

Dimension theory of tensor products of AF-rings

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Abstract. AF-rings are algebras over a field k which satisfy the Altitude Formula over k . This paper surveys a few works in the literature on the Krull and valuative dimensions of tensor products of AF-rings. The first section extends Wadsworth's classical results on the Krull dimension of AF-domains to the larger class of AF-rings. It also provides formulas for computing the valuative dimension with effect on the transfer of the (locally) Jaffard property. The second section studies tensor products of AF-rings over a zero-dimensional ring. Most results on algebras over a field are extended to these general constructions. The third section establishes formulas for the Krull and valuative dimensions of tensor products of pullbacks issued from AF-domains. Throughout, examples are provided to illustrate the scope and limits of the results.

Introduction

Throughout this paper, all rings and, for a given field k , all k -algebras are assumed to be commutative with identity element and *have finite transcendence degree over k* . For a ring A , we shall use $\text{Spec}(A)$ and $\text{Max}(A)$ to denote, respectively, the sets of all prime ideals and maximal ideals of A . Also, we will denote by $A[n]$ the polynomial ring $A[X_1, \dots, X_n]$ and by $p[n]$ the prime ideal $p[X_1, \dots, X_n]$ in $A[n]$, for any $p \in \text{Spec}(A)$ and positive integer n .

A finite-dimensional domain R is said to be Jaffard if $\dim(R[p]) = n + \dim(R)$ for all $n \geq 1$; equivalently, if $\dim(R) = \dim_v(R)$, where $\dim(R)$ denotes the Krull dimension of R and $\dim_v(R)$ denotes its valuative dimension (i.e., the supremum of dimensions of the valuation overrings of R). Since this notion does not carry over to localizations, R is said to be locally Jaffard if R_p is Jaffard for each $p \in \text{Spec}(R)$ (equivalently, $\text{ht}(p[n]) = \text{ht}(p)$, $\forall p \in \text{Spec}(A)$). The class of Jaffard domains contains most of the well-known classes of rings involved in dimension theory such as Noetherian domains, Prüfer domains, universally catenarian domains, and stably strong S-domains. Analogous definitions are given in Cahen's paper [16] for a finite-dimensional arbitrary ring (i.e., possibly with zero-divisors). We assume familiarity with these concepts as in [1, 2, 7, 11, 12, 13, 17, 19, 20, 25, 26, 27, 29], and any unreferenced material is standard as in [21, 28, 30].

Let k be a field and let $t(A)$ denote the transcendence degree over k of a k -algebra A . If A is not a domain, then by definition $t(A) := \max \{ t(A/p) \mid p \in \text{Spec}(A) \}$.

Definition 0.1. A k -algebra A is an AF-ring if it satisfies the Altitude Formula over k ; that is, $\text{ht}(p) + t(A/p) = t(A_p)$, for each $p \in \text{Spec}(A)$.

Examples and basic properties of AF-rings are provided at the beginning of the next section. In 1977, Sharp proved in [32] that

$$\dim(K_1 \otimes_k K_2) = \min (t(K_1), t(K_2))$$

for any field extensions K_1 and K_2 of k . In 1978, Sharp and Vámos generalized this result to the tensor product of a finite number of field extensions of k [33]. In

1979, Wadsworth extended their results on field extensions to the larger class of AF-domains [35]; namely, he proved that if D_1 and D_2 are AF-domains, then

$$\dim(D_1 \otimes_k D_2) = \min(\dim(D_1) + t(D_2), t(D_1) + \dim(D_2)).$$

Moreover, he established a formula for $\dim(D \otimes_k R)$ which holds for an AF-domain D , with no restriction on the ring R . He also proved that for any prime ideal p of an AF-ring A and, for any $n \geq 1$, $\text{ht}(p[n]) = \text{ht}(p)$ (i.e., A is locally Jaffard). In [22], Girolami studied the class of AF-domains with respect to the class of k -algebras which are stably strong S -domains and examined the behavior of the notion of AF-domain for certain pullback constructions. An upper bound was then given for the valuative dimension of the tensor product of two k -algebras; more exactly, if A_1 and A_2 are k -algebras, then

$$\dim_v(A_1 \otimes_k A_2) \leq \min(\dim_v(A_1) + t(A_2), t(A_1) + \dim_v(A_2)).$$

This paper surveys a few works in the literature on the Krull and valuative dimensions of tensor products of AF-rings. The first section extends Wadsworth's classical results on the Krull dimension of AF-domains to the larger class of AF-rings. It also provides formulas for computing the valuative dimension with effect on the transfer of the (locally) Jaffard property. The second section studies tensor products of AF-rings over a zero-dimensional ring. Most results on algebras over a field are extended to these general constructions. The third section establishes formulas for the Krull and valuative dimensions of tensor products of pullbacks issued from AF-domains. Throughout, examples are provided to illustrate the scope and limits of the results.

The three main papers involved in this survey are [8, 9, 10], which were co-authored with Samir Bouchiba (University of Meknes) and Florida Girolami (University of Rome) and published in 1997/1999.

1 Tensor products of AF-rings over a field

This section is devoted to [8]. First it extends some classical results (on the Krull dimension) known for the class of AF-domains to the class of AF-rings over a field. Then it provides formulas for computing the valuative dimension of tensor products emanating from AF-rings with effect on the possible transfer of the notion of (locally) Jaffard ring to these constructions.

Throughout this section k will denote a field and by a ring we mean a k -algebra. Also, algebras (resp., tensor products) are taken over (resp., relative to) k . For the reader's convenience, we first recall some basic properties of AF-rings.

Remark 1.1 ([22, 35]). Let \mathcal{A} denote the class of AF-rings over k and let n be a positive integer. Then:

- (i) Any finitely generated algebra and its integral extensions belong to \mathcal{A} .
- (ii) If $A \in \mathcal{A}$, then $S^{-1}A \in \mathcal{A}$, for every multiplicative subset S of A .
- (iii) If $A \in \mathcal{A}$, then $A[n] \in \mathcal{A}$.
- (iv) If $A_1, \dots, A_n \in \mathcal{A}$, then $A_1 \otimes \dots \otimes A_n \in \mathcal{A}$.
- (v) If $A_1, \dots, A_n \in \mathcal{A}$, then $A_1 \times \dots \times A_n \in \mathcal{A}$.
- (vi) If $A \in \mathcal{A}$, then A is locally Jaffard.
- (vii) The class \mathcal{A} is not stable under factor rings. However, if A is a catenarian AF-domain, then $A/p \in \mathcal{A}$, $\forall p \in \text{Spec}(A)$.

1.1 Krull dimension

This subsection aims at extending Wadsworth's results on AF-domains to the class of AF-rings. The first technical result links the transcendence degree of a localization of a tensor product to the transcendence degrees of its respective components.

Lemma 1.2. *Let A_1, \dots, A_n be AF-rings and let $P \in \text{Spec}(A_1 \otimes \dots \otimes A_n)$. Then*

$$t((A_1 \otimes \dots \otimes A_n)_P) = \sum_{1 \leq i \leq n} t(A_{i_{p_i}})$$

where $p_i := P \cap A_i$ for $i = 1, \dots, n$.

As an immediate consequence of this lemma, we obtain the following known result for AF-domains.

Corollary 1.3. *Let D_1, \dots, D_n be AF-domains and let $P \in \text{Spec}(D_1 \otimes \dots \otimes D_n)$. Then*

$$t((D_1 \otimes \dots \otimes D_n)_P) = t(D_1 \otimes \dots \otimes D_n) = \sum_{1 \leq i \leq n} t(D_i).$$

The following simple statement has important consequences on some of the following results.

Lemma 1.4. *Let A be an AF-ring and $p \in \text{Spec}(A)$. Let p_o be a minimal prime ideal of A contained in p such that $\text{ht}(p) = \text{ht}(p/p_o)$. Then, $t(A_p) = t(A_{p_o})$.*

In order to proceed with the main results, let us recall from [35] the following functions: Given two rings A and B with $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$, consider the function

$$\delta(p, q) = \max \{ \text{ht}(P) \mid P \in \text{Spec}(A \otimes B) \text{ with } P \cap A = p \text{ and } P \cap B = q \}.$$

Given a ring A , $p \in \text{Spec}(A)$ and d, s integers with $0 \leq d \leq s$, consider the two functions

$$\Delta(s, d, p) = \text{ht}(p[s]) + \min(s, d + t(A/p))$$

$$D(s, d, A) = \max \{ \Delta(s, d, p) \mid p \in \text{Spec}(A) \}.$$

The main result of this section provides a formula for the Krull dimension of a tensor product.

Theorem 1.5. *Let A be an AF-ring and B an arbitrary ring. Then:*

- (i) $\delta(p, q) = \Delta(t(A_p), \text{ht}(p), q)$, for any $p \in \text{Spec}(A)$ and $q \in \text{Spec}(B)$.
- (ii) $\dim(A \otimes B) = \max \{ D(t(A_p), \text{ht}(p), B) \mid p \in \text{Spec}(A) \}$.

Notice that (i) is the most important part of the above theorem. Its proof relies on the above two lemmas after reduction -via localization techniques- to the case where B is a field. Then, the result upon $\dim(A \otimes B)$ derives directly from the definitions of δ , Δ , and D .

In case both A and B are AF-rings, we get the following more explicit formula for the Krull dimension.

Corollary 1.6. *Let A and B be two AF-rings. Then:*

$$\dim(A \otimes B) = \max \{ \min(\text{ht}(p) + t(A_q), t(A_p) + \text{ht}(q)) \mid p \in \text{Spec}(A), q \in \text{Spec}(B) \}.$$

The general case of n AF-rings ($n \geq 2$) can be proved by induction on n via Corollary 1.6 and Lemma 1.2. Namely, we have the following result.

Corollary 1.7. *Let A_1, \dots, A_n be AF-rings. Then:*

$$\dim(A_1 \otimes \dots \otimes A_n) = \max \left\{ \min \left(\text{ht}(p_i) + \sum_{j \neq i} t(A_{p_j}) \right)_{1 \leq i \leq n} \mid p_i \in \text{Spec}(A_i) \right\}.$$

Notice that $D(s, d, A)$ is a nondecreasing function of the first two arguments and, hence, one can restrict the formulas in the above three results to the maximal ideals.

Wadsworth's well-known result [35, Theorem 3.8] on the Krull dimension of the tensor product of n AF-domains reads as follows: Let D_1, \dots, D_n be AF-domains with $n \geq 2$. Then

$$\dim(D_1 \otimes \cdots \otimes D_n) = \sum_{1 \leq i \leq n} t(D_i) - \max \{ t(D_i) - \dim(D_i) \}_{1 \leq i \leq n}.$$

This formula does not hold in general for AF-rings, as shown by the following example.

Example 1.8. Let $R_1 := k[X_1, X_2, X_3]_{(X_1)}$, $R_2 := k[X_1, X_2]$, $A_1 := R_1 \times R_2$, and $A_2 := k[X_1, X_2]_{(X_1)}$. Clearly, A_1 is an AF-ring with $\dim(A_1) = 2$ and $t(A_1) = 3$; and A_2 is an AF-domain with $\dim(A_2) = 1$ and $t(A_2) = 2$. By Corollary 1.6, one can check that $\dim(A_1 \otimes A_2) = 3 \not\leq t(A_1) + t(A_2) - 1 = 4$.

The second main result of this section establishes necessary and sufficient conditions for a tensor product of AF-rings to satisfy Wadsworth's aforementioned formula.

Theorem 1.9. *Let A_1, \dots, A_n be AF-rings. Then, the following assertions are equivalent:*

- (i) $\dim(A_1 \otimes \cdots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}$;
- (ii) *For each $i = 1, \dots, n$ there exists $M_i \in \text{Max}(A_i)$ such that $\text{ht}(M_{i_o}) = \dim(A_{i_o})$ for some $i_o \in \{1, \dots, n\}$ and, for all $i \neq i_o$, $t(A_{iM_i}) = t(A_i)$ & $t(A_i/M_i) \leq t(A_{i_o}/M_{i_o})$.*

Next, we give some applications of this result.

Corollary 1.10. *Let A_1, \dots, A_n be AF-rings such that, for each $i = 1, \dots, n$, there exists $M_i \in \text{Max}(A_i)$ with $\text{ht}(M_i) = \dim(A_i)$ and $t(A_{iM_i}) = t(A_i)$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}.$$

Corollary 1.11. *Let A_1, \dots, A_n be AF-rings such that, for each $i = 1, \dots, n$ and for each $M_i \in \text{Max}(A_i)$, $t(A_{iM_i}) = t(A_i)$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}.$$

The above corollary recovers Wadsworth's aforementioned result.

Corollary 1.12 ([35, Theorem 3.8]). *Let D_1, \dots, D_n be AF-domains. Then*

$$\dim(D_1 \otimes \cdots \otimes D_n) = \sum_{1 \leq i \leq n} t(D_i) - \max \{ t(D_i) - \dim(D_i) \}_{1 \leq i \leq n}.$$

Next, a sufficient condition involves the minimal prime ideals.

Corollary 1.13. *Let A_1, \dots, A_n be AF-rings such that, for each $i = 1, \dots, n$ and for each minimal prime ideal p_i of A_i , $t(A_i/p_i) = t(A_i)$. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}.$$

Corollary 1.14. *Let A_1, \dots, A_n be equidimensional AF-rings. Then*

$$\dim(A_1 \otimes \cdots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}.$$

It is known [22, Corollary 3.3] that if A is an AF-ring, then

$$\dim(A \otimes A) = \dim_v(A \otimes A) \leq \dim(A) + t(A) = \dim_v(A) + t(A).$$

This result follows also from Corollary 1.6. Now, applying Theorem 1.9 to $A \otimes A$ we obtain:

Corollary 1.15. *Let A be an AF-ring. Then, the following assertions are equivalent:*

- (i) $\dim(A \otimes A) = \dim(A) + t(A)$;
- (ii) $\exists M, N \in \text{Max}(A)$ with $\text{ht}(M) = \dim(A)$, $t(A_N) = t(A)$, and $t(A/N) \leq t(A/M)$.

Next, we provide an example of an AF-ring A with $\dim(A \otimes A) \not\leq \dim(A) + t(A)$.

Example 1.16. Let K be a field extension of k with $t(K) = 2$ and let $A := K \times k[X]$, where X is an indeterminate over k . Then, A is an AF-ring with $\dim(A) = 1$ and $t(A) = 2$. The maximal ideals of A are $(0) \times k[X]$ and $K \times N$ with $N \in \text{Max}(k[X])$. Moreover, $\text{ht}((0) \times k[X]) = 0$; $t(A_{(0) \times k[X]}) = 2 = t(A)$; $\text{ht}(K \times N) = 1 = \dim(A)$; $t(A/((0) \times k[X])) = 2$; and $t(A/(K \times N)) = 0$. So, by Corollary 1.15, we obtain $\dim(A \otimes A) \not\leq \dim A + t(A) = 3$.

We will conclude this subsection by an illustrative example which requires the following technical lemma.

Lemma 1.17. *Let A be an AF-ring such that there exist $p, q \in \text{Spec}(A)$ with $t(A_p) \neq t(A_q)$. Then, for any AF-ring B , $A \otimes B$ is not the tensor product of a finite number of AF-domains.*

Example 1.18. For each integer $n \geq 1$, there exist two AF-rings A_1 and A_2 such that:

- (i) $\dim(A_1 \otimes A_2) = n$;
- (ii) $A_1 \otimes A_2$ is not the tensor product of a finite number of AF-domains;
- (iii) If, in addition, there exists a non-finitely generated separable extension of k , then neither A_1 nor A_2 is a finite direct product of AF-domains.

Indeed, let K be a separable extension of k . Let $V_1 := K(X)[Y]_{(Y)} = K(X) + M_1$, $V := K(Y)[X]_{(X)} = K(Y) + M$, and $V_2 := K[Y]_{(Y)} + M = K + M_2$. Then, V_1 and V_2 are, respectively, one-dimensional and two-dimensional valuation domains of $K(X, Y)$. Since V_1 and V_2 are incomparable, $T := V_1 \cap V_2$ is a two-dimensional Prüfer domain with only two maximal ideals, M_1 and M_2 , such that $T_{M_1} = V_1$ and $T_{M_2} = V_2$ [31, Theorem 11.11]. Let $I := M_1 M_2$ and $R := T/I$. Then, R is a zero-dimensional ring (and, a fortiori, an AF-ring) with only two prime ideals, $p_1 := M_1/I$ and $p_2 := M_2/I$. Further, $t(R/p_1) = 1$ and $t(R/p_2) = 0$. By Corollary 1.15, we have

$$\dim(R \otimes R[n]) = \dim((R \otimes R)[n]) = \dim(R \otimes R) + n = t(R) + n = 1 + n.$$

Moreover, by Lemma 1.17, $R \otimes R[n]$ is not the tensor product of a finite number of AF-domains; so it suffices to take $A_1 := R$ and $A_2 := R[n - 1]$.

Now assume that K is not finitely generated over k . So $K \otimes K$ is a reduced [36, Theorem 39], zero-dimensional [32, Theorem 3.1], and non-Noetherian [34, Theorem 11] ring. Then $\text{Spec}(K \otimes K)$ is infinite [34, Lemma 0]. Next, let $A := K \otimes R$. Since A is an integral extension of R , it is zero-dimensional. Moreover, there exist two prime ideals of A , P_1 and P_2 such that $P_1 \cap R = p_1$ and $P_2 \cap R = p_2$ with $t(A/P_1) = 1$ and $t(A/P_2) = 0$. Since K is the quotient field of R/p_2 and $\text{Spec}(K \otimes K)$ is infinite, by [35, Proposition 3.2], $\text{Spec}(A)$ is infinite. So

A is not a finite direct product of AF-domains and the same holds for $A[n]$. By Corollary 1.15, $\dim(A \otimes A[n]) = \dim((A \otimes A)[n]) = \dim(A \otimes A) + n = t(A) + n = 1 + n$. Moreover, by Lemma 1.17, $A \otimes A[n]$ is not the tensor product of a finite number of AF-domains. So it suffices to take $A_1 := A$ and $A_2 := A[n-1]$, completing the proof of the example.

1.2 Transfer of the (locally) Jaffard property

In this subsection, we first establish the transfer of the locally Jaffard property in some context of tensor products. Then, we give some formulas for computing the valuative dimension of the tensor product of an AF-ring and an arbitrary ring. We conclude with the fact that the tensor product of an AF-ring and a Jaffard ring is not necessarily a Jaffard ring.

Next, we announce the main result of this subsection. Notice that the tensor product of two AF-rings is locally Jaffard (since it is an AF-ring).

Theorem 1.19. *Let A be an AF-ring and B a locally Jaffard ring. Then, $A \otimes B$ is a locally Jaffard ring.*

The proof of this result lies on a very important lemma which correlates the height of a prime ideal of $A \otimes B$ to its traces on A and B via the transcendence degrees; namely, we have:

Lemma 1.20. *Let A be an AF-ring and B an arbitrary ring. Let $P \in \text{Spec}(A \otimes B)$ with $p := P \cap A$ and $q := P \cap B$. Then*

$$\text{ht}(P) + t((A \otimes B)/P) = t(A_p) + \text{ht}(q[t(A_p)]) + t(B/q).$$

Next, we give some applications of Theorem 1.19. The first one establishes a formula for the valuative dimension of $A \otimes B$ where A is an AF-ring. To this purpose, one should first examine the variation of the function D between B and its associated polynomial rings.

Lemma 1.21. *Let A be an AF-ring, B an arbitrary ring, and $p \in \text{Spec}(A)$. Then, for any $n \geq 1$, we have*

$$D(t(A_p), \text{ht}(p), B[n]) = D(t(A_p) + n, \text{ht}(p) + n, B).$$

The next result provides a formula for the valuative dimension.

Corollary 1.22. *Let A be an AF-ring and B a ring with finite valuative dimension ≥ 1 . Then, for any $r \geq \dim_v(B) - 1$, we have*

$$\dim_v(A \otimes B) = \max \{ \text{ht}(q[r]) + \min(t(A_p), \text{ht}(p) + t(B/q)) \}$$

where (p, q) ranges over $\text{Spec}(A) \times \text{Spec}(B)$.

The special case where A is an AF-domain yields a more simplified formula.

Corollary 1.23. *Let A be an AF-domain and B a ring with finite valuative dimension ≥ 1 . Then, for any $r \geq \dim_v(B) - 1$, we have*

$$\dim_v(A \otimes B) = \max \{ \text{ht}(q[r]) + \min(t(A), \dim(A) + t(B/q)) \}$$

where q ranges over $\text{Spec}(B)$.

The next two results feature special contexts where the tensor product is Jaffard.

Corollary 1.24. *Let A be an AF-domain and B a ring such that $\dim_v(B) \leq t(A) + 1$. Then, $A \otimes B$ is a Jaffard ring.*

Recall that, for any ring B of valuative dimension 2, the ring $B[X]$ is locally Jaffard [16, Proposition 1(ii)]. Also, [1, Example 3.2] is an example of a Jaffard ring B that is not locally Jaffard but $B[X]$ is locally Jaffard.

Corollary 1.25. *Let A be an AF-domain and B a Jaffard ring such that $B[X]$ is locally Jaffard. Then, $A \otimes B$ is a Jaffard ring.*

We close this section with an example where the tensor product of an AF-domain and a Jaffard ring is not necessarily a Jaffard ring. This shows that a similar result to Theorem 1.19 does not hold, in general, for the transfer of the Jaffard property.

Example 1.26. We deduce this example from [1]. Let Z_1, Z_2, Z_3, Z_4 be four indeterminates over k . Let $L := k(Z_1, Z_2, Z_3, Z_4)$ and $V_1 := k(Z_1, Z_2, Z_3)[Z_4]_{(Z_4)}$. Then, V_1 is a one-dimensional valuation ring of L with maximal ideal $M_1 := Z_4V_1$. Let V' be a one-dimensional valuation overring of $k(Z_4)[Z_2, Z_3]$ of the form $V' := k(Z_4) + M'$ and $V'_2 = k[Z_4]_{(Z_4)} + M' = k + M'_2$, where $M'_2 = Z_4k[Z_4]_{(Z_4)} + M'$. So, V'_2 is a two-dimensional valuation ring. Now, let $V = k(Z_2, Z_3, Z_4)[Z_1]_{(Z_1)} = k(Z_2, Z_3, Z_4) + M$, with $M = Z_1V$, and $M_2 = M'_2 + M$. Then, $V_2 := V'_2 + M = k + M_2$ is a three-dimensional valuation ring.

We claim that V_1 and V_2 are incomparable. Otherwise, $V_2 \subset V_1$ and hence $V_1 = (V_2)_M$. So, M is a divided prime ideal of V_2 . That is, $Z_4V_1 = M_1 = MV_2M$. So $1 = Z_4Z_4^{-1} \in MV = M$, the desired contradiction. Now, V_1 and V_2 have the same quotient field L . By [31, Theorem 11.11], $S := V_1 \cap V_2$ is a three-dimensional Prüfer domain with only two maximal ideals, N_1 and N_2 , such that $S_{N_1} = V_1$ and $S_{N_2} = V_2$. Next, let $F := k(Z_1)$, $f : V_1 \rightarrow k(Z_1, Z_2, Z_3)$ be the natural ring homomorphism, and $D := f^{-1}(F) = F + M_1$. Let $g : S \rightarrow S/N_1 \cong V_1/N_1 \cong k(Z_1, Z_2, Z_3)$ be the natural ring homomorphism and $B := g^{-1}(F)$. We have $B = D \cap S = D \cap V_2$ and $\dim(B) = \dim(S) = 3$. Moreover, by [1, Theorem 2.11], we obtain

$$\dim_v(B) = \max \{ \dim_v(S), \dim_v(F) + \dim_v(S_{N_1}) + \text{t.d.}(S/N_1 : F) \} = 3.$$

Therefore, B is Jaffard. Since $B = D \cap V_2$ and V_1, V_2 are incomparable, it follows that $B_{n_1} = D$ and $B_{n_2} = V_2$, where $\{n_1, n_2\} = \text{Max}(B)$. Moreover, $\text{ht}(n_1[s]) = \text{ht}(n_1B_{n_1}[s]) = \text{ht}(M_1[s])$, for any positive integer s . Since V_1 is Jaffard, by [5, Theorem 1.7], $\text{ht}_{D[s]}(M_1[s]) = \text{ht}_{V_1}(M_1) + \min(s, 2)$. Then $\text{ht}(n_1) = 1$, $\text{ht}(n_1[X_1]) = 2$, and $\text{ht}(n_1[X_1, X_2]) = 3$; $\text{t}(B/n_1) = \text{t}(D/M_1) = 1$, and $\text{t}(B/n_2) = \text{t}(V_2/M_2) = 0$.

Let $A := k(X)$. By Theorem 1.5, we have

$$\dim(A \otimes B) = D(\text{t}(A), 0, B) = \max \{ \text{ht}(q[X_1]) + \min(1, \text{t}(B/q)) \mid q \in \text{Spec}(B) \}.$$

For $q := n_1$, it yields $\text{ht}(n_1[X_1]) + \min(1, \text{t}(B/n_1)) = 2 + 1 = 3$; for $q := n_2$, it yields $\text{ht}(n_2[X_1]) + \min(1, \text{t}(B/n_2)) = \text{ht}(n_2) = 3$, and $\text{ht}(q[X_1]) + \min(1, \text{t}(B/q)) \leq 3$ for every prime ideal q of B contained in n_2 . Consequently, $\dim(A \otimes B) = 3$. By Corollary 1.22, $\dim_v(A \otimes B) = \max \{ \text{ht}(q[X_1, X_2]) + \min(1, \text{t}(B/q)) \mid q \in \text{Spec}(B) \}$. For $q := n_1$, it is $\text{ht}(n_1[X_1, X_2]) + \min(1, \text{t}(B/n_1)) = 3 + 1 = 4$. Therefore, $\dim_v(A \otimes B) = 4 \neq \dim(A \otimes B)$. Consequently, $A \otimes B$ is not a Jaffard ring, completing the proof of the example.

2 Tensor products of AF-rings over a zero-dimensional ring

This section is devoted to [9]. Its purpose is to extend all the known results on the dimension of tensor products of AF-rings over a field to the general case of AF-rings over a zero-dimensional ring.

Throughout this section, R denotes a zero-dimensional ring, and algebras (resp., tensor products), when not specifically marked, are taken over (resp., relative to)

R . We denote by (A, λ_A) an algebra A and its associated ring homomorphism $\lambda_A : R \rightarrow A$; and, by λ_A^* , the associated spectral map $\text{Spec}(A) \rightarrow \text{Spec}(R)$. Notice that for any prime ideal P of A , $\lambda_A^{-1}(P)$ is a maximal ideal of R . So, we define the transcendence degree of the algebra A over R as follows

$$\text{t. d.}(A :_{\lambda_A} R) = \max \{ \text{t. d.}(A/P : R/\lambda_A^{-1}(P)) \mid P \in \text{Spec}(A) \}.$$

We write $\text{t}(A : R)$ or just $\text{t}(A)$ as an abbreviation for $\text{t. d.}(A :_{\lambda_A} R)$, when there is no ambiguity. All along this section, we consider only algebras (A, λ_A) such that $\text{t}(A) < \infty$, which also ensures that $\dim(A) < \infty$. If A is an integral domain, p_A denotes $\text{Ker}(\lambda_A)$.

First of all, observe that the transcendence degree of an algebra A depends on its R -module structure, as shown by the next example.

Example 2.1. Let $R := k(X) \times k$ and $A := k(X)$, where k is a field. Consider the two ring homomorphisms $\lambda_1 : R \rightarrow A$ and $\lambda_2 : R \rightarrow A$ defined by $\lambda_1(x, y) = x$ and $\lambda_2(x, y) = y$. Then $\text{t. d.}(A :_{\lambda_1} R) = \text{t. d.}(k(X) : k(X)) = 0$ whereas $\text{t. d.}(A :_{\lambda_2} R) = \text{t. d.}(k(X) : k) = 1$.

The following lemma provides simple generalizations of well-known facts for algebras over a field.

Lemma 2.2. *Let (A, λ_A) be an algebra, $P \in \text{Spec}(A)$, and $p := \lambda_A^{-1}(P)$. Then:*

- (i) $\text{ht}(P) + \text{t}(A/P : R) \leq \text{t}(A_p : R) = \text{t}((A/pA)_{P/pA} : R)$.
- (ii) $\text{ht}(P) = \text{ht}(P/pA)$.
- (iii) $\text{ht}(P[n]) = \text{ht}((P/pA)[n])$, for each $n \geq 1$.
- (iv) *If A is locally Jaffard, then so is A/qA , for each $q \in \text{Spec}(R)$ with $qA \neq A$.*

2.1 Tensorially compatible algebras

Let (A_1, λ_1) and (A_2, λ_2) be algebras. For $i = 1, 2$, we denote by $\mu_i : A_i \rightarrow A_1 \otimes A_2$ the canonical A_i -algebra homomorphism. The algebra $A_1 \otimes A_2$, when not specifically indicated, has $\lambda_{A_1 \otimes A_2} = \mu_1 \circ \lambda_1 = \mu_2 \circ \lambda_2$ as its associated ring homomorphism. Finally, let

$$\Gamma(A_1, A_2) := \{ (P_1, P_2) \in \text{Spec}(A_1) \times \text{Spec}(A_2) \mid \lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) \}.$$

We are interested in algebras (A_1, λ_1) and (A_2, λ_2) such that $A_1 \otimes A_2 \neq 0$, and call such algebras *tensorially compatible*. The next result provides some elementary and useful characterizations of tensorially compatible algebras. For a more general result, we refer the reader to [24, Corollary 3.2.7.1].

Proposition 2.3. *Let (A_1, λ_1) and (A_2, λ_2) be algebras. Then, the following assertions are equivalent:*

- (i) (A_1, λ_1) and (A_2, λ_2) are tensorially compatible;
- (ii) $\lambda_1^*(\text{Spec}(A_1)) \cap \lambda_2^*(\text{Spec}(A_2)) \neq \emptyset$;
- (iii) $\exists P_1 \in \text{Spec}(A_1)$ such that $\lambda_1^{-1}(P_1)A_2 \neq A_2$;
- (iv) $\exists P_2 \in \text{Spec}(A_2)$ such that $\lambda_2^{-1}(P_2)A_1 \neq A_1$;
- (v) $\exists p \in \text{Spec}(R)$ such that $pA_1 \neq A_1$ and $pA_2 \neq A_2$;
- (vi) $\text{Ker}(\lambda_1) + \text{Ker}(\lambda_2) \neq R$.

A similar result holds for any finite number of algebras, as shown below.

Proposition 2.4. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be algebras. Then, the following assertions are equivalent:*

- (i) $A_1 \otimes \cdots \otimes A_n \neq 0$;
- (ii) $\lambda_1^*(\text{Spec}(A_1)) \cap \lambda_2^*(\text{Spec}(A_2)) \cap \cdots \cap \lambda_n^*(\text{Spec}(A_n)) \neq \emptyset$;
- (iii) $\exists p \in \text{Spec}(R)$ such that $pA_i \neq A_i$, for each $i = 1, 2, \dots, n$.

The next result establishes an analogue to [35, Proposition 2.3].

Proposition 2.5. *Let (A_1, λ_1) and (A_2, λ_2) be algebras and $(P_1, P_2) \in \text{Spec}(A_1) \times \text{Spec}(A_2)$ with $\lambda_1^{-1}(P_1) = \lambda_2^{-1}(P_2) = p$. Let $\Omega := \{Q \in \text{Spec}(A_1 \otimes A_2) \mid \mu_i^{-1}(Q) = P_i, i = 1, 2\}$. Then*

- (i) Ω is lattice isomorphic to $\text{Spec}\left(\frac{A_{P_1}}{P_1 A_{P_1}} \otimes_{\frac{R}{p}} \frac{A_{P_2}}{P_2 A_{P_2}}\right)$.
- (ii) $Q \in \Omega$ is minimal in Ω if and only if $t((A_1 \otimes A_2)/Q) = t(A_1/P_1) + t(A_2/P_2)$.
- (iii) If $Q_o \in \text{Spec}(A_1 \otimes A_2)$ and $\mu_i^{-1}(Q_o) \supseteq P_i$ ($i = 1, 2$), then $\exists Q \in \Omega$ such that $Q \subseteq Q_o$.

Follow two applications of the above result, which extend two known results on algebras over a field [35] to R -algebras.

Corollary 2.6. *Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible algebras and let $Q \in \text{Spec}(A_1 \otimes A_2)$. Then*

$$\text{ht}(Q) \geq \text{ht}(\mu_1^{-1}(Q)) + \text{ht}(\mu_2^{-1}(Q)).$$

Corollary 2.7. *Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible algebras. Then*

$$t(A_1 \otimes A_2) = \max \{t(A_1/P_1) + t(A_2/P_2) \mid (P_1, P_2) \in \Gamma\} \leq t(A_1) + t(A_2).$$

Let (A_1, λ_1) and (A_2, λ_2) be tensorially compatible algebras. Clearly, $t(A_1 \otimes A_2) = t(A_1) + t(A_2)$ if and only if there exists $(P_1, P_2) \in \Gamma$ with $t(A_1) = t(A_1/P_1)$ and $t(A_2) = t(A_2/P_2)$. The second condition holds, for instance, if A_1 and A_2 are integral domains or if $\text{Spec}(R)$ is reduced to only one prime ideal. In general, the equality fails as it is shown by the next example. Moreover, when R is a field, we have $\dim(A_1 \otimes A_2) \geq \dim(A_1) + \dim(A_2)$ [35, Corollary 2.5]. This is not true, in general, in the zero-dimensional case, as shown below.

Example 2.8. Let $R := \mathbb{R} \times \mathbb{R}$, $A_1 := \mathbb{R}$ and $A_2 := \mathbb{R} \times \mathbb{R}[X]$. Consider the two ring homomorphisms $\lambda_1 : R \rightarrow A_1$ and $\lambda_2 : R \rightarrow A_2$ defined by $\lambda_1(x, y) = x$ and $\lambda_2(x, y) = (x, y)$. Clearly, A_1 and A_2 are tensorially compatible. We claim that

$$t(A_1 \otimes A_2) \not\leq t(A_1) + t(A_2) \text{ and } \dim(A_1 \otimes A_2) \not\leq \dim(A_1) + \dim(A_2).$$

Indeed, one can easily see that $t(A_1) = t(\mathbb{R} :_{\lambda_1} R) = t.\text{d.}(\mathbb{R} : \mathbb{R}) = 0$, and $t(A_2) = t(\mathbb{R} \times \mathbb{R}[X] :_{\lambda_2} R) = \max \{t.\text{d.}(\mathbb{R} : \mathbb{R}), t.\text{d.}(\mathbb{R}[X] : \mathbb{R})\} = 1$. Moreover, by Corollary 2.7, $t(A_1 \otimes A_2) = t(A_1) + t(A_2/(0 \times \mathbb{R}[X])) = t.\text{d.}(\mathbb{R} : \mathbb{R}) + t.\text{d.}(\mathbb{R} : \mathbb{R}) = 0$. It follows that

$$\dim(A_1 \otimes A_2) \leq t(A_1 \otimes A_2) = 0 \not\leq t(A_1) + t(A_2) = \dim(A_1) + \dim(A_2) = 1$$

completing the proof of the example.

2.2 Krull dimension

This subsection investigates the Krull dimension of tensor products of AF-rings over zero-dimensional rings. We first extend Wadsworth's definition of AF-rings over fields to AF-rings over zero-dimensional rings. Recall that R denotes a zero-dimensional ring and algebras are taken over R .

Definition 2.9. Under the above notation, an algebra (A, λ_A) is an AF-ring if

$$\text{ht}(P) + t(A/P) = t(A_P), \forall P \in \text{Spec}(A).$$

It is worthwhile observing that this notion of AF-ring is independent of the structure of algebra defined by the ring homomorphism λ_A . Indeed, let A be an algebra and let λ and λ' be two ring homomorphisms defining two different structures of algebra over R on A . Let $P \in \text{Spec}(A)$ and $\pi : A \rightarrow A/P$ be the canonical ring homomorphism. Let $p := \text{Ker}(\pi \circ \lambda) = \lambda^{-1}(P)$ and $q := \text{Ker}(\pi \circ \lambda') = \lambda'^{-1}(P)$. One can regard R/p and R/q as subfields of A/P . Let $k := R/p \cap R/q$. On one hand, we have

$$t(A/P :_{\lambda} R) = t.d.(A/P : R/p) = t.d.(A/P : k) - t.d.(R/p : k)$$

and

$$t(A/P :_{\lambda'} R) = t.d.(A/P : R/q) = t.d.(A/P : k) - t.d.(R/q : k).$$

On the other hand, we have

$$\begin{aligned} t(A_P :_{\lambda} R) &= \max \{ t.d.(A/Q : R/p) \mid Q \in \text{Spec}(A) \text{ and } Q \subseteq P \} \\ &= \max \{ t.d.(A/Q : k) \mid Q \in \text{Spec}(A) \text{ and } Q \subseteq P \} - t.d.(R/p : k) \end{aligned}$$

and

$$\begin{aligned} t(A_P :_{\lambda'} R) &= \max \{ t.d.(A/Q : R/q) \mid Q \in \text{Spec}(A) \text{ and } Q \subseteq P \} \\ &= \max \{ t.d.(A/Q : k) \mid Q \in \text{Spec}(A) \text{ and } Q \subseteq P \} - t.d.(R/q : k). \end{aligned}$$

It follows that $t(A_P :_{\lambda} R) - t(A/P :_{\lambda} R) = t(A_P :_{\lambda'} R) - t(A/P :_{\lambda'} R)$. That is, (A, λ) is an AF-ring if and only if (A, λ') is an AF-ring.

Next, we provide some examples and basic properties of AF-rings.

Lemma 2.10. *Let \mathcal{R} be the class of AF-rings (over R) and let n be a positive integer. Then:*

- (i) $A \in \mathcal{R} \Leftrightarrow A/pA$ is an AF-ring over the field R/p , $\forall p \in \text{Spec}(R)$ with $pA \neq A$.
- (ii) Any finitely generated R -algebras and its integral extensions belong to \mathcal{R} .
- (iii) If $A \in \mathcal{R}$, then $S^{-1}A \in \mathcal{R}$, for every multiplicative subset S of A .
- (iv) If $A \in \mathcal{R}$, then $A[n] \in \mathcal{R}$.
- (v) If $A_1, \dots, A_n \in \mathcal{R}$ and are tensorially compatible, then $A_1 \otimes \dots \otimes A_n \in \mathcal{R}$.
- (vi) If $A_1, \dots, A_n \in \mathcal{R}$, then $A_1 \times \dots \times A_n \in \mathcal{R}$.
- (vii) If $A \in \mathcal{R}$, then A is locally Jaffard.

Next, we establish adequate analogues of the main results stated in Section 1 on the dimension of tensor products of AF-rings over a field. The first result provides a formula for the Krull dimension of the tensor product $A \otimes B$, where A is an AF-ring.

Theorem 2.11. *Let A be an AF-ring and B an algebra with $A \otimes B \neq 0$. Then*

$$\dim(A \otimes B) = \max \{ \text{ht}(Q[t(A_P)]) + \min(t(A_P), \text{ht}(P) + t(B/Q)) \mid (P, Q) \in \Gamma(A, B) \}.$$

It is worthwhile noting that $\dim(A \otimes B)$ depends on the R -module structure of A and B , as shown by the next example.

Example 2.12. Let (A, λ_A) be an AF-ring and (B, λ_B) an algebra with $\dim(A \otimes B) \neq 0$. Let $p \in \text{Spec}(R)$ and $\pi : R \rightarrow R/p$ be the canonical ring homomorphism. On one hand, let $\lambda_1 : R \times R \times R \rightarrow R/p \times A$ and $\lambda_2 : R \times R \times R \rightarrow R/p \times B$ be ring homomorphisms defined by $\lambda_1(x, y, z) = (\pi(x), \lambda_A(y))$ and $\lambda_2(x, y, z) =$

$(\pi(x), \lambda_B(z))$. It is easily seen that $\Gamma(R/p \times A, R/p \times B) = \{((0) \times A, (0) \times B)\}$. By Theorem 2.11, the dimension of the tensor product of the $R \times R \times R$ -algebras $((R/p \times A), \lambda_1)$ and $((R/p \times B), \lambda_2)$ is equal to 0. On the other hand, let $\lambda'_2 : R \times R \times R \rightarrow R/p \times B$ be a ring homomorphism defined by $\lambda'_2(x, y, z) = (\pi(x), \lambda_B(y))$. By Theorem 2.11, the dimension of the tensor product of $((R/p \times A), \lambda_1)$ and $((R/p \times B), \lambda'_2)$ is equal to $\dim(A \otimes B) \neq 0$.

The next corollary handles the special case of domains.

Corollary 2.13. *Let (A, λ_A) be an AF-domain and (B, λ_B) an algebra with $\dim(A \otimes B) \neq 0$. Set $t := t(A)$, $d := \dim(A)$, and $p_A := \text{Ker}(\lambda_A)$. Then*

$$\dim(A \otimes B) = \max \left\{ \text{ht}(\overline{Q}[t]) + \min(t, d + t(B/Q)) \mid \overline{Q} = \frac{Q}{p_A B} \in \text{Spec} \left(\frac{B}{p_A B} \right) \right\}.$$

If, in addition, B is a domain, then

$$\dim(A \otimes B) = \max \left\{ \text{ht}(Q[t]) + \min(t, d + t(B/Q)) \mid Q \in \text{Spec}(B) \right\}.$$

The next main result extends Theorem 1.9 to the zero-dimensional case, by establishing necessary and sufficient conditions for a tensor product of AF-rings to satisfy Wadsworth's formula on AF-domains over a field [35, Theorem 3.8].

Theorem 2.14. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be tensorially compatible AF-rings. Then, the following assertions are equivalent:*

- (i) $\dim(A_1 \otimes \dots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}$;
- (ii) For each $i = 1, \dots, n$ there exists $M_i \in \text{Max}(A_i)$ with $\lambda_1^{-1}(M_1) = \dots = \lambda_n^{-1}(M_n)$ such that $\text{ht}(M_{i_o}) = \dim(A_{i_o})$ for some $i_o \in \{1, \dots, n\}$ and, for all $i \neq i_o$, $t(A_{iM_i}) = t(A_i)$ & $t(A_i/M_i) \leq t(A_{i_o}/M_{i_o})$.

Corollary 2.15. *Let $(A_1, \lambda_1), \dots, (A_n, \lambda_n)$ be tensorially compatible AF-rings. If anyone of the following conditions holds:*

- (i) For each $i = 1, \dots, n$ there exists $M_i \in \text{Max}(A_i)$ with $\lambda_1^{-1}(M_1) = \dots = \lambda_n^{-1}(M_n)$ such that $\text{ht}(M_i) = \dim(A_i)$ and $t(A_{iM_i}) = t(A_i)$
- (ii) If M_1, \dots, M_n are maximal ideals, respectively, of A_1, \dots, A_n with $\lambda_1^{-1}(M_1) = \dots = \lambda_n^{-1}(M_n)$, then $t(A_{iM_i}) = t(A_i)$ for $i = 1, \dots, n$
- (iii) If P_1, \dots, P_n are minimal prime ideals, respectively, of A_1, \dots, A_n with $\lambda_1^{-1}(P_1) = \dots = \lambda_n^{-1}(P_n)$, then $t(A_i/P_i) = t(A_i)$ for $i = 1, \dots, n$
- (iv) A_1, \dots, A_n are equicodimensional,

then

$$\dim(A_1 \otimes \dots \otimes A_n) = \sum_{1 \leq i \leq n} t(A_i) - \max \{ t(A_i) - \dim(A_i) \}_{1 \leq i \leq n}.$$

The special case of AF-domains reads as follows.

Corollary 2.16. *Let $(D_1, \lambda_1), \dots, (D_n, \lambda_n)$ be tensorially compatible AF-domains. Then*

$$\dim(D_1 \otimes \dots \otimes D_n) = \sum_{1 \leq i \leq n} t(D_i) - \max \{ t(D_i) - \dim(D_i) \}_{1 \leq i \leq n}.$$

The special case of $A \otimes A$ is given below.

Corollary 2.17. *Let (A, λ_A) be an AF-ring. Then, the following assertions are equivalent:*

- (i) $\dim(A \otimes A) = \dim(A) + t(A)$;
- (ii) $\exists M, N \in \text{Max}(A)$ with $\lambda_A^{-1}(M) = \lambda_A^{-1}(N)$ such that $\text{ht}(M) = \dim(A)$, $t(A_N) = t(A)$, and $t(A/N) \leq t(A/M)$.

2.3 Transfer of the (locally) Jaffard property

Theorem 1.19 states that if A is an AF-ring over a field k and B is a locally Jaffard k -algebra, then $A \otimes B$ is locally Jaffard. The main result of this subsection extends this result to AF-rings over a zero-dimensional ring.

Theorem 2.18. *Let A be an AF-ring (over R) and B a locally Jaffard R -algebra with $A \otimes B \neq 0$. Then, $A \otimes B$ is locally Jaffard.*

The next result asserts that Girolami's inequality on the valuative dimension [22, Proposition 3.1] holds in the zero-dimensional case.

Proposition 2.19. *Let A and B be tensorially compatible algebras. Then*

$$\dim_v(A_1 \otimes_k A_2) \leq \min(\dim_v(A_1) + t(A_2), t(A_1) + \dim_v(A_2)).$$

The next result handles the case where one of the two algebras is an AF-ring.

Corollary 2.20. *Let A be an AF-ring and B an algebra with $\dim_v(B) \geq 1$ and $A \otimes B \neq 0$. Then, for any $r \geq \dim_v(B) - 1$, we have*

$$\dim_v(A \otimes B) = \max \{ \text{ht}(Q[r]) + \min(t(A_P), \text{ht}(P) + t(B/Q)) \mid (P, Q) \in \Gamma(A, B) \}.$$

If A is an AF-domain, we get the following two results.

Corollary 2.21. *Let (A, λ_A) be an AF-domain and (B, λ_B) an algebra with $\dim_v(B) \geq 1$ and $A \otimes B \neq 0$. Then, for any $r \geq \dim_v(B) - 1$, we have*

$$\dim_v(A \otimes B) = \max \{ \text{ht}(Q[r]) + \min(t(A), \dim(A) + t(B/Q)) \}$$

where Q ranges over the prime ideals of B such that $\lambda_B^{-1}(Q) = \text{Ker}(\lambda_A)$.

Corollary 2.22. *Let A be an AF-domain and B an algebra with $A \otimes B \neq 0$. If $\dim_v(B) \leq t(A) + 1$, then $A \otimes B$ is a Jaffard ring.*

We conclude this section with the following observation. Let A_{red} denote the reduced ring associated to a ring A . Then, $t(A : R) = t(A_{red} : R_{red})$, for any R -algebra A . Further, if (A, λ_A) and (B, λ_B) are R -algebras, then $(A \otimes_R B)_{red} = (A_{red} \otimes_{R_{red}} B_{red})_{red}$ by [24, Corollary 4.5.12]. Thus, one may assume that R is absolutely flat and (A, λ_A) and (B, λ_B) are reduced R -algebras.

3 Tensor products of pullbacks issued from AF-domains

This section is devoted to [10], which establishes formulas for the Krull and valuative dimensions of tensor products of pullbacks issued from AF-domains. To this purpose, we use previous investigations of the prime ideal structure of various pullbacks, as in [1, 3, 4, 6, 12, 14, 15, 16]. Moreover, in [23], a dimension formula for the tensor product of two particular pullbacks was established and a conjecture for more general pullbacks was raised; in this section, this conjecture is resolved.

Throughout, k will be a field and \mathcal{C} will denote the class of (commutative) k -algebras with finite transcendence degree over k . Algebras (resp., tensor products), when not specifically marked, will be taken over (resp., relative to) k .

Let T be a domain, M a maximal ideal of T , K its residue field, $\varphi : T \rightarrow K$ the canonical surjection, and D a proper subring of K . Let R be the pullback issued from the following diagram of canonical homomorphisms:

$$\begin{array}{ccc} R := \varphi^{-1}(D) & \longrightarrow & D \\ (\square) & \downarrow & \downarrow \\ & T & \xrightarrow{\varphi} K = T/M \end{array}$$

Recall, from [18], that $M = (R : T)$ and $D \cong R/M$; and for $p \in \text{Spec}(R)$, if $M \not\subseteq p$, then $\exists! q \in \text{Spec}(T)$ such that $q \cap R = p$ and $T_q = R_p$. However, if $M \subseteq p$, then $\exists! q \in \text{Spec}(D)$ such that $p = \varphi^{-1}(q)$ and R_p is a pullback determined by the following diagram

$$\begin{array}{ccc} R_p & \longrightarrow & D_q \\ \downarrow & & \downarrow \\ T_M & \longrightarrow & K \end{array}$$

with $\text{ht}(p) = \text{ht}(M) + \text{ht}(q)$. Recall also, from [1, 14, 15], that

$$\dim(R) = \max \{ \dim(T), \dim(D) + \dim(T_M) \}$$

and

$$\dim_v(R) = \max \{ \dim_v(T), \dim_v(D) + \dim_v(T_M) + \text{t. d.}(K : D) \}.$$

As for the dimension of the polynomial ring, we have the following lower bound which turned to be useful for the current study

$$\dim(R[n]) \geq \dim(D[n]) + \dim(T_M[n]) + \min(n, \text{t. d.}(K : D))$$

where the equality holds if T is locally Jaffard with $\text{ht}(M) = \dim(T)$.

3.1 Krull dimension

Recall that a pullback R of type \square is an AF-domain if and only if T and D are AF-domains and $\text{t. d.}(K : D) = 0$ [22]. A combination of this result with the main result of this subsection allows one to compute dimensions of tensor products for a large class of algebras (that are not necessarily AF-domains).

The main theorem of this section relies on the following preliminaries which are important on their own. The next two lemmas deal with extensions of prime ideals of R to polynomial rings over pullbacks.

Lemma 3.1. *Let R be a pullback of type \square and n a positive integer. For any $p \in \text{Spec}(R)$ with $M \subseteq p$, we have*

$$\text{ht}(p[n]) = \text{ht}(p[n]/M[n]) + \text{ht}(M[n]).$$

Lemma 3.2. *Let R be a pullback of type \square such that T_M and D are locally Jaffard and let n be a positive integer. For any $p \in \text{Spec}(R)$ with $M \subseteq p$, we have*

$$\text{ht}(p[n]) = \text{ht}(p) + \min(n, \text{t. d.}(K : D)).$$

The next two lemmas deal with the extensions of prime ideals to the tensor products.

Lemma 3.3. *Let $A, B \in \mathcal{C}$ such that B is a domain. For any $p \in \text{Spec}(A)$, we have*

$$\text{ht}(p \otimes B) = \text{ht}(p[\text{t}(B)]).$$

Lemma 3.4. *Let $A, B \in \mathcal{C}$ such that B is an AF-domain. For any $P \in \text{Spec}(A \otimes B)$ with $p := P \cap A$, we have*

$$\text{ht}(P) = \text{ht}(p \otimes B) + \text{ht}\left(\frac{P}{p \otimes B}\right).$$

Let us fix notation for the rest of this section. Let R_1 and R_2 be two pullbacks of type \square issued from the k -algebras ($\in \mathcal{C}$), respectively, $(T_1, D_1, K_1 = T_1/M_1)$ and $(T_2, D_2, K_2 = T_2/M_2)$. For $i = 1, 2$, set $d_i := \dim(T_i)$, $d'_i := \dim(D_i)$, $t_i := \text{t}(T_i)$, $r_i := \text{t}(K_i)$, and $s_i := \text{t}(D_i)$. Also, we will use the functions $\delta(p, q)$ and $D(s, d, A)$ as defined in Section 1.

Lemma 3.5. *Assume T_1 and T_2 are AF-domains. For any $(p_1, p_2) \in \text{Spec}(R_1) \times \text{Spec}(R_2)$ with $M_1 \not\subseteq p_1$ and $M_2 \not\subseteq p_2$, we have*

$$\delta(p_1, p_2) = \min(\text{ht}(p_1) + t_2, t_1 + \text{ht}(p_2)) \leq \min(d_1 + t_2, t_1 + d_2).$$

Lemma 3.6. *Assume T_1 and T_2 are AF-domains. For any $P \in \text{Spec}(R_1 \otimes R_2)$ with $M_1 \subseteq p_1 := P \cap R_1$ and $M_2 \not\subseteq p_2 := P \cap R_2$, we have*

$$\text{ht}(P) = \text{ht}(M_1[t_2]) + \text{ht}\left(\frac{P}{M_1 \otimes R_2}\right).$$

Next, we state the main theorem of this subsection.

Theorem 3.7. *Assume T_1, T_2, D_1 , and D_2 are AF-domains such that $\text{ht}(M_1) = d_1$ and $\text{ht}(M_2) = d_2$. Then*

$$\dim(R_1 \otimes R_2) = \max\{\text{ht}(M_1[t_2]) + D(s_1, d'_1, R_2), \text{ht}(M_2[t_1]) + D(s_2, d'_2, R_1)\}.$$

It is an open problem to compute $\dim(R_1 \otimes R_2)$ if only T_1 (or T_2) is assumed to be an AF-domain. However, if both are not AF-domains, then the above formula does not hold in general [35, Examples 4.3].

The formula stated in the above theorem matches Wadsworth's formula in the particular case where R_1 and R_2 are AF-domains. Indeed, for $i := 1, 2$, if R_i is an AF-domain, then so are T_i and D_i and $r_i = s_i$. Moreover, by [1], $n_i := \dim(R_i) = d_i + d'_i$. So, the above theorem yields

$$\begin{aligned} \dim(R_1 \otimes R_2) &= \max\{\text{ht}(M_1[t_2]) + \dim(D_1 \otimes R_2), \text{ht}(M_2[t_1]) + \dim(R_1 \otimes D_2)\} \\ &= \max\{d_1 + \min(n_2 + s_1, t_2 + d'_1), d_2 + \min(n_1 + s_2, t_1 + d'_2)\} \\ &= \max\{\min(n_2 + r_1 + d_1, t_2 + d'_1 + d_1), \min(n_1 + r_2 + d_2, t_1 + d'_2 + d_2)\} \\ &= \min(t_1 + n_2, t_2 + n_1), \text{ as desired.} \end{aligned}$$

3.2 Valuative dimension

Recall for convenience, that the valuative dimension is stable under adjunction of indeterminates; i.e., $\dim_v(R[n]) = \dim_v(R) + n$, for any ring R and any positive integer n [25, Theorem 2]. However, the problem of computing the valuative dimension of the tensor product of two algebras is still elusively open. In [22], Girolami established a very useful upper bound for such an invariant; more exactly, she proved that if A_1 and A_2 are algebras, then

$$\dim_v(A_1 \otimes_k A_2) \leq \min(\dim_v(A_1) + t(A_2), t(A_1) + \dim_v(A_2)).$$

The goal of this subsection is to compute the valuative dimension for a large class of tensor products of algebras arising as pullbacks issued from AF-domains (and where the pullbacks are not necessarily AF-domains). To this purpose, the next two preliminary results establish the transfer of the notion of AF-domain to a polynomial ring over an arbitrary domain and over a pullback, respectively.

Lemma 3.8. *Let A be a domain ($\in \mathcal{C}$) and let n be a positive integer. Then, $A[n]$ is an AF-domain if and only if $\text{ht}(p[n]) + t(A/p) = t(A)$, $\forall p \in \text{Spec}(A)$.*

Lemma 3.9. *Let R be a pullback of type \square such that T and D are AF-domains. Then, the polynomial ring $R[t(K) - t(D)]$ is an AF-domain.*

Next, we present the main result of this subsection. Similarly to the previous subsection, we consider two pullbacks R_1 and R_2 of type \square issued, respectively, from $(T_1, D_1, K_1 = T_1/M_1)$ and $(T_2, D_2, K_2 = T_2/M_2)$.

Theorem 3.10. *Assume T_1, T_2, D_1 , and D_2 are AF-domains such that $\text{ht}(M_1) = \dim(T_1)$ and $\text{ht}(M_2) = \dim(T_2)$. Then*

$$\dim_v(R_1 \otimes R_2) = \min(\dim_v(R_1) + t(R_2), \dim_v(R_2) + t(R_1)).$$

3.3 Some applications and examples

This subsection presents some applications of Theorem 3.7 and Theorem 3.10. The first result features mild assumptions, on the transcendence degrees, for a tensor product of pullbacks issued from AF-domains to inherit the Jaffard property. As above, we consider two pullbacks R_1 and R_2 of type \square issued, respectively, from $(T_1, D_1, K_1 = T_1/M_1)$ and $(T_2, D_2, K_2 = T_2/M_2)$; and, for $i = 1, 2$, we set $t_i := t(T_i)$, $r_i := t(K_i)$, and $s_i := t(D_i)$.

Theorem 3.11. *Assume T_1, T_2, D_1 , and D_2 are AF-domains such that M_1 is the unique maximal ideal of T_1 with $\text{ht}(M_1) = \dim(T_1)$ and M_2 is the unique maximal ideal of T_2 with $\text{ht}(M_2) = \dim(T_2)$. Then, the following assertions are equivalent:*

- (i) $R_1 \otimes R_2$ is a Jaffard ring;
- (ii) Either " $r_1 - s_1 \leq t_2$ and $r_2 - s_2 \leq s_1$ " or " $r_2 - s_2 \leq t_1$ and $r_1 - s_1 \leq s_2$."

The next result states, under weak assumptions, a formula for the Krull dimension similar to the one of Theorem 3.7. One may regard this result as an analogue of [6, Theorem 5.4] (also [1, Proposition 2.7] and [15, Corollary 1]) in the special case of tensor products of pullbacks issued from AF-domains.

Theorem 3.12. *Assume T_1, T_2, D_1 , and D_2 are AF-domains such that $\text{ht}(M_1) = \dim(T_1)$ and $\text{ht}(M_2) = \dim(T_2)$. Suppose that either $s_1 \leq r_2 - s_2$ or $s_2 \leq r_1 - s_1$. Then*

$$\dim(R_1 \otimes R_2) = \max \{ \text{ht}(M_1[t_2]) + \dim(D_1 \otimes R_2), \text{ht}(M_2[t_1]) + \dim(R_1 \otimes D_2) \}.$$

The next result handles the spacial case when $R_1 = R_2$.

Corollary 3.13. *Let R be a pullback of type \square such that T is an AF-domain with $\text{ht}(M) = \dim(T)$ and D is a Jaffard domain. Set $t := t(T)$. Then*

$$\dim(R \otimes R) = \text{ht}(M[t]) + \dim(D \otimes R)$$

If, in addition, $t.d.(K : D) \leq t(D)$, then

$$\dim(R \otimes R) = \dim_v(R \otimes R) = t + \dim_v(R).$$

We close with some illustrative examples. The first example illustrates the fact that, in Theorem 3.7 and Corollary 3.13, the assumption " $\text{ht}(M_i) = \dim(T_i)$ ($i = 1, 2$)" is not superfluous.

Example 3.14. Let K be an algebraic extension of k , $T := S^{-1}K[X, Y]$, where $S := K[X, Y] \setminus ((X) \cup (X - 1, Y))$, and $M := S^{-1}(X)$. Consider the pullback R of type \square issued from $(T, k(Y), T/M = K(Y))$. Since $S^{-1}K[X, Y]$ is an AF-domain and $k(Y) \subset K(Y)$ is algebraic, then R is an AF-domain [22]. So, $\dim(R \otimes R) = \dim(R) + t(R) = 2 + 2 = 4$ by [35, Corollary 4.2]. Now, $\text{ht}(M[2]) = \text{ht}(M) = 1$ and $\dim(k(Y) \otimes R) = \min(2, 1 + 2) = 2$. It follows that $\text{ht}(M[2]) + \dim(k(Y) \otimes R) = 3 \neq \dim(R \otimes R)$.

Next, we show how one can use Theorem 3.7 to compute the Krull dimension of the tensor product of two algebras for a large class of algebras (which are not necessarily AF-domains).

Example 3.15. Consider two pullbacks R_1 and R_2 of type \square issued, respectively, from $(k(X, Y)[Z]_{(Z)}, k(X), k(X, Y))$ and $(k(X)[Z]_{(Z)}, k, k(X))$. We have Clearly, $\dim(R_1) = \dim(R_2) = 1$ and $\dim_v(R_1) = \dim_v(R_2) = 2$. So, R_1 and R_2 are not AF-domains. By Theorem 3.7, $\dim(R_1 \otimes R_2) = 4$. Now, notice that Wadsworth's formula fails here since $\min(\dim(R_1) + t(R_2), \dim(R_2) + t(R_1)) = 3$.

A combination of Theorem 3.7 and Theorem 3.12 allows one to compute the Krull dimension of the tensor product for more general algebras, as shown by the next example.

Example 3.16. Consider two pullbacks R_1 and R_2 of type \square issued, respectively, from $(k(X)[Y]_{(Y)}, k, k(X))$ and $(k(X, Y, Z)[T]_{(T)}, R_1, k(X, Y, Z))$. We have, $\dim(R_1) = 1$ and $\dim_v(R_1) = 2$. So, R_1 is not an AF-domain and, by Theorem 3.7, we obtain $\dim(R_1 \otimes R_1) = 3$. Moreover, $\dim(R_2) = 2$ and $\dim_v(R_2) = 4$. The conditions of Theorem 3.7 do not hold for the pullbacks R_1 and R_2 . We may, however, appeal to Theorem 3.12 to get

$$\begin{aligned} \dim(R_1 \otimes R_2) &= \max \{ \text{ht}(M_1[4]) + \dim(k \otimes R_2), \text{ht}(M_2[2]) + \dim(R_1 \otimes R_1) \} \\ &= \max \{ 2 + 2, 2 + 3 \} \\ &= 5 \end{aligned}$$

where $M_1 := Yk(X)[Y]_{(Y)}$ and $M_2 := Tk(X, Y, Z)[T]_{(T)}$.

Next, we show how one can use Corollary 3.13 to construct examples of non-AF-domains R where the tensor product $R \otimes R$ is Jaffard.

Example 3.17. Let R be the pullback issued from $(k(X, Y, Z)[T]_{(T)}, k(X, Y), k(X, Y, Z))$. Clearly R is not an AF-domain since $\dim(R) \neq \dim_v(R)$. Moreover, note that $\text{t. d.}(k(X, Y, Z) : k(X, Y)) \not\leq \text{t}(k(X, Y))$. By Corollary 3.13, $\dim(R \otimes R) = \dim_v(R \otimes R) = 5$. That is, $R \otimes R$ is a Jaffard ring.

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