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Grushin Problems and Control Theory: Formulation and Examples

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Abstract. In this paper we give a new formulation of an abstract control problem in terms of a Grushin problem, so that we will reformulate all notions of controllability, observability and stability in a new form that gives readers an easy interpretation of these notions.

1 Introduction

Grushin problem is a simple linear algebraic tool which has proved itself very useful in the mathematical study of spectral problems arising in electromagnetism and quantum mechanics. This approach appears constantly under different names and guises in many works of pure and applied mathematics.
The key observation goes back to Schur and his complement formula:
If we have for matrices

$$\begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix},$$

then P is invertible if and only if E_{-+} is invertible and

$$P^{-1} = E - E_+ E_{-+}^{-1} E_-, \; E_{-+}^{-1} = -R_+ P^{-1} R_-.$$

This tools was developed by J. Sjöstrand and M. Zworski [16], Hager and Sjöstrand [8], Hellfer and Sjöstrand [10].
The aim of this paper is to reformulate abstract control problems studied in control theory by Weiss [22] and Ammari and Tucsnak [1] in a form of Grushin problems and give some regularity results arising in the two theory.
More concisely, let $U, \; X$ be two Hilbert spaces and consider the abstract control problem

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \\ y(t) = B^* z(t) \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \longrightarrow X$ generates a C_0-semigroups of contractions $T(t)_{t\geq 0}$, $B \in \mathcal{L}(U, X)$ is an admissible control operator, $u \in L^2_{\text{loc}}(0, +\infty; U)$. The transfer function of (1.1) is given by $H(\lambda) \in \mathcal{L}(U)$ such that

$$\hat{y}(\lambda) = H(\lambda)\hat{u}(\lambda),$$

where ˆ denotes the Laplace transformation . For these concepts, see [19].
Suppose that $H(\lambda)$ is invertible in $\mathcal{L}(U)$, therefore system (1.1) can be written as a well-posed

Grushin problem as:
for $\lambda \in \rho(A)$

$$\begin{cases} (\lambda - A)u + Bu_- & = v \\ B^*u & = v_+ \end{cases} . \tag{1.2}$$

Thus, (1.2) is well-posed if

$$\begin{bmatrix} \lambda - A & B \\ B^* & 0 \end{bmatrix}^{-1} = \begin{bmatrix} E & E+ \\ E_- & E_{-+} \end{bmatrix},$$

we refer to E_{-+} as the effective Hamiltonien of $\lambda - A$. We prove that the inverse of the transfer function of system (1.1) is the effective Hamiltonien of $\lambda - A$ in (1.2).

The paper is organized as follows. In the second section we give some preliminary results dealing to system theory, and we investigate some spectral properties of transfer function, moreover we show how regularity property (in the Weiss sense) of system (1.1) is stable under iterations of Grushin problems. Our main results and statements are given in section 3. The last section is devoted to some applications.

2 Some background

In this section we gather, for easy reference, some basic facts about admissible control and observation operators, about well-posed and regular linear systems, their transfer functions, well-posed triples of operators and closed-loop systems. For proofs and for more details we refer to the literature.
We assume that X is a Hilbert space and $A : D(A) \longrightarrow X$ is the generator of a strongly continuous semigroup $\mathbb{T}$ on X. We define the Hilbert space X_1 as $D(A)$ with the norm $\|z\|_1 = \|(\beta I - A)z\|$, where $\beta \in \rho(A)$ is fixed (this norm is equivalent to the graph norm). The Hilbert space X_{-1} is the completion of X with respect to the norm $\|z\|_{-1} = \|(\beta I - A)^{-1}z\|$. This space is isomorphic to $D(A^*)^*$, and we have

$$X_1 \subset X \subset X_{-1},$$

densely and with continuous embeddings. $\mathbb{T}$ extends to a semigroup on X_{-1}, denoted by the same symbol. The generator of this extended semigroup is an extension of A, whose domain is X, so that $A : X \longrightarrow X_{-1}$. We assume that U is a Hilbert space and $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\mathbb{T}$, defined as in Weiss [24]. This means that if z is the solution of $z(t) = Az(t) + Bu(t)$, which is an equation in X_{-1}, with $z(0) = z_0 \in X$ and $u \in L^2(\mathbb{R}_+, U)$, then $z(t) \in X$ for all $t \geq 0$. In this case, z is a continuous $X-$valued function of t. We have

$$z(t) = \mathbb{T}_t + c(t)u, \tag{2.1}$$

where $c(t) \in \mathcal{L}(L^2(\mathbb{R}_+, U); X)$ is defined by

$$c(t)u = \int_0^t \mathbb{T}_{t-s}Bu(s)ds. \tag{2.2}$$

The above integration is done in X_{-1}, but the result is in X. The Laplace transform of z is

$$\hat{z}(s) = (sI - A)^{-1}[z_0 + B\hat{u}(s)].$$

B is called bounded if $B \in \mathcal{L}(U, X)$ (and unbounded otherwise). If B is an admissible control operator for $\mathbb{T}$, then $(sI - A)^{-1}B \in \mathcal{L}(U, X)$ for all s with $\mathfrak{Re}(s)$ sufficiently large. Moreover, there exist positive constants δ, ω such that

$$\|(sI - A)^{-1}B\|_{\mathcal{L}(U,X)} \leq \frac{\delta}{\sqrt{Res}}, \quad \forall Res > \omega,$$

and if $\mathbb{T}$ is normal then the last inequality implies admissibility, see [22].
We assume that Y is another Hilbert space and $C \in \mathcal{L}(X_1, Y)$ is an admissible observation operator for $\mathbb{T}$, defined as in Weiss [25]. This means that for every $T > 0$ there exists a $K_T \geq 0$ such that

$$\int_0^T \|C\mathbb{T}_t z_0\|^2 dt \leq K_T^2 \|z_0\|^2 \quad \forall\, z_0 \in D(A). \tag{2.3}$$

C is called bounded if it can be extended such that $C \in \mathcal{L}(X, Y)$.
We regard $L^2_{loc}(\mathbb{R}_+; Y)$ as a Fréchet space with the seminorms being the L^2 norms on the intervals $[0, n]$, $n \in \mathbb{N}$. Then the admissibility of C means that there is a continuous operator $\Psi : X \longrightarrow L^2_{loc}([0, \infty), Y)$ such that

$$(\Psi z_0)(t) = C\mathbb{T}_t z_0 \quad \forall\, z_0 \in D(A). \tag{2.4}$$

The operator Ψ is completely determined by (2.4), because $D(A)$ is dense in X. Now we introduce two extensions of C as following:

Definition 2.1. Let X and Y be Hilbert spaces with $\mathbb{T}$ a C_o-semigroup on X and suppose that $C \in \mathcal{L}(X_1, Y)$. Then the Lebesgue extension of C (with respect to $\mathbb{T}$), $C_L : D(C_L) \longrightarrow Y$ defined by

$$C_L x = \lim_{t \to 0} C \frac{1}{t} \int_0^t \mathbb{T}_s x ds \tag{2.5}$$

with $D(C_L) = \{x \in X | \text{the limit in (2.5) exists}\}$.

Weiss showed in [25] that C_L is an extension of C, in particular,

$$X_1 \hookrightarrow D(C_L) \hookrightarrow X.$$

The significance of the Lebesgue extension, C_L, is that it makes it possible to give a simple pointwise interpretation of the output map (2.4) for every x in the original state space X. For every $x_0 \in X$, there holds $\mathbb{T}_t x_0 \in D(C_L)$ for almost every $t \geq 0$ and

$$(\Psi x_0)(t) = C_L \mathbb{T}_t x_0.$$

A similar Λ-extension of C was introduced by Weiss [22]

$$C_\Lambda x_0 = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} x_0, \tag{2.6}$$

for $\lambda \in \mathbb{C}$ with $\mathfrak{Re}(\lambda)$ sufficiently large and for all

$$x_0 \in D(C_\Lambda) = \{x_0 \in X |\, \text{the limit in(2.6) exists}\}.$$

Definition 2.2. Let U, X, Y, V and W be Hilbert spaces such that $W \subset X \subset V$ and let $B \in \mathcal{L}(U, V)$ and $C \in \mathcal{L}(W, Y)$ and let $\mathbb{T} = (\mathbb{T}_t)_{t \geq 0}$ be a C_0-semigroup on X. Suppose that B is an admissible control operator for $\mathbb{T}$ with respect to V and that C is an admissible observation operator for $\mathbb{T}$ with respect to W. Then we define the transfer functions of the triple (A, B, C) to be the solutions, $H : \rho(A) \longrightarrow \mathcal{L}(U, Y)$ of

$$\frac{H(s) - H(\beta)}{s - \beta} = -C(sI - A)^{-1}(\beta I - A)^{-1}B \tag{2.7}$$

for $s, \beta \in \rho(A)$, $s \neq \beta$, where $\rho(A)$ is the resolvent set of A.

We remark that, since B is an admissible control operator for $\mathbb{T}$, $(\beta I - A)^{-1}B$ is an $\mathcal{L}(U,X)$-valued analytic function and since C is an admissible observation operator for $\mathbb{T}$, $C(sI-A)^{-1}$ is a $\mathcal{L}(X,Y)$-valued analytic function. Both $(\beta I - A)^{-1}B$ and $C(sI-A)^{-1}$ are analytic on some right half-plane $\mathbb{C}_\alpha = \{s \in \mathbb{C} : Re(s) > \alpha\}$. Consequently the transfer functions always exist as $\mathcal{L}(U,Y)$-valued functions which are analytic in some $\mathbb{C}_\alpha$. They differ only by an additive constant, $D \in \mathcal{L}(U,Y)$ (often called feedthrought operator). The point is that they don't need necessarily be bounded on any $\mathbb{C}_\alpha$. We impose this as an extra assumption on the triple (A,B,C) and call this well-posedness.

Definition 2.3. Under the same assumptions as in Definition (2.2), we say that the triple (A,B,C) is well-posed if B is an admissible control operator for $\mathbb{T}$ with respect to the Hilbert space V, C is an admissible observation operator for $\mathbb{T}$ with respect to the Hilbert space W and its transfer function is bounded on some half-plane $\mathbb{C}_\alpha$.

Next, we give some notions of controllability and observability. For more details, see [19]. Let $A : \mathcal{D}(A) \longrightarrow X$ generates a C_0-semigroup $\mathbb{T}_t$ on X, $B \in \mathcal{L}(U,X)$, and $z_0 \in X$.

Definition 2.4. (Controllability) The system (A,B) is said to be **exactly controllable** in time $T > 0$ if for every $z_0, z_1 \in X$ there exists $u \in L^2(0,T;U)$ such that the solution of the system (A,B) given by the Duhammel formula verify $z(T) = z_1$.

The fact that (A,B) is exactly controllable in $T > 0$ is equivalent to the fact that the operator $c(t)$ defined by (2.2) is surjective, that's

$$\operatorname{Im} c(t) = X.$$

Definition 2.5. (Observability) Let A be a generator of C_0-semigroup $\mathbb{T}_t$ on X, and $C \in \mathcal{L}(X,U)$. The system (A, C) is said to be **exactly observable** in time $T > 0$ if there exists $\delta > 0$ such that

$$\int_0^T \|C\mathbb{T}_t z\|_U^2 dt \geq \delta \|z\|_X^2, \quad \forall z \in X. \tag{2.8}$$

Remark 2.6. For every $T > 0$, we denote by

$$(\Psi_T z)(t) = \begin{cases} C\mathbb{T}_t z & t \in [0,T] \\ 0 & t > T. \end{cases} \tag{2.9}$$

Since C is bounded, then $\Psi_T \in \mathcal{L}(X, L^2((0,\infty),U))$ for every $T > 0$, and we remark that (A, C) is exactly observable in time $T > 0$ if and only if there exists $\delta > 0$ such that

$$\|\Psi_T z\|_{L^2(0,\infty;U)} \geq \delta \|z\|_X \quad \forall z \in X.$$

The following theorem gives the links between these concepts.

Theorem 2.7. *Let A be a generator of a semigroup on X and $B \in \mathcal{L}(U,X)$. Then, the following assertions are equivalent :*

(i) (A,B) *is exactly controllable on* $[0,T]$.

(ii) (A^*,B^*) *is exactly observable in time* $T > 0$.

Proof. We set $c(t)u := \int_0^t \mathbb{T}_{t-s}Bu(s)ds$. Then, (A, B) is exactly controllable if and only if $\operatorname{Im} c(T) = X$, which is equivalent to saying that $c(T)^*$ is bounded below, i.e., there exists $\delta > 0$ such that

$$\|c(T)^*z\| \geq \delta\|z\|, \quad \forall\, z \in X. \tag{2.10}$$

Compute $c(T)^*$. For all $u \in L^2((0,\infty), U)$ and $z \in X$:

$$\begin{aligned}\langle c(T)u, z\rangle &= \langle \int_0^T \mathbb{T}_{T-s}Bu(s)ds, z\rangle_X \\ &= \int_0^T \langle u(s), B^*\mathbb{T}^*_{T-s}z\rangle_U ds \\ &= \langle u, \Lambda_T \Psi^d_T z\rangle_{L^2((0,\infty),U)}\end{aligned}$$

where $\Lambda_T u(t) := \begin{cases} u(T-t) & t \in [0,T] \\ 0 \quad t > T \end{cases}$ for all $u \in L^2((0,\infty), U)$.

Thus, $c(T)^* = \Lambda_T \Psi^d_T$, and inequality (2.10) becomes

$$\|\Psi^d_T z\| \geq \|\Lambda_T \Psi^d_T z\| \geq \delta\|z\| \quad \forall\, z \in X$$

since Λ_T is un unitary. And therefore, (A^*, B^*) is exactly observable by Remark 2.6. □

Now, we introduce the notion of Grushin problem and Schur Complement.

Definition 2.8. Let $P : H_1 \longrightarrow H_2$ be a linear operator where $H_1,\ H_2$ are two Hilbert spaces. Then, a **Grushin problem** for P is a system

$$\begin{cases} Pu + R_-u_- = v \\ R_+u = v_+ \end{cases} \tag{2.11}$$

where $R_- : H_- \longrightarrow H_2,\ R_+ : H_1 \longrightarrow H_+$, $(u, u_-) \in H_1 \times H_-$ are unknown and $(v, v_+) \in H_2 \times H_+$ are given. In matrix form we can write

$$\mathcal{P} := \begin{bmatrix} P & R_- \\ R_+ & 0 \end{bmatrix} : H_1 \oplus H_- \longrightarrow H_2 \oplus H_+.$$

We say that the Grushin problem is well posed if we have the inverse

$$\mathcal{E} = \begin{bmatrix} E & E_+ \\ E_- & E_{-+} \end{bmatrix} : H_2 \oplus H_+ \longrightarrow H_1 \oplus H_-.$$

In this case we will refer to E_{-+} as **the effective Hamiltonian** of P.

For the concepts of Grushin problems and Schur complements we refer readers to [7], [8],[10].

3 Reformulation of abstract control problem

We will connect the theory of well-posed linear system with well-posed Grushin problems.
Let $A : D(A) \subset X \longrightarrow X$ skew-adjoint and then generates C_0-group of isometries on X and $B \in \mathcal{L}(U, X)$ where U is another Hilbert space identified with its dual. Thus, we prove that

Proposition 3.1. *If the following abstract control problem with observation (or equivalently the triple of operators (A, B, B^*))*

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = B^*z(t), \\ z(0) = z_0 \end{cases} \tag{3.1}$$

is well-posed with A as above and $B \in \mathcal{L}(U, X)$ with invertible transfer function $H(\lambda)$ then, the following Grušhin problem is well-posed:
For all $(v, v_+) \in X \times U$, there exists $(u, u_-) \in X \times U$ solution of

$$\begin{cases} (\lambda I - A)u + Bu_- &= v \\ \quad B^*u &= v_+ \end{cases} \tag{3.2}$$

and the effective Hamiltonien is given by

$$E_{-+}(\lambda)^{-1} = -B^*(\lambda I - A)^{-1}B = -H(\lambda). \tag{3.3}$$

Note that the transfer function is define as $\hat{y}(\lambda) = H(\lambda)\hat{u}(\lambda)$, where $\hat{}$ denotes the Laplace transform with $z_0 = 0$.

Proof. Suppose that (3.1) is well-posed and the associated transfer function $H(\lambda) \in \mathcal{L}(U)$ is invertible. Since $(\lambda - A)^{-1}B$ takes its values in $X = D(B^*)$ for all $\lambda \in \rho(A)$, then $H(\lambda)$ is given explicitly by the desired formula

$$H(\lambda) = B^*(\lambda - A)^{-1}B, \quad \forall \lambda \in \rho(A). \tag{3.4}$$

For given $(v, v_+) \in X \times U$; and $\lambda \in \rho(A)$, is there $(u, u_-) \in X \times U$ such that

$$\begin{cases} (\lambda I - A)u + Bu_- &= v \\ \quad B^*u &= v_+ \end{cases}$$

from the first equation, we can write

$$u = (\lambda - A)^{-1}v - (\lambda - A)^{-1}Bu_-,$$

and the second equation becomes

$$H(\lambda)u_- = B^*(\lambda - A)^{-1}v - v_+,$$

and therefore, since $H(\lambda)$ is invertible we get that (3.2) is well-posed. □

Now, we consider abstract control problems with feedthrought operators $D \neq 0$, in the form

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \\ y(t) = B^*z(t) + Du(t), \\ z(0) = z_0 \end{cases} \tag{3.5}$$

where $A : D(A) \subset X \longrightarrow X$, skew-adjoint and $B \in \mathcal{L}(U, X)$ and $D \in \mathcal{L}(U)$. It was showed in Weiss [22], that with $z_0 = 0$ and $\mathfrak{Re}(\lambda)$ sufficiently large, the transfer function of (3.5) is given by

$$H(\lambda) = D + B^*(\lambda I - A)^{-1}B. \tag{3.6}$$

and it satisfies the equation

$$\frac{H(s) - H(\beta)}{s - \beta} = -C(sI - A)^{-1}(\beta I - A)^{-1}B, \tag{3.7}$$

for any $s, \beta \in \rho(A)$ with $s \neq \beta$. Thus, the connection between the transfer function and the effectif Hamiltonien is given by the following lemma.

Lemma 3.2. *for $\lambda \in \rho(A)$, suppose that the following Grushin problem is well-posed: for given $(v, v_+) \in X_{-1} \times U$, there exists unique $(u, u_-) \in D(A) \times U$ such that*

$$\begin{cases} (\lambda I - A)u + & Bu_- = v, \\ B^*u + & Du_- = v_+ \end{cases} \tag{3.8}$$

therefore, the effectif Hamiltonien of $(\lambda I - A)$ is given by

$$E_{-+}(\lambda)^{-1} = D - B^*(\lambda I - A)^{-1}B. \tag{3.9}$$

in the sense that its invertibility controls the existence of the resolvent.

In fact, feedthrought operator play an important role in the study of regularity of such abstract control problems with observation in the Weiss sense as we will see in section 3, thus we have the characterization of regularity obtained in Weiss [22]:

Theorem 3.3 (Weiss [22]). *An abstract linear system is regular if and only if its transfer function has a strong limit at $+\infty$ (along the real axis), and we have*

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H(\lambda)v = Dv, \quad \forall\, v \in U.$$

Let us now consider abstract problems (3.1) with unbounded control and observation operators, that's with the same assumption on A and $\tilde{A}$ is an extension of A with domain $D(A)$ on X_{-1} denoted also by A exception that $B \in \mathcal{L}(U, X_{-1})$ assumed to be admissible and U is an Hilbert space identified with its dual. We assume that its transfer function is invertible as an element of $\mathcal{L}(U)$(here we have not explicitly its desired expression (3.4) but the only thing we know that it checks relation (3.7)).

Under a suitable construction of a well-posed Grushin problem, we prove some properties of transfer function of (3.1).

Proposition 3.4. *Let $\mathcal{O}_c$ be a connected open of $\rho(A)$ and $\lambda \in \mathcal{O}_c$. Then, $(H(\lambda))_{\lambda \in \mathcal{O}_c}$ is a family of Fredholm operators depends holomorphically on λ. Moreover, if $H(\lambda_0)^{-1}$ exists at some points $\lambda_0 \in \mathcal{O}_c$, then $\mathcal{O}_c \ni \lambda \longrightarrow H(\lambda)^{-1}$ is meromorphic.*

Proof. From the reformulation of abstract control problem with observation on a well-posed Grushin problem, with B_L^* the Lebesgue extension of B in the place of this later, we proved that the link is the invertibility of the family of transfer functions $(H(\lambda)_{\lambda \in \mathcal{O}_c}$ where $\mathcal{O}_c$ is a connected open of $\rho(A)$, and consequently is Fredholm of index 0.
For $\lambda_0 \in \mathcal{O}_c \subset \rho(A)$ (and therefore $\lambda_0 - A$ is Fredholm), we can always take U with finite dimensional.

let $n_+ = \dim\ker(\lambda_0 - A) = \dim\operatorname{coker}(\lambda_0 - A), \quad n_+ = n_- = n$ and choose $B : \mathbb{C}^n \longrightarrow X$. In this case

$$E_{-+}^{\lambda_0} = H(\lambda_0)^{-1} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$$

is finite matrix with index $n_+ - n_- = 0$.
The invertibility of $H(\lambda_0),\ \lambda_0 \in \mathcal{O}_C$ is equivalent to the well-posedness of $\mathcal{A}(\lambda)$ where

$$\mathcal{A}(\lambda) = \begin{bmatrix} \lambda - A & B \\ B^* & 0 \end{bmatrix}.$$

This shows that there exists a locally finite covering of $\mathcal{O}_c$, $\{O_j\}$, such that for $\lambda \in O_j,\ H(\lambda)$ is invertible, more precisely when $f_j(\lambda) \neq 0$, where f_j is holomorphic in O_j(indeed, we can define $f_j(\lambda_0) :=$ det $H(\lambda_0)^{-1}$ where $H(\lambda_0)^{-1}$ exists for $\lambda_0 \in O_j$). Since O_c is connected and since $H(\lambda_1)$ is invertible for at least one $\lambda_1 \in \mathcal{O}_c$ shows that all f_j's are not identically zero. That means that det $H(\lambda)^{-1}$ is non-vanishing holomorphic in some neighbourhood of λ_0, $V(\lambda_0)$, and consequently $H(\lambda)$ is a family of meromorphic operators in $V(\lambda_0)$, where λ_0 was arbitrary in $\mathcal{O}_c$.

□

Proposition 3.5. *Let g be holomorphic function on $\mathcal{O}_c$ connected open of $\rho(A)$. Then for any curve γ homologous to 0 in $\mathcal{O}_c$, and on which $(\lambda - A)^{-1}$ exists, the operator $\frac{1}{2\pi i}\int_\gamma(\lambda - A)^{-1}g(\lambda)d\lambda$ (that's the spectral projection of A onto $\mathcal{O}_c$) is of trace class and we have*

$$\operatorname{tr}\int_\gamma(\lambda - A)^{-1}g(\lambda)d\lambda = \operatorname{tr}\int_\gamma \partial_\lambda H(\lambda)^{-1}H(\lambda)g(\lambda)d\lambda. \tag{3.10}$$

Proof. Basic idea: writing $\partial_\lambda A(\lambda) = \dot{A}(\lambda)$, we have

$$\dot{\mathcal{E}}(\lambda) = -\mathcal{E}(\lambda)\dot{\mathcal{A}}(\lambda)\mathcal{E}(\lambda)$$

where $\mathcal{E}(\lambda)$ as in the previous proposition and $\mathcal{E}(\lambda)$ is given by

$$\begin{bmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{-+}(\lambda) \end{bmatrix}$$

which gives

$$E_-(\lambda)E_+(\lambda) = -\dot{\mathcal{E}}_{-+}(\lambda)$$

we recall that

$$(\lambda - A)^{-1} = E(\lambda) - E_+(\lambda)\dot{\mathcal{E}}_{-+}(\lambda)E_-(\lambda).$$

Since $E_{-+}(\lambda)^{-1}$ is a finite matrix, then

$$\int_\gamma(\lambda - A)^{-1}g(\lambda)d\lambda = -\int_\gamma E_+(\lambda)E_{-+}(\lambda)^{-1}E_-(\lambda)g(\lambda)d\lambda$$

is an operator of trace class

□

3.1 Some regularity results

In this section we show how the property of regularity in the Weiss sense [22] is conserved along the iterations of Grushin problems.
Consider the system of evolution equations

$$\begin{cases} \dot{z}(t) & = Az(t) + Bu(t), \quad z(0) = z_0, \\ y(t) & = B^* z(t) \end{cases} \tag{3.11}$$

where

(i) $A : D(A)(\subset X_{-1}) \longrightarrow X_{-1}$ is an unbounded positive self-adjoint operator in the Hilbert space X,

(ii) $B \in \mathcal{L}(U, X_{-1})$,

(iii) $B^* \in \mathcal{L}(X_1, U)$ is defined as

$$(B^* x, u)_U = \langle x, Bu \rangle_{X_1 \times X_{-1}} \quad \forall x \in X_1.$$

Assume that (3.11) is well-posed and its transfer function $H(s) \in \mathcal{L}(U)$ is uniquely determined by the pair (A, B) and assumed to be invertible and checks the following relation

$$\frac{H(\lambda) - H(\mu)}{\lambda - \mu} = -B^*(\lambda I - A)^{-1}(\mu I - A)^{-1} B, \quad \forall \lambda,\, \mu \in \rho(A).$$

Suppose that system (3.11) is regular; that's:

$$\lim_{\lambda \in \mathbb{R}, \lambda \to +\infty} H(\lambda) u = Du \quad \forall\, u \in U,$$

where $D \in \mathcal{L}(U)$ called feedthrought operator. For more details we refer to [2],[22]....Thus, for $\lambda \in \rho(A)$, the associated Grushin problem is

$$\begin{cases} (\lambda I - A)u_1 + Bu_2 & = v_1 \\ B^* u_1 & = v_2. \end{cases} \tag{3.12}$$

In matrix form, (3.12) is written as

$$\mathcal{A}(\lambda) = \begin{bmatrix} \lambda I - A & B \\ B^* & 0 \end{bmatrix} : X_1 \oplus U \longrightarrow X_{-1} \oplus U.$$

Hence, (3.12) is well-posed if and only if $\mathcal{A}(\lambda)$ is invertible with

$$\mathcal{A}(\lambda)^{-1} = \begin{bmatrix} E(\lambda) & E_+(\lambda) \\ E_-(\lambda) & E_{-+}(\lambda) \end{bmatrix}.$$

In the Grushin problem context, $E_{-+}(\lambda)$ is called the effective Hamiltonien of $(\lambda I - A)$, and is also the Schur complement of $(\lambda I - A)$ and we have

$$E_{-+}(\lambda)^{-1} = -B^*(\lambda I - A)^{-1} B, \quad \forall \lambda \in \rho(A)$$

which is invertible. System (3.12) can be iterated in the following way:
Assume that there exists two operators

$$N_- : V_- \longrightarrow U, \quad N_+ : U \longrightarrow V_+,$$

with V_-, V_+ are two Hilbert space such that the following Grushin problem is well-posed

$$\begin{cases} E_{-+}(\lambda)u_3 + N_- u_4 & = v_3 \\ N_+ u_3 & = v_4 \end{cases} \tag{3.13}$$

that's

$$\mathcal{E} = \begin{bmatrix} E_{-+}(\lambda) & N_- \\ N_+ & 0 \end{bmatrix} : U \oplus V_- \longrightarrow U \oplus V_+$$

is invertible with the inverse

$$\mathcal{F} = \begin{bmatrix} F(\lambda) & F_+(\lambda) \\ F_-(\lambda) & F_{-+}(\lambda) \end{bmatrix},$$

then the new Grushin problem

$$\begin{cases} (\lambda I - A)u + BN_- \tilde{u} & = \tilde{v} \\ N_+ B^* u & = \tilde{v}_- \end{cases} \tag{3.14}$$

with the inverse given by

$$\mathcal{G} = \begin{bmatrix} E - E_+ F E_- & E_+ F_+ \\ F_- E_- & -F_{-+}(\lambda) \end{bmatrix}.$$

Thus, the corresponding evolution problem to (3.14) is

$$\begin{cases} \dot{z}(t) = (\lambda I - A)z(t) + BN_- v(t), \quad z(0) = z_1 \\ y_2(t) = N_+ B^* z(t) \end{cases} \tag{3.15}$$

which still regular with transfer function given

$$H_1(\lambda) = N_+ H(\lambda) N_-.$$

4 Application of Grushin problem in control theory

Let us starting by recalling some definitions and properties mentioned in [3].

Definition 4.1. Suppose that $\Sigma = (\mathbb{T}, \Phi, \mathbb{L}, \mathbb{F})$ is an abstract linear system. If A is the generator of $\mathbb{T}$, B is the control operator of Σ and C is the observation operator of Σ, then we say that (A, B, C) is the triple associated with Σ. A triple of operators (A, B, C) will be called well-posed if there is an abstract linear system Σ such that (A, B, C) is the triple associated with Σ.

In the following two remarks we try to clarify what well-posedness of triple of operators means in terms of differential equations. See [3].

Remark 4.2. Suppose that U, X and Y are Hilbert spaces, A is the generator of a semigroup on X, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. If C_L is the Lebesgue extension of C, and if the operator $C_L(\beta I - A)^{-1}B$ is well defined for some (and hence any) $\beta \in \rho(A)$, then (A, B, C) is well-posed if and only if the system of equations

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0 \\ y(t) = C_L z(t) \end{cases} \tag{4.1}$$

is well-posed in a certain natural sense. If the triple is well-posed, but $C_L(\beta I - A)^{-1}B$ does not exist,then (4.1) is no longer well-posed.

Remark 4.3. Let U, X, Y, A, B, C and C_L be as in the previous remark, but we do not assume that $C_L(\beta I - A)^{-1}B$ makes sense. Then (A, B, C) is well-posed if and only if the following (more complicated) system of equations is well-posed:

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0 \\ y(t) = C_L[z(t) - (\beta I - A)^{-1}Bu(t)] \end{cases}$$

in the same sense as (4.1).

In this section, we show how a well-posed Grushin problem of type (3.2) gives a Hautus test Criteria and then exact observability and exponential stability of system of type (4.1).
Before starting, we recall some properties and definitions, for more details see Miller [13] and Hautus [9]. The exact observability property is dual to the exact controllability property, as it has been shown in Dolecki and Russell [8].
few papers in the area of controllability and observability of systems governed by partial differential equations have considered a frequency domain approach, related to the classical Hautus test in the theory of finite dimensional systems (see Hautus [9]). Roughly speaking, a frequency domain test for the observability of (4.1) is formulated only in terms of the operators A, B^* and of a parameter (the frequency). This means that the time t does not appear in such a test and that we do not have to solve an evolution equation. In the case of a bounded observation operator B^*, such frequency domain methods have been proposed in Liu [11]. In the case of an unbounded observation operator B^*, a Hautus type test has been recently obtained in Miller [13]. Thus we have

Proposition 4.4. *The system (3.11) is exactly observable in time $T > 0$ if and only if there exists a constant $\delta > 0$ such that*

$$\|(\lambda I - A)z\|_X^2 + \|B^* z\|_U^2 \geq \delta \, \|z\|_X^2 \,, \quad \forall \, z \in D(A), \lambda \in \mathbb{R}. \tag{4.2}$$

We shall refer to (4.2) as the (infinite-dimensional) Hautus test.

A new result in this paper reads as follows.

Theorem 4.5. *Let $A : D(A) \longrightarrow X$, $B \in \mathcal{L}(U, X)$ such that X, U, $D(A) \subset X$ be complex Hilbert spaces and assume that*

$$\operatorname{Im} B \subset D(A), \tag{4.3}$$

$$\mathcal{A}(\lambda) = \begin{bmatrix} \lambda - A & B \\ B^* & 0 \end{bmatrix} : D(A) \times U \longrightarrow X \times U,$$

and that B^* has a uniformly bounded right inverse.
If for $Q = \lambda - A$, $\quad ||Q_{\operatorname{Im} B}||_{\mathcal{L}(X)} = \mathcal{O}(1)$, then

$$\mathcal{A}(\lambda)\begin{bmatrix} u \\ u_- \end{bmatrix} = \begin{bmatrix} v \\ v_+ \end{bmatrix}$$

gives that

$$\|v\|_X^2 + \|v_+\|_U^2 \geq C(\|u\|_X^2 + \|u_-\|_U^2). \tag{4.4}$$

Remark 4.6. (i) In Theorem 4.5, we remark that we don't need to have a well-posed Grushin problem in order to get inequality (4.4).

(ii) Since B^* has a uniformly bounded right inverse, then B^* is surjective and according to N. K. Nikolski [14], the system (A, B^*) is exactly controllable in any time $\tau > 0$ with A is skew-adjoint operator.

(iii) From the point of view of applications it is often sufficient to have an explicit description of the space accessible states

$$\operatorname{Im} c(\tau) = c(\tau)L^2(0, \tau; U)$$

instead of strong demand restrictive exact controllability $\operatorname{Im} c(\tau) = X$.

(iv) If A is skew-adjoint on X and $B \in \mathcal{L}(U, X)$ as in the previous theorem and for $\lambda = i\omega$, $\omega \in \mathbb{R}$, therefore we have the following Hautus type estimation

$$\|(i\omega - A)u\|_X^2 + \|B^*u\|_U^2 \geq C\|u\|_X^2, \quad \forall\, u \in X. \tag{4.5}$$

and if we consider the following abstract control problem with observation with A and B as above in the previous theorem

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = 0 \\ y(t) = B^*z(t) \end{cases} \tag{4.6}$$

then, (4.6) is exactly observable and the C_0-groups of isometries $T(t)_{t\in\mathbb{R}}$ generated by A is exponentially stable via estimation (4.5). Hence, the setup of a Grushin problem (not necessary well-posed) give us exactly observable system.

Proof. of Theorem 4.5.

Let $\Pi : H \longrightarrow (\text{kernel}\, B^*)^\perp = \operatorname{Im} B$ be the orthogonal projection. Then

$$\begin{aligned} \|(I - \Pi)u\|_X^2 &\leq |\langle P(I - \Pi)u, (I - \Pi)u\rangle| \\ &= |\langle (I - \Pi)v - (I - \Pi)P\Pi u, (I - \Pi)u\rangle| \\ &\leq \|v\|_X\|(I - \Pi)u\|_X + \|P\Pi u\|_X\|(I - \Pi)u\|_X, \end{aligned}$$

with $P = (\lambda I - A)$. By assumption, there exists a uniformly bounded operator

$$P_+ : U \longrightarrow (\ker B^*)^\perp \subset X$$

such that $B^*P_+v_+ = v_+$, and consequently $\Pi u = P_+v_+$. Thus

$$\|(\lambda - A)\Pi u\|_X = \|(\lambda - A)|_{\operatorname{Im} B}P_+v_+\| = \mathcal{O}(1)\|v_+\|_U,$$

and hence

$$\|(I-\Pi)u\|_X \leq \|v\| + \mathcal{O}(1)\|v_+\|_U.$$

With $P_- = P_+^*$, also we have $P_- B^* u_- = u_-$, so that

$$u_- = P_-(v-(\lambda-A)u) = P_-v - P_-\Pi(\lambda-A)(I-\Pi)u - P_-(\lambda-A)\Pi P_+ v_+$$

and

$$\begin{aligned}\|u\|_X &\leq C(\|v\| + \|(\lambda-A)|_{\operatorname{Im} B}\|_{\mathcal{L}(X)}\|(I-\Pi)u\|_X + \|(\lambda-A)|_{\operatorname{Im} B}\|_{\mathcal{L}(X)}\|v_+\|_U \\ &\leq C(\|v\|_X + \|(I-\Pi)u\|_X + \|v_+\|_U.\end{aligned}$$

It 's easy to prove that $\|\Pi u\|_X = \|P_+ v_+\| \leq C\|v_+\|$ and therefore

$$\|v\|_X + \|v_+\|_U \geq C(\|u\|_X + \|u_-\|_U).$$

□

Example 4.7. We consider the following initial and boundary value problem:

$$\partial_t^2 u - \Delta u + G\partial_t u = 0, \quad \Omega \times (0,+\infty), \tag{4.7}$$

$$u = 0, \qquad \partial\Omega \times (0,+\infty), \tag{4.8}$$

$$u(.,0) = u^0 \in H^2(\Omega)\cap H_0^1(\Omega),\ \partial_t u(.,0) = u^1 \in H_0^1(\Omega), \qquad \Omega \tag{4.9}$$

where $G = (-\Delta)^{-1}$. If we introduce the following notations:

$$H = L^2(\Omega),\ \mathcal{D}(A_0) = H^2(\Omega)\cap H_0^1(\Omega)$$

$$A_0\varphi = -\Delta\varphi, \quad \forall \varphi \in \mathcal{D}(A_0).$$

The system (4.7)-(4.9) can be written in the following abstract form:

$$\begin{cases} \dot{z}(t) = \mathcal{A}_d z(t) \\ z(0) = z^0, \end{cases} \tag{4.10}$$

where

$$z(t) = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathcal{A}_d = \begin{pmatrix} 0 & I \\ -\Delta & -G \end{pmatrix}, \quad z^0 = \begin{pmatrix} u^0 \\ u^1 \end{pmatrix}.$$

A_d can be written in the form $\mathcal{A}_d = \mathcal{A}_0 + BB^*$ with

$$\mathcal{A}_0 = \begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ G^{\frac{1}{2}} \end{pmatrix}, \qquad B^* = \begin{pmatrix} 0 & G^{\frac{1}{2}} \end{pmatrix}.$$

Then, let

$$B : U = L^2(\Omega) \longrightarrow (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

thus, it's easy to check that B is onto and that the range of B is contained in $D(A_0) = (H^2(\Omega)\cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Then according to the above proposition, we have

$$\exists \delta' > 0; \quad \left\|(i\omega - \mathcal{A}_0)\begin{pmatrix} z \\ y \end{pmatrix}\right\|_X^2 + \left\|B^*\begin{pmatrix} z \\ y \end{pmatrix}\right\|_U^2 \geq \delta' \left\|\begin{pmatrix} z \\ y \end{pmatrix}\right\|_X^2,$$

$$\forall\, \omega \in \mathbb{R},\ \begin{pmatrix} z \\ y \end{pmatrix} \in D(\mathcal{A}_0).$$

which is equivalent to (with $y = iwz$)

$$\left\|(\omega^2 - A_0)z\right\|_H^2 + \left\|\omega G^{\frac{1}{2}} z\right\|_U^2 \geq \delta\|\omega z\|^2, \quad \forall \omega \in \mathbb{R},\ z \in \mathcal{D}(A_0). \tag{4.11}$$

Now, we introduce other types of controllability of system (4.6). Before that we recall the notion of Riesz basis and for more details, we refer readers to [19].

Definition 4.8. A sequence $(\varphi_n)_{n\geq 1}$ in a Hilbert space X forms a Riesz basis if

(i) $\overline{\text{span}}\{\varphi_n\} = X$ and

(ii) There exist positive constants m and M such that for an arbitrary integer n and scalar $(a_n)_{n\geq 1}$ one has

$$m\sum_{n\geq 1}|a_n|^2\|\varphi_n\|^2 \leq \|\sum_{n\geq 1}a_n\varphi_n\|^2 \leq M\sum_{n\geq 1}|a_n|^2\|\varphi_n\|^2.$$

From the definition, one can easily see that an orthonormal complete sequence in a Hilbert space is a Riesz basis. Hence, Riesz basis is such a basis that is equivalent to orthonormal basis under bounded invertible transform, that's, for any given Riesz basis $(\varphi_n)_{n\geq 1}$ in X, there exist a bounded invertible operator T such that

$$T\varphi_n = e_n, \quad n \geq 1$$

where $(e_n)_{n\geq 1}$ is an orthonormal basis. Also once we have a Riesz basis $(\varphi_n)_{n\geq 1}$ for X, then we can identify X with ℓ^2 via

$$x = \sum_{n\geq 1}a_n\varphi_n \in X \longleftrightarrow \sum_{n\geq 1}|a_n|^2 < \infty.$$

As we said in Remark 4.6 about the characterization of $\operatorname{Im} c(t)$, the following theorem of Nikolski [14] gives an explicit description of $\operatorname{Im} c(t)$ in the case where the generator A has a Riesz basis of eigenvectors.

Theorem 4.9. *Let $(\varphi_n)_{n\geq 1}$ be a Riesz basis in X consisting of eigenvectors of A and $(\psi_n)_{n\geq 1}$ its biorthogonal and assume that*

$$A\varphi_n = -\lambda_n\varphi_n, \quad n \geq 1$$

then, if the family $(\mathcal{E})_{n\geq 1}$ defined by

$$\mathcal{E}_n(t) = e^{-\bar{\lambda}_n t}B^*\psi_n$$

is also a Riesz basis in $L^2(0,t;U)$ then

$$\operatorname{Im} c(t) = \{\sum_{n\geq 1}b_n\varphi_n \ , \sum_{n\geq 1}|b_n|^2\frac{1}{\|\mathcal{E}_n\|^2_{L^2}} < \infty\}.$$

In the case where A is as in the previous theorem then, each state $x \in X$ is defined formally by its Fourier series

$$x \sim \sum_{n\geq 1} < x, \psi_n > \varphi_n$$

where $(\psi_n)_{n\geq 1}$ is the biorthogonal sequence. It's natural to search an explicit description of the control space $\operatorname{Im} c(t)$ in the form of "Fourier multipliers".

Definition 4.10. Let $(\omega_n)_{n\geq 1}$ be a positive sequence of reel number. We put

$$X(\omega_n) = \{x \in X / \exists y \in X \text{ s.t } < x, \psi_n >= \frac{1}{\omega_n} < y, \psi_n >, \, n \geq 1\}.$$

The system (A, B) is said to be exactly controllable in time $T > 0$ up to a renomalization if there exist $\omega_n > 0$, $n \geq 1$ such that

$$S(t)X(\omega_n) \subset X(\omega_n); \quad t \geq 0$$

$$BU = \operatorname{Im} B \subset X(\omega_n) \tag{4.12}$$

and $(A|_{X(\omega_n)}, B)$ is exactly controllable.

The following proposition link the Hautus test criteria obtained in Theorem 4.5 with condition (4.12) introduced in the above definition of controllability up to a renormalization.

Proposition 4.11. *Let $A : D(A) \longrightarrow X$ as in Theorem 4.9 and $B \in \mathcal{L}(U, X)$.*
With the renormalization

$$X(\omega_n) = \{\sum_{n\geq 1} a_n \varphi_n; \quad \sum_{n\geq 1} |a_n|^2 \frac{1}{\|\mathcal{E}_n\|_{L^2}^2} < \infty\} \tag{4.13}$$

we assume that we have

$$\operatorname{Im} B \subset X(\omega_n), \tag{4.14}$$

$$\mathcal{A}(\lambda) = \begin{bmatrix} \lambda - A & B \\ B^* & 0 \end{bmatrix} : X(\omega_n) \times U \longrightarrow X \times U,$$

and that B^* has a uniformly bounded right inverse.
If for $Q = \lambda - A$, $\quad ||Q_{\operatorname{Im} B}||_{\mathcal{L}(X)} = \mathcal{O}(1)$, then

$$\mathcal{A}(\lambda) \begin{bmatrix} u \\ u_- \end{bmatrix} = \begin{bmatrix} v \\ v_+ \end{bmatrix}$$

gives that

$$\|v\|_X^2 + \|v_+\|_U^2 \geq C(\|u\|_X^2 + \|u_-\|_U^2). \tag{4.15}$$

Proof. The proof is the same as Theorem 4.5. □

References

[1] K. Ammari and M. Tucsnak, Stabilization of second order evolution equations by a class of unbounded feedbacks, *ESAIM: Control, Optimisation and Calculus of Variations.*, **6**(2001), 361-386.

[2] R.F. Curtain, The Salamon-Weiss of well-posed infinite dimensional linear system, *IMA J. Math. Control Inform.*, **14**(1997) 207-223.

[3] R. F. Curtain and G. Weiss, *Well posedness of triples of operators (in the sense of linear systems theory)*, In Distributed parameter systems, proceedings of the conference in Vorau, Austria. (F. Kappel, K. Kunisch, and W. Schappacher, Eds). Birkhäuser, Basel.(1989) 41-59.

[4] R.F. Curtain and H.J. Zwart, *An introduction to infinite dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[5] S. Dolecki and D. L. Russell, A general theory of observation and control., *SIAM J. Control Optimization.*, **15** (1977), 185-220.

[6] K-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolutions Equations*. Graduate Texts in Mathematics. Springer, 2000.

[7] V. V. Grushin Les problèmes aux limites dégénérés et les opérateurs pseudo-différentiels. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, 737-743.

[8] M. Hager et J. Sjöstrand , Eigenvalue asymptotics for randomly perturbed non-selfadjoint operators, *math.SP*,.2006

[9] M. Hautus, Controllability and observability conditions of linear autonomous systems, *Nederl. Akad. Wet., Proc., Ser. A*, **72** (1969), 443-448.

[10] B. Helffer et J. Sjöstrand , Résonances en limite semi-classique. *Mémoires de la S.M.F.*, **114**(1986).

[11] K. Lui, Locally distributed control and damping for the conservative systems, *SIAM J. Control and optimization.*, **35** (1997), 1574-1590.

[12] K. Liu, Z. Liu, and B. Rao, Exponential stability of an abstract nondissipative linear system., *SIAM J. Control Optimization*, **40** (2001), 149-165.

[13] L. Miller, Controllability cost of conservative systems : resolvent condition and transmutation, *J. Funct. Anal.*, **218**(2005), 425-444.

[14] N. K. Nikolski, Contrôlabilitè à une renormalisation près et petits opérateurs de contrôle,*J. Mathématiques Pures et Appliqués*, **77**(1998), 439-479.

[15] K. Ramdani, T. Takahashi, G. Tenenbaum and M. Tucsnak, A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator, *J. Funct. Anal.*, **226** (2005), 193-229.

[16] J. Sjöstrand and M. Zworski, Elementary linear algebra for advanced spectral problems, *Ann. Inst. Fourier, Grenoble.*,**57**(2007), 2095-2141.

[17] O. Staffans and G. Weiss, Transfer functions of regular linear systems Part II: The system operator and the Lax-Philips semigroup.,*Transactions of the American Mathematical Society.*, **354** (2002), 3229-3262.

[18] J. J, Sylvester, On the relation between the minor determinants of lin- early equivalent quadratic functions, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science, Fourth Series.*, **1** (1851), 295-305.

[19] M. Tucsnak and G. Weiss, *Observation and control for operators semigroups*, Birkhäuser. 2009.

[20] M. Tucsnak and G. Weiss, How to get a conservative well-posed linear system out of thin air. Part II. Controllability and stability, *SIAM J. Control and optimization.*, **0** (2003), 1-29.

[21] M. Tucsnak and G. Weiss, How to get a conservative well-posed linear system out of thin air. Part I. Well-posedness and energy balance, *SIAM J. Control and optimization.*, **9** (2003), 247-274.

[22] G. Weiss, Transfer functions of regular linear systems. Part I: Characterizations of regularity, *Trans. Amer. Math. Soc.*, **342** (1994), 827-854.

[23] G. Weiss, Regular linear systems with feedback, *Math. Control Signals Systems.*, **7** (1994), 23-57.

[24] G. Weiss, Admissibility of unbounded control operators, *SIAM J. Control and Optim.*,**27**(1989), 527-545.

[25] G. Weiss, Admissible observation operators for linear semigroups, *Israel J. Math.*, **65** (1989), 17-43.

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