, $\{\pi[\Lambda(f_k)]\}$ becomes a Banach frame for $\mathcal{X}^*$ with respect to some associated Banach space $\mathcal{Z}_0$.

Suppose that $(\{f_k\},\{f_k^*\})$ has the reconstruction property for $\mathcal{X}$ where $\{f_k^*\} \subset \mathcal{X}^* \setminus \{0\}$. Let $\mathcal{V}$ be a Banach space and let $\mathcal{V}_d = \{\{\xi_k\} \subset \mathcal{V} : \sup\limits_n \sup\limits_{\substack{f\in\mathcal{X}\\ \|f\|\leq 1}} \|\sum_{k=1}^n f_k^*(f)\xi_k\| < \infty\}$ be its associated Banach space of sequences with norm given by $\|\{\xi_k\}\|_{\mathcal{V}_d} = \sup\limits_n \sup\limits_{\substack{f\in\mathcal{X}\\ \|f\|\leq 1}} \|\sum_{k=1}^n f_k^*(f)\xi_k\|_{\mathcal{V}}$. The following proposition provides the existence of a Banach Λ-frame for the operator space $\mathcal{B}(\mathcal{X},\mathcal{V})$ with respect to $\mathcal{V}_d$ (associated with a reconstruction system).

Proposition 6.3. *Suppose that $\{f_k^*\} \subset \mathcal{X}^* \setminus \{0\}$ has the reconstruction property for $\mathcal{X}$ with respect to $\{f_k\} \subset \mathcal{X}$. Then, $\{f_k\}$ is a Banach Λ-frame for the operator space $\mathcal{B}(\mathcal{X},\mathcal{V})$ with respect to $\mathcal{V}_d$.*

Proof. Let $\Lambda \in B(\mathcal{X},\mathcal{V})$ be arbitrary. For each $n \in \mathbb{N}$, define $\Lambda_n : \mathcal{X} \to \mathcal{V}$ by

$$\Lambda_n(f) = \sum_{k=1}^n f_k^*(f)\Lambda(f_k), f \in \mathcal{X}.$$

Then

$$\begin{aligned}
&\lim_{n\to\infty} \Lambda_n(f)\\
&= \lim_{n\to\infty} \sum_{k=1}^n f_k^*(f)\Lambda(f_k)\\
&= \Lambda\left(\sum_{k=1}^\infty f_k^*(f)f_k\right)\\
&= \Lambda(f).
\end{aligned}$$

Thus, $\sup\limits_{1\leq n<\infty} \|\Lambda_n(f)\| < \infty$, for all $f \in \mathcal{X}$. Therefore, by the theorem of Banach-Steinhaus, we have $\sup\limits_{1\leq n<\infty} \|\Lambda_n\| < \infty$.
Fix $\Lambda \in \mathcal{B}(\mathcal{X},\mathcal{V})$. Then,

$$\begin{aligned}
\|\Lambda\| &= \sup_{\substack{f\in\mathcal{X}\\ \|f\|\leq 1}} \|\Lambda(f)\|\\
&= \sup_{\substack{f\in\mathcal{X}\\ \|f\|\leq 1}} \left\|\Lambda\left(\sum_{k=1}^\infty f_k^*(f)f_k\right)\right\|\\
&= \sup_{\substack{f\in\mathcal{X}\\ \|f\|\leq 1}} \left\|\lim_{n\to\infty}\sum_{k=1}^n f_k^*(f)\Lambda(f_k)\right\|\\
&\leq \sup_{1\leq n<\infty} \|\Lambda_n\|\\
&= \|\{\Lambda(f_n)\}\|_{\mathcal{V}_d}.
\end{aligned} \tag{6.2}$$

Also for all $f \in \mathcal{X}$, we have

$$\begin{aligned}
&\left\| \sum_{k=1}^{n} f_k^*(f)\Lambda(f_k) \right\| \\
&\leq \|\Lambda\| \left\| \sum_{k=1}^{n} f_k^*(f) f_k \right\| \\
&= \|\Lambda\| \|P_n(f)\| \\
&\leq B\|\Lambda\|\|f\|, \text{for all } \Lambda \in B(\mathcal{X}, \mathcal{V}), \qquad (6.3)\\
&\text{where} \quad B = \sup_{1 \leq n < \infty} \|P_n\|.
\end{aligned}$$

Therefore, by using (6.3) we obtain $\sup\limits_{\substack{f \in \mathcal{X} \\ \|f\| \leq 1}} || \sum_{k=1}^{n} f_k^*(f)\Lambda(f_k)| | \leq B\|\Lambda\|$.

This gives

$$\begin{aligned}
&\|\{\Lambda(f_k)\}\|_{\mathcal{V}_d} \\
&= \sup_n \sup_{\substack{f \in \mathcal{X} \\ \|f\| \leq 1}} \left\| \sum_{k=1}^{n} f_k^*(f)\Lambda(f_k) \right\|_{\mathcal{V}} \\
&\leq B\|\Lambda\|, \text{for all } \Lambda \in B(\mathcal{X}, \mathcal{V}). \qquad (6.4)
\end{aligned}$$

By using (6.2) and (6.4) with $A = 1$, we have

$$A\|\Lambda\| \leq \|\{\Lambda(f_k\}\|_{\mathcal{V}_d} \leq B\|\Lambda\|, \text{ for all } \Lambda \in B(\mathcal{X}, \mathcal{V}).$$

Hence $\{\Lambda(f_k)\}$ is a Banach Λ-frame for the operator space $\mathcal{B}(\mathcal{X}, \mathcal{V})$ with respect to $\mathcal{V}_d$. This completes the proof. □

Remark 6.4. The Banach Λ-frame $\{f_k\}$ in Proposition 6.3 is associated with $(\{f_k\}, \{f_k^*\})$. We call $\{f_k\}$ an *associated Banach* Λ*-frame* for $\mathcal{X}^*$.

References

[1] P.G.Casazza, The art of frame theory, *Taiwanese J. Math.*, 4 (2) (2000), 129–201.

[2] P.G.Casazza and G. Kutynoik, Finite Frames, Birkhäuser, 2012.

[3] P.G. Casazza, Approximation Properties, in Handbook on the Geometry of Banach spaces Vol I, W.B. Johnson and J. Lindenstrauss editors (2001), 271–316.

[4] P.G. Casazza, D. Han and D.R. Larson, Frames for Banach spaces, *Contemp. Math.*, 247 (1999), 149–182.

[5] P.G. Casazza, O. Christensen, The reconstruction property in Banach spaces and a perturbation theorem, *Canad. Math. Bull.*, 51 (2008), 348–358.

[6] P.G. Casazza, O. Christensen, Perturbation of operators and applications to frame thoery, *J. Fourier Anal. Appl.*, 3 (5) (1997), 543–557.

[7] O. Christensen and C. Heil, Pertubation of Banach frames and atomic decompositions, *Math. Nachr.*, 185 (1997), 33–47.

[8] O. Christensen, Frames and bases (An introductory course), Birkhäuser, Boston, 2008.

[9] R.R. Coifman and G. Weiss, Expansions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.*, 83 (1977), 569–645.

[10] I. Daubechies, A. Grossmann and Y. Meyer, Painless non-orthogonal expansions, *J. Math. Phys.*, 27 (1986), 1271–1283.

[11] R.J. Duffin and A.C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.*, 72 (1952), 341–366.

[12] H. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representationsand their atomic decompositions I, *J. Funct Anal.*, 86 (2) (1989), 307–340.

[13] H. Feichtinger and K. Gröchenig, Banach spaces related to integrable group representationsand their atomic decompositions II, *Monatsh. Math.*, 108 (2-3) (1989), 129–148.

[14] D. Gabor, Theory of communicatons, *J. Inst. Elec. Engg.*, 93 (1946), 429–457.

[15] K. Gröchenig, Describing functions: Atomic decompositions versus frames, *Monatsh. Math.*, 112 (1991), 1–41.

[16] D. Han and D.R. Larson, Frames, bases and group representations, *Mem. Amer. Math. Soc.*, 147 (697) (2000), 1–91.

[17] C.Heil and D. Walnut, Continuous and discrete wavelet transforms, *SIAM Rev.,* 31 (4) (1989), 628–666.

[18] J.R. Holub, Pre-Frame operators, Besselian Frames,and Near- Riesz bases in Hilbert spaces, *Proc. Amer. Math. Soc.*, 122 (1994), 779–785.

[19] W.B. Johnson and J. Lindenstrauss (Eds), Handbook of the Geometry of Banach Spaces, Vol. I, (2001).

[20] W.B. Johnson and J. Lindenstrauss (Eds), Handbook of the Geometry of Banach Spaces, Vol. II (2003).

[21] W.B. Johnson, H.P. Rosenthal, and M. Zippin, On bases, finite dimensional decompositions and weaker structures in Banach spaces, *Israel J. Math.*, 9 (1971), 77–92.

[22] S. K. Kaushik, S. K. Sharma, and Khole Timothy Poumai, On Schauder Frames in Conjugate Banach Spaces, *J. Math.*, (2013), Article ID 318659.

[23] N.J. Kalton and C.V. Wood, Orthonormal systems in Banach spaces and their applications, *Math. Proc. Camb. Phil. Soc.*, 79 (1976), 493–510.

[24] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Springer-Verlag Heidelberg - New York (1977).

[25] R. Liu and B. Zheng, A characterization of Schauder frames which are near-Schauder bases, *J. Fourier Anal. Appl.*, 16 (2010), 791–803.

[26] R. Liu, On shrinking and boundedly complete Schauder frames of Banach spaces, *J. Math. Anal. Appl.*, 365 (2010), 385–398.

[27] Béla de Sz. Nagy, Expansion theorems of Paley-Wiener type, *Duke Math. J.*, 14 (4) (1947), 975–978.

[28] A. Pelczynski, Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis, *Studia Math.*, 40 (1971), 239–242.

[29] A. Pelczynski and P. Wojtaszczyk, Banach spaces with finite dimensional expansions of identity and universal bases of finite dimensional subspaces, *Studia Mathematica XL*, (1971), 91–108.

[30] I. Singer, Bases in Banach Spaces. II, Springer, New York (1981).

[31] L.K. Vashisht, On Φ-Schauder frames, *TWMS J. App. and Eng. Math.(JAEM)*, 2 (1) (2012), 116–120.

[32] L.K. Vashisht and G. Khattar, On $\mathfrak{I}$-reconstruction property, *Adv. Pure Math.*, 3 (3) (2013), 324–330.

[33] K. Yosida, Functional Analysis, Springer; Reprint of the 6^{th} ed. Berlin, Heidelberg, New York 1980 edition (April 30, 1996)

[34] R. Young, A introduction to non-harmonic Fourier series, Academic Press, New York (revised first edition 2001).

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