Palestine Journal of Mathematics

Vol. 4(1) (2015) , 12–20 © Palestine Polytechnic University-PPU 2015

Spectral Method for Mixed Initial-Boundary Value Problem

Amor Boutaghou

Communicated by Fathi Allan

MSC 2010 Classifications: Primary 65D05, 65D32; Secondary 65D32, 65N35.

Keywords and phrases: Spectral, orthogonal polynomials, quadrature formula, Eigenvalues.

Abstract The aim of this paper is to investigate the Legendre spectral method of one-dimensional inhomogeneous mixed initial-boundary value problem in a finite regular set ?, we use some techniques to convert the problem to a system of ordinary differential equations and by an analysis matricial we find a general term defines all ordinary defferential equations of this system, we solve this general term we get the desired approximate solution, we also present the error estimate.

1 Introduction

The differential equations play a very important role in all fields of science like mathematics and Mathematical Physics and other Sciences, and a long time ago the scientists and researchers face difficulties in resolving many of these equations, for this we turned in recent years, especially after the emergence of the computer to search for approximate solution instead of the exact solution for these problems, these methods gave his fruit.

The main motivation in this paper is the numerical analysis of discretization of the inhomogeneous mixed initial-boundary value problem using spectral element method, this method is associated with quadrature formulas which allow for a complete discretization of the right-hand side and of the linear form involved in the variational formulation, see also [3, 5, 7, 6, 10].

the concerned problem refers to the equation:

$$\begin{cases} \partial_t u - \partial_x^2 u = f \quad , x \in \Lambda, \ \ t > 0 \\ u(-1,t) = u(1,t) = 0 \ , \ \ t > 0 \\ u(x,0) = g(x), \qquad\quad x \in \Lambda \end{cases} \tag{1.1}$$

where $\Lambda = (-1, 1)$, which $u(x,t)$ represents the temperature at point x and time t,the discretization consists therefore the space variable and the time variable,

Then the problem (1.1) is a problem of one space variable, by using the orthogonal matrix we reduce this problem to a system of ordinary differential equations.

In this work we construct approximate solution to the boundary value problem (1.1) in the following form

$$u_N(t,x) = \sum_{n=0}^{N} a_n(t) l_n(x) \tag{1.2}$$

Where $l_n(x)$, $0 \le n \le N$, are the Lagrangian interpolates at the points $x_i \in \overline{\Lambda} = [-1,1]$, $0 \le i \le N$, these interpolates satisfy the property $l_n(x_j) = \delta_{nj}$, $1 \le n, j \le N-1$, the points $x_j, 0 \le j \le N$ are the collocation points on the Gauss-Lobatto Legendre grid. The grid made by $x_j, 0 \le j \le N$, is denoted by Λ_{N+1}. The choice of the form (1.2) for the solution, added to some techniques give a linear system which can be written in a matricial form as $\Gamma Da - Aa = \Gamma G$, where A is a square matrix and Γ is a diagonal invertible matrix and the operator $D = \frac{d}{dt}$. We write $a = Pv$ where P is an orthogonal matrix such that $P^{-1}\left(\Gamma^{-1}A\right)P = C$ is a diagonal matrix, then we obtain a system of $N-1$ ordinary differential equations, we can use the Lagrange method to solve for each component $v_j(x)$ of v, finally we conclude the expression of functions $a_n(t)$ and for which we obtain the desired approximate solution, see also [1, 2, 8, 12, 11].

2 Orthogonal polynomials

We work in the model domain Λ and we use the Legendre polynomials L_n, $n \ge 0$: each polynomial L_n has a degree n, it is orthogonal to the other ones in

$$L^2(\Lambda) = \left\{ f : \Lambda \to R \text{ measurable } / \int_\Lambda (f(x))^2 \, dx < \infty \right\}$$

and satisfies the following property

$$\int_{-1}^{1} L_n(x)L_m(x)dx = \frac{2}{2n+1}\delta_{nm} \tag{2.1}$$

$$h_n'(x) = -n(n+1)L_n(x), h_n(x) = \left(1-x^2\right)L_n'(x), n \geq 0 \tag{2.2}$$

$$h_n(x) = \frac{n(n+1)}{2n+1}(L_{n-1}(x) - L_{n+1}(x)) \tag{2.3}$$

2.1 The continuous problem

To introduce the variational formulation for the continuous problem (1.1), we need the subspace of the variational space with zero Dirichlet trace:

$$A_0(\Lambda) = \left\{v \in H^1(\Lambda),\ v = 0 \text{ on } \partial\Lambda,\ t > 0,\ v = g,\ t = 0\right\} \tag{2.4}$$

We introduce the product in $L^2(\Lambda)$:

$$(f, v) = \int_\Lambda f(t,x)v(t,x)dx \tag{2.5}$$

Then the variational formulation of continuous problem (1.1) is: find $u \in A_0(\Lambda)$, such that,

$$\forall v \in A_0(\Lambda),\quad a(u,v) = \langle f, v\rangle \tag{2.6}$$

where

$$a(u,v) = \int_\Lambda \left(\partial_t u - \partial_x^2 u\right) v dx$$

integrating by parts leads to,

$$a(u,v) = \int_\Lambda \left(\partial_t uv + \partial u_x \partial v_x\right) dx \tag{2.7}$$

3 Discrete space and form

Let us denoted by N the parameter of discretization for the problem (1.1), in spectral method N represents the degree of polynomials. The approximate space is essentially generated by the finite dimensional subspace of $L^2(\Lambda)$, $P_N^\Delta(\Lambda)$ is the approximate space of the space $A_0(\Lambda)$,where

$$P_N^\Delta(\Lambda) = \left\{ q_N \in P_N(\Lambda) \,/\, q_N(x,t) = 0 \text{ on } \partial\Lambda,\ q_N(x,0) = \sum_{j=1}^{N-1} g(x_j)l_j(x) \right\}$$

Where $P_N(\Lambda)$ is the set of polynomials of degree less than or equal to N. We consider also the exact quadrature formula and introduce a bilinear form a_N with approach to the form a and we approximate the scalar $(.,.)$ for $(.,.)_N$.

3.1 The Discrete problem

Firstly we observe that the Lagrange polynomials $l_m(x), 0 \leq m \leq N$, form a basis of $P_N^\Delta(\Lambda)$, then the exact solution u of problem (1.1) is approached by the solution u_N^I belonging to $P_N^\Delta(\Lambda)$ with $(u_N^I - g_N) \in P_N^\nabla(\Lambda)$, where

$$P_N^\nabla(\Lambda) = \{q_N \in P_N(\Lambda) \,/\, q_N(x,t) = 0 \text{ on } \partial\Lambda\}$$

and the variational problem is:

$$\left\{ \begin{array}{l} \text{find } u_N^I \in P_N^\Delta(\Lambda),\ \text{s.t} \\ \forall v_N \in P_N^\nabla(\Lambda),\ a_N(u_N^I, v_N) = (f_N, v_N)_N \end{array} \right. \tag{3.1}$$

where

$$a_N(u_N^I, v_N) = \sum_{l=0}^{N} \left(\partial_t u_N^I v_N + \partial_x u_N \partial_x v_N\right)(x_l, t)\,\rho_l$$

where x_l, ρ_l, $0 \leq l \leq N$ are defined in propositions 1, $u_N^I = u_N + g_N$, $u_N \in P_N^{\nabla}(\Lambda)$ and the problem (3.1) is equivalent to the following problem: Find u_N^I in $P_N^{\Delta}(\Lambda)$ with $u_N = u_N^I - g_N$ in $P_N^{\nabla}(\Lambda)$ such that, $\forall v_N \in P_N^{\nabla}(\Lambda)$

$$a_N(u_N,\ v_N) = b_N(g_N,\ v_N) \tag{3.2}$$

Where

$$b_N(g_N,\ v_N) = (f_N,\ v_N) - a_N(g_N, v_N) \tag{3.3}$$

3.2 Existence and uniqueness of solution

Quadrature formula

Proposition 3.1. *There exists a unique set of* $N-1$ *nodes* x_j, $1 \leq j \leq N-1$ *in* Λ *and with the condition* $x_0 = -1$, $x_N = 1$, *there exists* $N+1$ *positive weights* ρ_j, $0 \leq j \leq N$, *such that the following exactness property holds:*

$$\forall \varphi \in P_{2N-1}(\Lambda),\quad \int_{-1}^{1} \varphi(x)dx = \sum_{j=0}^{N} \varphi(x_j)\,\rho_j \tag{3.4}$$

where x_j , $1 \leq j \leq N-1$ *are the roots of the polynomial* L'_N *and the weights are given by:*

$$\begin{cases} \rho_0 = \rho_N = \frac{1}{N(N+1)} \\ \rho_j = \frac{\rho_0}{P_N^2(z_j)} \qquad 1 \leq j \leq N-1 \end{cases} \tag{3.5}$$

Proof. See [4, 5] □

Definition 3.2. We define the discrete product for all polynomials v_N, u_N in $P_N^{\Delta}(\Lambda)$ as:

$$(v_N, u_N)_N = \sum_{l=0}^{N} u_N(x_l, t)\, v_N(x_l, t)\, \rho_l$$

Lemma 3.3. *The polynomial* $h_{N-1} \in P_N^0(\Lambda)$ *verifies the double inequality:*

$$\|h_{N-1}\|_{L^2(\Lambda)}^2 \leq (h_{N-1},\ h_{N-1})_N \leq \frac{3}{2}\|h_{N-1}\|_{L^2(\Lambda)}^2 \tag{3.6}$$

such that the subspace $P_N^0(\Lambda) = \{p_n \in P_N(\Lambda) \,/\, p_N(-1) = p_N(1) = 0\}$, *where* $P_N(\Lambda)$ *are the space of polynomials with degrees* $\leq N$ *on* Λ

Proof. See[1]

Proposition 3.4. *For all polynomial* $h_n \in P_n^0(\Lambda)$ *we have*

$$n\,\|h_n\|_{L^2(\Lambda)} \leq \|h'_n\|_{L^2(\Lambda)} \tag{3.7}$$

□

Proof. See[1]. □

Also the lagrange's polynomials $l_j(x)$, $j = \overline{1, N-1}$ can be written in the following form

$$l_j(x) = \sum_{k=0}^{N-1} \gamma_{kj} h_k(x)$$

using (2.2), then we get,

$$l_j(x) = \sum_{k=0}^{N} \lambda_{kj} L_k(x) \tag{3.8}$$

Proposition 3.5. *The set of polynomials* $\{L_n(s)\}$, $n = 0..N$ *forms a basis to the polynomial space* $P_N(\Lambda)$, *then any polynomial* $\varphi_N \in P_N(\Lambda)$ *can be written as:* $\varphi_N(s) = \sum_{n=0}^{N} b_n L_n(s)$ *and we have the following inequality:*

$$c_3 Log(N+1) \leq \|\varphi_N\|_{L^2(\Lambda)}^2 \leq c_4 Log(2\exp(2)(N+1)) \tag{3.9}$$

where $(c_3, c_4) = (\min(b_n^2), \max(b_n^2))$.

Proof. See[1]. □

Definition 3.6. For a positive integer m the Sobolev space $H^m(\Lambda)$ is defined by:

$$H^m(\Lambda) = \left\{ \varphi \in L^2(\Lambda) : 1 \le k \le m,\ \frac{d^k}{dz^k}\varphi \in L^2(\Lambda) \right\} \tag{3.10}$$

with the norm:

$$\|\varphi\|^2_{H^m(\Lambda)} = \int_J \sum_{k=0}^{m} (\frac{d^k}{dz^k}\varphi)^2(x)dx \tag{3.11}$$

Proposition 3.7. *The bilinear form $a_N(\cdot,\cdot)$ in (3.2) satisfies the following properties of continuity:*

$$\forall v_N \in P_N^\nabla(\Lambda), \forall u_N \in P_N^\nabla(\Lambda), |a_N(u_N, v_N)| \le \frac{3\max(C,1)}{2} \|u_N\|_{H^1(\Lambda)} \cdot \|v_N\|_{H^1(\Lambda)} \tag{3.12}$$

and ellipticity:

$$\forall u_N \in P_N^\nabla(\Lambda), \quad |a_N(u_N, u_N)| \ge \min(C,1)\|u_N\|^2_{H_0^1(\Lambda)} \tag{3.13}$$

Proof. continuity

$$a_N(u_N, v_N) = \sum_{l=0}^{N} \partial_t u_N(x_l,t)v_N(x_l,t)\rho_l + \sum_{l=0}^{N} \partial_x u_N(x_l,t)\partial_x v_N(x_l,t)\rho_l$$

We consider the solution and its derivatives are bounded then there exists two real positive constants C and C_1 such that

$$C|u_N(x_l,t)| \le |\partial_t u_N(x_l,t)| \le C_1|u_N(x_l,t)| \tag{3.14}$$

We use lemmas 3.3, the exact quadrature formula and the Schwartz inequality then we obtain the desired results also and ellipticity:

$$a_N(u_N, u_N) = \sum_{l=0}^{N} \partial_t u_N(x_l,t)u_N(x_l,t)\rho_l + \sum_{l=0}^{N} \partial_x u_N(x_l,t)\partial_x u_N(x_l,t)\rho_l$$

using the exactingness quadrature formula we can write,

$$a_N(u_N, u_N) = \sum_{l=0}^{N} \partial_t u_N(x_l,t)u_N(x_l,t)\rho_l + \int_{-1}^{1} \partial_x u_N(x,t)\partial_x u_N(x,t)dx \tag{3.15}$$

then from (3.14) we can write:

$$|a_N(u_N, u_N)| \ge C\sum_{l=0}^{N} u_N(x_l,t)u_N(x_l,t)\rho_l + \int_{-1}^{1} \partial_x u_N(x,t)\partial_x u_N(x,t)dx$$

Using (3.6) we can write .

$$a_N(u_N, u_N) \ge \min(C,1)\|u_N\|_{H_0^1(\Lambda)}$$

then for this inequality yields the desired result . □

Proposition 3.8. *(the inequality of stability) For any continuous function g on Λ ,the problem (3.2) has a unique solution u_N in $P_N^\nabla(\Lambda)$, and this solution verifies the inequality of stability:*

$$\|u_N(x,t)\|_{H_0^1(\Lambda)} \le \gamma\left(\|f_N(x,t)\|_{L^2(\Lambda)} + \|g_N(x)\|_{L^2(\Lambda)}\right) \tag{3.16}$$

Proof. Using (3.2) we can write

$$a_N(u_N,\ u_N) = (f_N,\ u_N) - a_N(g_N, u_N) \le |(f_N,\ u_N)| + |a_N(g_N, u_N)| \tag{3.17}$$

using Schwartz inequality we can write

$$\begin{aligned} |(g_N,\ u_N)| + |a_N\,(g_N, u_N)| \ \leq\ & \frac{3}{2}\,\|f_N(x,t)\|_{L^2(\Lambda)}\,\|u_N(x,t)dt\|_{L^2(\Lambda)} \\ & +\alpha_2\,\|\partial_t g_N(x)\|_{L^2(\Lambda)}\,\|u_N(x,t)\|_{L^2(\Lambda)} \\ & +\alpha_3\,\|\partial_x g_N(x)\|_{L^2(\Lambda)}\,\|\partial_x u_N(x,t)\|_{L^2(\Lambda)} \end{aligned}$$

the quantities $\|\partial_x g_N(x)\|_{L^2(\Lambda)}$, $\|\partial_x v_N(x,t)\|_{L^2(\Lambda)}$ are bounded then there exists a positive number γ such that,

$$a_N(u_N,\ u_N) \leq |(f_N,\ v_N)|+|a_N\,(g_N, v_N)| \leq \gamma\left(\|f_N(x,t)\|_{L^2(\Lambda)} + \|g_N(x)\|_{L^2(\Lambda)}\right)\|u_N(x,t)\|_{H_0^1(\Lambda)}$$

using (3.13) , yields the desired result. □

4 Numerical experiment

At the points x_k, $1 \leq k \leq N-1$ the problem (1.1) is equivalent to,

$$\begin{cases} \sum\limits_{n=1}^{N-1} l_n(x_k)a_n'(t) - l_n''(x_k)a_n(t)) = \sum\limits_{n=1}^{N-1} f_n(t)l_n(x_k) + g''\,(x_k) \ \text{ in } \ \Lambda \cap \Lambda_{N+1} \\ u_N\,(r_k,-1) = u_N\,(r_k,1) \end{cases} \tag{4.1}$$

Since the functions

$$-l_n''(x),\ 1 \leq n \leq N-1$$

are polynomials with degree $N-2$, we multiply both sides by $l_m(x_k)\rho_k$ and applying the sum, by using the quadrature formula, when m varies from 1 to $N-1$, we obtain a linear system, then we can write this system in a matricial form:

$$\Gamma Da - Aa = \Gamma G \tag{4.2}$$

where A is a square symmetric define positive matrix with order $N-1$, its elements have the form:

$$\alpha_{mn} = -l_n''(x_m)\rho_m,\ n = \overline{1, N-1}\,\},m = \overline{1, N-1}$$

Γ is a diagonal invertible matrix its elements are defined as:

$$\gamma_{mn} = \begin{cases} \rho_m & ,n = m \\ 0, & n \neq m \end{cases} \quad ,\ m,n = \overline{1, N-1}$$

G is a known vector where:

$$G = (f_1(t)+g''(x_1),\ f_2(t)+g''(x_2), f_3(t)+g''(x_3), ..., f_{N-2}(t)+g''(x_{N-2}), f_{N-1}(t)+g''(x_{N-1}))^t$$

and the vector a is an unknown vector where

$$a = (a_1(t), a_2(t), a_3(t), ..., a_{N-1}(t), a_{N-1}(t))^t$$

the operator,

$$D = \frac{d}{dt}$$

multiplying (4.2) by the invertible matrix Γ^{-1} of Γ then we find

$$Da - \Gamma^{-1}Aa = G \tag{4.3}$$

the matrix $\Gamma^{-1}A$ has positive eigenvalues and there exists an orthogonal invertible matrix P such that,

$$P^{-1}\left(\Gamma^{-1}A\right)P = C$$

where C is a diagonal matrix, the elements of the diagonal are the eigenvalues $\alpha_i, i = \overline{1, N-1}$ of the matrix $\Gamma^{-1}A$, if we consider the vector v such that

$$a = Pv$$

then the system (4.3) becomes

$$PDv - (\Gamma^{-1}A)Pv = G \tag{4.4}$$

multiplying (4.4) by the matrix P^{-1} we obtain,

$$Dv - Cv = P^{-1}G \tag{4.5}$$

The matricial form (4.5) has $N-1$ linear ordinary differential equations defined as

$$\begin{aligned} v_k'(t) - \alpha_k v_k(t) &= h_k & (4.6)\\ \text{where} \quad h_k(t) &= \sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(t) + g_j''(x_k)\right), \ 1 \le k \le N-1 & (4.7) \end{aligned}$$

$p^{-1}(i,j)$ are the elements of the inverse matrix P^{-1}. To solve the equations (4.6) we use Lagrange's method $[12]$, we may write the solution in the closed form :

$$v_k(t) = \int_0^t e^{-\alpha_k(s-t)} h_k(s)ds + c_k e^{-\alpha_k t}$$

where c_k is constant to be determined, using the boundary conditions then (4.8) may be written in the following form:

$$v_k(t) = \int_0^t e^{-\alpha_k(s-t)} h_k(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g(x_k)\right) e^{-\alpha_k t} \tag{4.8}$$

Finally we obtain the functions,

$$a_n(t) = \sum_{j=1}^{N-1} p_{nj} v_j(t) \tag{4.9}$$

where p_{nj} , $1 \le n, j \le N$ are the elements of the matrix P, and the approximation solution is:

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} p_{nj}\left(\int_0^t e^{-\alpha_k(s-t)} h_k(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g(x_k)\right) e^{-\alpha_k t}\right) l_n(x)$$

If the time t defined in the interval $I = [0, T]$, we can consider the solution in the form

$$u(x,t) = \sum_{n=1}^{N-1}\sum_{j=1}^{N-1} u_{nj} l_n(x) l_j(t), \quad a_n(t) = \sum_{j=1}^{N-1} u_{nj} l_j(t) \tag{4.10}$$

using (4.9) and (4.10) then we determine the coefficients

$$u_{nj} = \sum_{j=1}^{N} p_{nj}\left(\int_0^{t_j} e^{-\alpha_k(s-t_j)} h_k(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g(x_k)\right) e^{-\alpha_k t_j}\right)$$

and the approximate solution is

$$\begin{aligned} u_N(t,x) &= \sum_{n=1}^{N-1}\sum_{m=1}^{N}\left(\sum_{j=1}^{N} p_{nj}\left(\int_0^{t_j} e^{-\alpha_k(s-t_j)} h_k(s)ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g(x_k)\right) e^{-\alpha_k t_j}\right)\right) l_m(x) l_n(t) + t(1,x)\\ t(1,x) &= \sum_{n=1}^{N-1} g(x_n) l_n(x) \end{aligned}$$

and using (4.7) we get

$$\begin{aligned} u_N(t,x) = &\sum_{n=1}^{N-1}\sum_{m=1}^{N}\left(\sum_{j=1}^{N} p_{nj}\left(\int_0^{t_j} e^{-\alpha_k(s-t_j)}\left(\sum_{j=1}^{N-1} p^{-1}(k,j)\left(f_j(s) + g_j''(x_k)\right)\right) ds\right.\right.\\ &\left.\left. + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g(x_k)\right) e^{-\alpha_k t_j}\right)\right) \times l_m(x) l_n(t) + t(1,x) \end{aligned}$$

Numerical integration

The function

$$q_k(s) = e^{-\alpha_k(s-t)} h_k(s) \tag{4.11}$$

is explicit but we can not always calculate its primitive explicitly, in this case we use the polynomial interpolation and seek numerical approximation of the integral. Then the Lagrange polynomial interpolation is

$$q_{Nj}(s) = \sum_{n=0}^{N} q_j(t_n) l_j(s)$$

where t_n defined by

$$t_n = \frac{T}{2} x_n - 1,\, N = \overline{0, N}\ ,\, x_n,\ 0 \le n \le N$$

are the collocation points on the Gauss-Lobatto Legendre grid, then the approximation of the integral (4.8)

$$v_{Nj}(t) = \int_0^t q_{Nj}(s) ds + \left(\sum_{j=1}^{N-1} p_{kj}^{-1} g\left(x_k\right) \right) e^{-\alpha_k t}$$

then we obtain

$$a(t) = \sum_{j=1}^{N-1} p_{nj}(t_n) v_{Nj}(t)$$

where p_{nj}, $1 \le n, j \le N$ are the entries of the matrix P , using (1.2) we get the approximate solution

$$u(x,t) = \sum_{n=1}^{N-1} \sum_{j=1}^{N-1} p_{nj} v_{Nj}(t) l_n(x)$$

4.1 Error estimation

Definition 4.1. The polynomial space $P_N^{\Delta}\left(\Lambda\right)$ is dense in the space of continuous functions on Λ hence in $A_0\left(\Lambda\right)$ then any function $u \in A_0\left(\Lambda\right)$ admits the expansion

$$u\left(z,t\right) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \alpha\left(n,m\right) h_n(x) l_m(t) + \sum_{n=0}^{\infty} \gamma\left(n\right) h_n(x) \tag{4.12}$$

We know

$$T_n(t) = \frac{n(n+1)}{2n+1} (p_{n-1}(t) - p_{n+1}(t)) \tag{4.13}$$

where

$$p_n(t) = \{L_n((2t-T)/T)\ ,\ n \ge 0 \tag{4.14}$$

and using (4.13) then

$$u\left(z,t\right) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \gamma\left(n,m\right) p_m(t) h_n(z) \tag{4.15}$$

Proposition 4.2. *The following estimate holds between the exact solution u in $H_0^1\left(\Lambda\right)$ and the approximate solution $u_N \in P_N^{\nabla}\left(\Lambda\right)$ verify,*

$$\|u - u_N\|_{L^2(\Lambda)} \le 3cN^{-1} \left(\|(g - g_N)\|_{L^2(\Lambda)} + \|(f - f_N)\|_{L^2(\Lambda)} \right) \tag{4.16}$$

Proof. Using the ellipticity condition (3.7) and (3.13) we can write,

$$\begin{aligned} N^2\ \|u - u_N\|_{L^2(\Lambda)}^2 &\le a(u - u_N, u - u_N) = (f - f_N, u - u_N) - a(g - g_N, u - u_N) \\ &\le c\left(\left| \int_\Lambda (\ (f - f_N)\left(u - u_N\right)) dz \right| + \ \left| a(g - g_N, u - u_N) \right| \ \right) \end{aligned} \tag{4.17}$$

$$\left| \int_\Lambda (\ (f - f_N)\left(u - u_N\right)) dz \right| \le \|(f - f_N)\|_{L^2(\Lambda)} \|(u - u_N)\|_{L^2(\Lambda)} \tag{4.18}$$

$$\left| a(g - g_N, u - u_N) \right| \le \left| \int_\Lambda (\ \partial_x\left(g - g_N\right) \partial_x\left(u - u_N\right)) dz \right| + \left| \int_\Lambda (\ \partial_t\left(g - g_N\right)\left(u - u_N\right)) dz \right|$$

the function g is independent of the variable t then

$$\int_{\Lambda}(\ \partial_t\,(g-g_N)\,(u-u_N))dtdz=0$$

and

$$\left|\int_{\Lambda}(\ (g-g_N)\,\partial_x\,(u-u_N))dtdz\right|\leq\|\partial_x\,(g-g_N)\|_{L^2(\Lambda)}\,\|\partial_x\,(u-u_N)\|_{L^2(\Lambda)} \tag{4.19}$$

also we can prove that

$$n\,\|h_n\|_{L^2(\Lambda)}\leq\|h_n'\|_{L^2(\Lambda)}\leq 3n\,\|h_n\|_{L^2(\Lambda)} \tag{4.20}$$

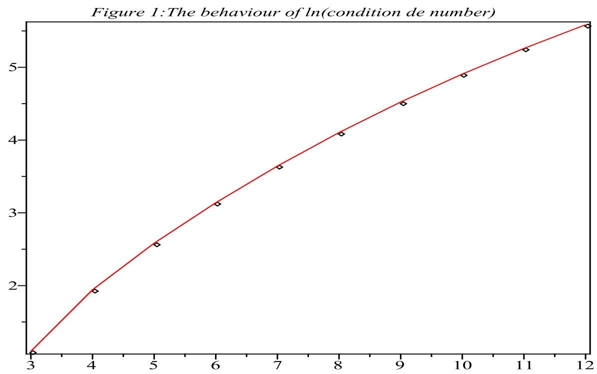
using (4.18), (4.19)and (4.20) we get

$$N^2\,\|u-u_N\|^2_{L^2(\Lambda)}\leq 3cN\left(\|(g-g_N)\|_{L^2(\Lambda)}+\|(f-f_N)\|_{L^2(\Lambda)}\right)\|(u-u_N)\|_{L^2(\Lambda)} \tag{4.21}$$

finally we find the desired results □

4.2 Figure illustration

The figure 1 and 2 present the behavior of the log condition number and the error , N varies from 3 to 12, and the figures 3 and 4 present the true and the approximate solution u_N and u respectively, the true solution is: $u(x,t)=\sin(\pi x)e^{-\frac{\pi^2}{50}t}$



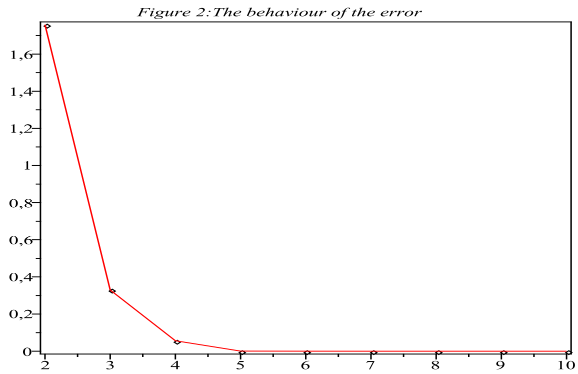


Figure 3:The graphe of the true solution

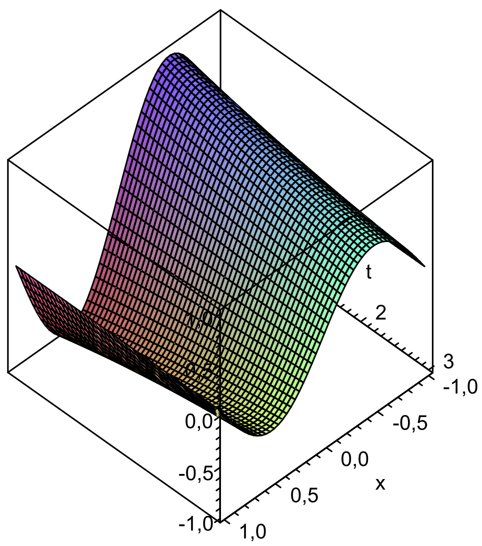
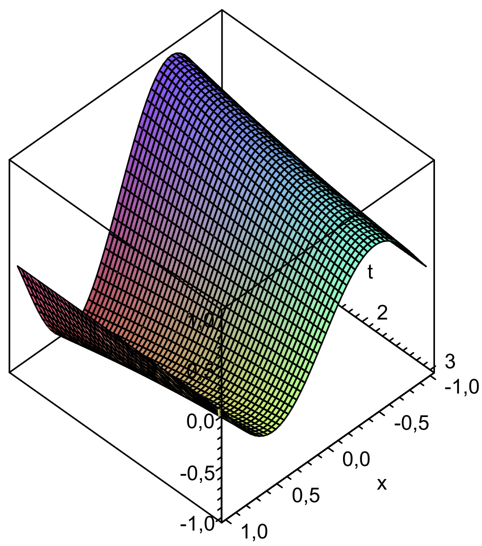


Figure 4:The graphe of the approximate solution



In table 1 we compare the numerical solution with the known analytical solution. In this table we present the errors in the numerical solution at a representative selection of degrees of approximate solution

Table 1
Error in the computed solution at various degrees of approximate solution.

N	3	4	5	6	7	8	9	10
$Error$	.330	$.563e-1$	$.621e-3$	$.1666e-3$	$.476e-6$	$.211e-6$	$.8e-8$	$.7e-8$

5 Conclusion

We know that many ordinary or partial differential equations do not admit exact solution, so we seek the approximate solution, in this article I have described a numerical method converges quickly to the solution of the problem, this method based on the properties of orthogonal polynomials and matrix analysis.

References

[1] A.Boutaghou and F.Z.Nouri,On Finite Spectral Method for Axi-symmetric Elliptic problem, Journal of Analysis and Applications,vol. 4 (2006), No 3, pp.149-168, SAS International Publications.

[2] A.Boutaghou and F.Z.Nouri, Stokes problem in the case of axi-symmetric data and homogeneous boundary conditions, Far East Journal of Applied Mathematics, ISSN 0972-0960, 21(2)(2005), p219-239.

[3] B. Mercier . stabilité et convergence des méthodes spectrales polynomiales : Application à l'équation d'avection. R.A.I.R.O. Anal. numér. 16 (1982) 97-100.

[4] C. Bernardy and Y-Maday, Approximation spectrale de problemes aux limites elliptiques, 1992, Springer Verlag France Paris.

[5] C. Bernardy, M. Dauge and Y. Maday, Numerical algorithms and tests due to Mejdi Azaiez, 1999 Spectral Methods for Axisymmetric Domains, Editions scientifiques et médicales Elsevier France.

[6] D. Gottlieb. The stability of pseudospectral Chebyshev methods. Math Comput. 36 (1981), 107-118.

[7] E. P. Stephan. M. Suri. On the convergences of the p-version of the boundary element Galerkin method. Math. Compt. 52(1989), 31-48.

[8] J. de Frutos, R. Munoz-Sola, Chebyshev pseudospectral collocation for parabolic problems with non constant coefficients. Appear in Houston J. Math (Proccedings ICOSAHOM 95).

[9] M. Dauge, Spectral-Fourier Method for axi-symmetric Problems, ICOSAHOM 95: Proceedings of the third International Conference on Spectral and High Order Methods, 1996, Houston Journal of mathematics, University of Houston.

[10] S. A. Orzag - Spectral methods for problems in complex geometries , J. Comput. Phys. 37 (1980), 70-92

[11] Stig Larsson and Vidar thomée, Partial differential equations with numerical methods, Springer.

[12] T.N.Phillips and A.R.Davies, On semi-infinite spectral elements for Poisson problems with re-entrant boundary singularities, 1988, Journal of Computationl and Apllied Mathematics 21 p173-188.

Author information

Amor Boutaghou, Département de Mathématiques, Université Dr: Yahias Fares Médéa Laboratoire de Mécanique, Physique et Modélisation Mathématique(LMP2M) de Médéa, Projet CNEPRU, Algeria.
E-mail: boutaghou_a@yahoo.com

Received: December 2, 2013.

Accepted: March 7, 2014