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CONSTRUCTION OF INVARIANT NON-SEPARABLE EXTENSIONS OF THE MEASURE DEFINED BY H-VALUED MEASURABLE G-PROCESS ON H^G

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Abstract For an arbitrary infinite additive group G and for an uncountable compact Hausdorff topological group H with $\text{card}(H) = \text{card}(H^{\aleph_0}) = \text{card}(H^G)$, H-valued measurable G-processes are constructed on the group H^G and some set-theoretical characteristics of their various $\mathcal{F}^*(H^G)$-invariant extensions are calculated, where $\mathcal{F}^*(H^G)$ denotes a group of transformations of H^G generated by the eventually neutral sequences and all permutations of G. More precisely, an orthogonal family of $\mathcal{F}^*(H^G)$-invariant extensions of the left-invariant probability Haar-Baire measure on H^G is constructed such that topological weights of metric spaces associated with such extensions are maximal. In addition, for such a family of measures in H^G, the $\mathcal{F}^*(H^G)$-invariant measure extension problem is studied.

1 Introduction

Let $(\Omega, \mathcal{F}, p)$ be a probability space and G be an infinite additive group. Further, let H be an uncountable compact Hausdorff topological group and $\mathcal{B}(H)$ the Borel σ-algebra generated by open subsets of H.

The minimal σ-algebra of subsets of H under which all continuous real-valued functions on H are measurable is called the Baire σ-algebra of subsets of H and is denoted by $B_0(H)$. It is obvious that $B_0(H) \subseteq B(H)$.

Let λ be a left-invariant probability Haar measure on H. Its restriction to the class $B_0(H)$ is called a left-invariant probability Haar-Baire measure on H.

Definition 1.1. Let S be a σ-algebra of subsets of H. A stochastic process $X = (X_g)_{g\in G}$: $\Omega \to H^G$ is called H-valued $(\mathcal{F}, S^G)$-measurable G-process on $(\Omega, \mathcal{F}, p)$ if a joint probability distribution

$$F^{(X)}_{(g_1,\cdots,g_n)}(B_1, \cdots, B_n) = p(\{\omega : X_{g_1}(\omega) \in B_1, \cdots, X_{g_n}(\omega) \in B_n\})$$

with $(g_1, \cdots, g_n) \in G^n$ and $B_k \in S (1 \leq k \leq n, n \in \mathbb{N})$, does not change when shifted simultaneously in groups G and $H^{(G)}$, where $H^{(G)}$ denotes a group of eventually neutral sequences defined by

$$H^{(G)} = \{(h_g)_{g\in G} : \text{card}\{g : h_g \neq e\} < \omega\}.$$

In other words, the following equality

$$F^{(X)}_{(g_1,\cdots,g_n)}(B_1, \cdots, B_n) = F^{(X)}_{(g_1+h,\cdots,g_n+h)}(h_1B_1, \cdots, h_nB_n))$$

holds for arbitrary $h \in G$ and $h_k \in H (1 \leq k \leq n)$.

Remark 1.2. Notice that the notion of H-valued $(\mathcal{F}, S^G)$-measurable G-process is a generalization of the notion of a G-process introduced in [8].

Example 1.3. Let H be a compact Hausdorff topological group and λ be a left-invariant probability Haar measure on H. Then the family of all coordinate projections $(Pr_g)_{g\in G}$ defined on a probability space $(H^G, B_0(H^G), \lambda^G)$ is H-valued $(B_0(H^G), B_0(H^G))$ measurable G-process, where λ^G denotes the G-power of the λ.

Remark 1.4. It can be shown that λ^G is a left-invariant probability Haar-Baire measure on H^G; in other words, the measure λ^G is the restriction of the left-invariant probability Haar measure (defined on the compact Hausdorff topological group H^G) to the Baire sigma algebra $B_0(H^G)$.

For $g \in G$, let $U_g : H^G \to H^G$ be defined by $U_g((h_f)_{f\in G}) = (h_{f+g})_{f\in G}$. We denote by $\mathcal{F}(H^G)$ a group of transformations of H^G generated by the groups $H^{(G)}$ and $\{U_g : g \in G\}$. A group of transformations of H^G generated by the group $H^{(G)}$ and the group of all permutations of H^G is denoted by $\mathcal{F}^*(H^G)$, where under permutation of H^G we understand a transformation $T : H^G \to H^G$ defined by $T((h_g)_{g\in G}) = (h_{f(g)})_{g\in G}$, where $f : G \to G$ is a usual permutation of the set G. It is obvious that $\mathcal{F}(H^G) \subseteq \mathcal{F}^*(H^G)$.

For an infinite additive group G and a compact Hausdorff topological group H with $\text{card}(H) = \text{card}(H^{\aleph_0}) = \text{card}(H^G)$, we plan to construct a maximal(in the sense of cardinality) family of orthogonal $\mathcal{F}^*(H^G)$ -invariant extensions of the left-invariant probability Haar-Baire measure λ^G on H^G such that topological weights of metric spaces associated with such extensions are maximal. In addition, for such a family of measures in H^G, we plan to study the $\mathcal{F}^*(H^G)$-invariant measure extension problem.

2 Some auxiliary notions and facts

Lemma 2.1. *Let G be an infinite additive group. Let μ be a left-invariant probability measure on a group H. Then the G-power μ^G of the measure μ is $\mathcal{F}^*(H^G)$-invariant probability measure on H^G.*

Proof. By using Fubini theorem, one can easily prove that the measure μ^G is $H^{(G)}$-invariant.

Let $X \subseteq H^G$ be a cylindrical set having a form

$$X = B_{g_1} \times \cdots \times B_{g_n} \times H^{G\setminus\{g_1,\cdots,g_n\}}, \tag{2.1}$$

where $B_{g_k} \in \text{dom}(\mu)$ for $1 \le k \le n$(as usual,dom(μ) denotes the domain of the measure μ). It is obvious that for each permutation f of the group H^G we have $\mu^G(f(X)) = \mu^G(X)$. Since the class of sets having the form (2.1) constitutes an algebra $A(H^G)$ which generates the σ-algebra $(\text{dom}(\mu))^G$, by using Charatheodory measure extension theorem we deduce that μ^G is invariant with respect to the group of all permutations of H^G. Now, following definition of the group $\mathcal{F}^*(H^G)$ we claim that the measure μ^G is $\mathcal{F}^*(H^G)$-invariant. □

Lemma 2.2. *Let G be an infinite additive group. Let $(\lambda_k)_{k\in\mathbb{N}}$ be an orthogonal family of left invariant extensions of the left-invariant probability Haar measure λ defined in a compact Hausdorff topological group H such that $L = dom(\lambda_k)$ for each $k \in \mathbb{N}$. Let $(\alpha_k)_{k\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{k\in\mathbb{N}} \alpha_k = 1$. Let λ_k^G be the G-power of the measure λ_k for $k \in \mathbb{N}$ and $\mu = \sum_{k\in\mathbb{N}} \alpha_k \lambda_k^G$. Then the family of coordinate projections $X = (Pr_g)_{g\in G}$ defined on a probability space (H^G, L^G, μ) is H-valued (L^G, L^G)-measurable G-process and the measure μ is $\mathcal{F}^*(H^G)$-invariant extension of the G-power of the left invariant probability Haar-Baire measure λ^G.*

Proof. **Step 1.** Let us show that the family of coordinate projections $X = (Pr_g)_{g\in G}$ defined on a probability space (H^G, L^G, μ) is H-valued (L^G, L^G)-measurable G-process. Indeed, for $n \in N$, $(g_1, \cdots, g_n) \in G^n$, $B_k \in L(1 \le k \le n)$,$h \in G$ and $h_k \in H(1 \le k \le n)$, we have

$$F^{(X)}_{(g_1,\cdots,g_n)}(B_1, \cdots, B_n) = \mu(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G} \in H^G \; \& \; (\omega_{g_1}, \cdots, \omega_{g_n}) \in \prod_{k=1}^n B_k\}) =$$

$$(\sum_{k\in\mathbb{N}} \alpha_k \lambda_k^G)(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G} \in H^G \; \& \; (\omega_{g_1}, \cdots, \omega_{g_n}) \in \prod_{k=1}^n B_k\}) =$$

$$\sum_{k\in\mathbb{N}} \alpha_k \lambda_k^G(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G} \in H^G \; \& \; (\omega_{g_1}, \cdots, \omega_{g_n}) \in \prod_{k=1}^n B_k\}) =$$

$$\sum_{k\in\mathbb{N}} \alpha_k \lambda_k^G(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G} \in H^G \; \& \; (\omega_{g_1+h}, \cdots, \omega_{g_n+h}) \in \prod_{k=1}^n h_k B_k\}) =$$

$$(\sum_{k\in\mathbb{N}}\alpha_k\lambda_k^G)(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G}\in H^G \;\&\; (\omega_{g_1+h},\cdots,\omega_{g_n+h})\in\prod_{k=1}^n h_kB_k\}) =$$

$$\mu(\{(\omega_g)_{g\in G} : (\omega_g)_{g\in G}\in H^G \;\&\; (\omega_{g_1+h},\cdots,\omega_{g_n+h})\in\prod_{k=1}^n h_kB_k\}) =$$

$$F^{(X)}_{(g_1+h,\cdots,g_n+h)}(h_1B_1,\cdots,h_nB_n).$$

Step 2. Let us show that the measure μ is $\mathcal{F}^*(H^G)$-invariant extension of the left invariant probability Haar-Baire measure λ^G. Indeed, following Lemma 2.1, we have that λ_k^G is $\mathcal{F}^*(H^G)$-invariant probability measure defined on the measurable space (H^G, L^G) for each $k\in\mathbb{N}$. The latter relation implies that the analogous property has the measure $\mu=\sum_{k\in\mathbb{N}}\alpha_k\lambda_k^G$. Since $B(H)\subseteq \mathrm{dom}(\lambda_k)=L$ and λ_k is a left invariant extension of the λ, we deduce that λ_k^G is an extension of λ^G for each $k\in\mathbb{N}$. Now it is obvious that μ also is an extension of λ^G. □

Let (E,G,S,μ) be an invariant measurable space with invariant (possibly infinite) measure. An element $X\in S$ is called μ-almost G-invariant if the condition

$$(\forall g)(g\in G\rightarrow \mu(g(X)\triangle X)=0)$$

is fulfilled.

Let (E,G,S,μ) be a space with an invariant measure and X be a μ-almost G-invariant subset of this space. Following [3], the function

$$\mu_X : S\rightarrow \overline{R}^+$$

defined by the formula

$$(\forall Z)(Z\in S\rightarrow \mu_X(Z)=\mu(X\cap Z))$$

is called a component of the measure μ associated with the set X.

Analogously, the component μ_X of the measure μ is an elementary component of μ if, for arbitrary $Z\in S$ with $\mu(Z)>0$, there exists a sequence $(g_k)_{k\in N}$ of elements of the group G such that

$$\mu(X\setminus\bigcup_{k\in N}g_k(Z))=0.$$

A G-invariant measure μ is nonelementary if it does not have any elementary component.

Also note that the function ϱ_μ, defined by

$$(\forall X)(\forall Y)(X\in S \;\&\; Y\in S\rightarrow \varrho_\mu(X,Y)=\mu(X\triangle Y)),$$

is a quasimetric defined on the class $\mathrm{dom}(\mu)=S$ of all μ-measurable subsets of the base space E;

The pair $(dom(\mu),\rho_\mu)$ is called a metric space associated with the measure μ.

The measure μ is called separable (nonseparable) if the topological weight $a(\mu)$ of the metric space $(dom(\mu),\rho_\mu)$ associated with the measure μ satisfies the condition

$$a(\mu)<\aleph_1\ (a(\mu)\geq\aleph_1),$$

where $\aleph_1$ denotes the first uncountable cardinal number.

Lemma 2.3. *([7, Theorem 11.7, p. 175]) Let H be an uncountable locally compact σ-compact topological group with $card(H^{\aleph_0})=card(H)$. Let λ be the Haar measure defined on the topological group H. Then there exists a maximal (in the sense of cardinality) orthogonal family $(\lambda_t)_{t\in T}$ of H-invariant non-elementary extensions of the Haar measure with $card(T)=2^{2^{card(H)}}$ such that :*

1) $(\forall i)(\forall j)(i\in T \;\&\; j\in T\rightarrow dom(\lambda_i)=dom(\lambda_j))$;

2) $(\forall i)(i\in T\rightarrow \alpha(\lambda_i)$ is maximal & $\alpha(\lambda_i)=2^{card(H)})$.

Definition 2.4. Let $(G,\cdot)$ be an arbitrary uncountable group and X its subset. We say that X is G-absolutely negligible (in G) if, for any σ-finite G-invariant (respectively, G-quasi-invariant) measure μ on G, there exists a G-invariant (respectively, G-quasi-invariant) measure μ' on G extending μ and satisfying the relation $\mu'(X)=0$.

Example 2.5. Definition 2.4 implies at once that if X is a G-absolutely negligible set in G and X does not belong to the domain of an initial measure μ, then μ is strictly extendible by using this X. The said above immediately leads to the following method of extending μ. Denote by ω the first infinite cardinal and suppose that a given group G admits a countable covering $\{X_n : n < \omega\}$ such that all sets $X_n (n < \omega)$ are G-absolutely negligible in G. If our measure μ is not identically equal to zero, then there exists at least one $n < \omega$ for which the set X_n does not belong to $(dom)(\mu)$. Consequently, our μ can be strictly extended with the aid of X_n. It is natural to ask what uncountable groups $(G, \cdot)$ admit a countable covering consisting of G-absolutely negligible sets. In this direction result of A.B. Kharazishvili (see, [4, Theorem 1, p. 259]) is an object of some interest, where absolutely negligible sets in uncountable groups are considered in connection with the measure extension problem (for σ-finite invariant or quasi-invariant measures) and it is proved that, for any uncountable solvable [1] group $(G, \cdot)$, there exists a countable covering of G consisting of G-absolutely negligible sets.

Example 2.6. Let H be an uncountable compact Hausdorff topological group with $card(H^{\aleph_0}) \neq card(H)$. We know that if H is uncountable then it's every subset $H^{'}$ with card$(H^{'}) <$ card(H) is H-absolutely negligible. Indeed, since cardinality of the factor group $H/H^{'}$ is uncountable, each H-invariant (respectively, H-quasi-invariant) measure $\lambda^{'}$ on H extending Haar measure λ on H with $H^{'} \in$ dom$(\lambda^{'})$ must satisfy the relation $\lambda^{'}(H^{'}) = 0$. In other case, we will get the contradiction with the σ-finiteness of $\lambda^{'}$. On the other hand, we know that if α is an infinite cardinal number such that $\alpha^{\aleph_0} > \alpha$, then, under Generalized Continuum Hypothesis, $\aleph_0$ is cofinal with α(see, [7], Lemma 11.1, p. 162). Since $card(H^{\aleph_0}) \neq card(H)$, we deduce that $card(H^{\aleph_0}) > card(H)$ and hence, $\aleph_0$ is cofinal with H. The latter relation means that H is presented as a union of increasing subsets $(H_n)_{n \in N}$ of H with card$(H_n) <$ card(H) for $n \in N$. So, under General Continuum Hypothesis, the group H can be presented as the union of a countable family of H-absolutely negligible subsets $(H_n)_{n \in \mathbb{N}}$ of H whose every element has cardinality less than the cardinality of the group H.

Example 2.7. Let H be a compact Hausdorff topological group of rotations of the plane $\mathbb{R}^2$ about its origin. Since each uncountable additive group (including H) is solvable, by Example 2.5 we deduce that the group H can be presented as the union of a countable family of H-absolutely negligible subsets $(H_n)_{n \in \mathbb{N}}$ of H.

3 Main Results

Theorem 3.1. *Let G be an infinite additive group and H be an uncountable compact Hausdorff topological group with card$(H) =$ card$(H^{\aleph_0}) =$ card(H^G). Let λ be the Haar measure defined on the topological group H. Then there exists an orthogonal family of probability measures $(\psi_t)_{t \in T}$ on H^G such that:*

(i) $(\forall i)(\forall j)(i \in T \ \& \ j \in T \rightarrow dom(\psi_i) = dom(\psi_j))$;

(ii) ψ_t is an $\mathcal{F}^*(H^G)$-invariant non-elementary extension of the left-invariant probability Haar-Baire measure λ^G for each $t \in T$;

(iii) card$(T) = 2^{2^{card(H^G)}}$;

(iv) $(\forall i)(i \in T \rightarrow \alpha(\psi_i) = 2^{card(H^G)})$.

Proof. Let $(\lambda_t)_{t \in T}$ be a maximal (in the sense of cardinality) orthogonal family of H-invariant non-elementary extensions of the Haar measure with $card(T) = 2^{2^{card(H)}}$ defined by Lemma 2.3. We put $\psi_t = \lambda_t^G$ for each $t \in T$. Now it is obvious the conditions (i)-(iv) hold true for the family of probability measures $(\psi_t)_{t \in T}$.

□

Corollary 3.2. *Let G be an infinite additive group and H be an uncountable compact Hausdorff topological group with $card(H^{\aleph_0}) = card(H)$. Further, let $(\psi_t)_{t \in T}$ be a family of probability measures on H^G defined by Theorem 3.1. Then the family of all coordinate projections $(Pr_g)_{g \in G}$ defined on a probability space $(H^G, dom(\psi_t), \psi_t)$ is H-valued $(dom(\psi_t), dom(\psi_t))$- measurable G-process for each $t \in T$.*

[1]A group $(G, \cdot)$ is called solvable, if we have some composition series for this group:

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_n = G,$$

where e denotes the neutral element of G, each G_m is a normal subgroup of G_{m+1} and all factor groups G_{m+1}/G_m are commutative.

Theorem 3.3. *Let G be an infinite additive group and H be an uncountable compact Hausdorff topological group with card(H) = card$(H^{\aleph_0})$ = card(H^G). Suppose that the group H can be presented as the union of a countable family of H-absolutely negligible subsets $(X_n)_{n\in\mathbb{N}}$ of H. Further, let $(\psi_t)_{t\in T}$ be a family of probability measures on H^G defined by Theorem 3.1. Then there exists a family $(\overline{\psi}_t)_{t\in T}$ of probability measures on H^G such that:*

(i) $(\forall i)(\forall j)(i \in T \ \& \ j \in T \to dom(\overline{\psi}_i) = dom(\overline{\psi}_j))$;

(ii) $\overline{\psi}_t$ is an $\mathcal{F}^*(H^G)$-invariant extension of the measure ψ_t for each $t \in T$;

(iii) $\overline{\psi}_t$ is an $\mathcal{F}^*(H^G)$-invariant non-elementary extension of the left-invariant probability Haar-Baire measure λ^G for each $t \in T$;

(iv) the family of all coordinate projections $(Pr_g)_{g\in G}$ defined on a probability space $(H^G, dom(\overline{\psi}_t), \overline{\psi}_t)$ is H-valued $(dom(\overline{\psi}_t), dom(\overline{\psi}_t))$- measurable G-process for each $t \in T$.

Proof. Let $(\lambda_t)_{t\in T}$ be the family of the left invariant extensions of the Haar measure which comes from Lemma 2.3. Suppose that $t_0 \in T$. For the family of H-absolutely negligible subsets $(X_n)_{n\in\mathbb{N}}$ of H, there is an index $n_0 \in \mathbb{N}$ such that $X_{n_0} \notin \mathrm{dom}(\lambda_{t_0})$. Indeed, if we assume the contrary, then we will get that $X_n \in \mathrm{dom}(\lambda_{t_0})$ for each $n \in \mathbb{N}$. Since X_n is H-absolutely negligible subset of H for each $n \in \mathbb{N}$, we get $\lambda_{t_0}(X_n) = 0$ for each $n \in \mathbb{N}$, which implies that $\lambda_{t_0}(H) \le \sum_{n\in N}\lambda_{t_0}(X_n) = 0$. The latter relation is the contradiction. Since $\mathrm{dom}(\lambda_{t_1}) = \mathrm{dom}(\lambda_{t_2})$ for each $t_1, t_2 \in T$, we deduce that $X_{n_0} \notin \mathrm{dom}(\lambda_t)$ for each $t \in T$. Now for each $X \in \mathrm{dom}(\lambda_t)$ and countable H-configurations[2] X_1, X_2 of the set X_{n_0}, we put $\overline{\lambda}_t((X \setminus X_1) \cup X_2) = \lambda_t(X)$. We set $\overline{\psi}_t = \overline{\lambda}_t^G$ for each $t \in T$. Now it is obvious to see that the conditions (i)-(iv) hold true for the family of probability measures $(\overline{\psi}_t)_{t\in T}$. □

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[2] A subset $X \subseteq H$ is called a countable H-configuration of $Y \subseteq H$ if there is a countable family of elements $(h_k)_{k\in N}$ of H such that $X \subseteq \cup_{k\in N} h_k Y$.