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# Codes on $s$-periodic random error of length $b$

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**Abstract** In coding theory, several kinds of errors which depend on the nature of the communication channel are to be dealt with and codes are constructed to detect and correct such errors. This paper considers a new kind of error which will be termed as '*$s$-periodic random error of length $b$*'. Linear codes that can detect such errors have been studied. Further, codes capable of correcting such errors have also been dealt with. The paper obtains lower and upper bounds on the number of parity-check digits required for such codes. An example of such a code is also provided.

## 1 Introduction

Detection and correction of errors (which occur during the process of communication) is one of the most important investigations in coding theory. In this direction, numerous works were done by various mathematicians/researchers and codes (like Hamming codes, Golay codes, BCH codes) were developed to combat different types of errors. They have been found applications in numerous areas of practical interest. There is a long history towards the growth of the subject. It is well known that the nature of errors differs from channel to channel depending upon the behaviour of channels. Though the nature of errors are broadly classified into two categories- random and burst, there are channels where the nature of occurrence of error is periodic. It was in this spirit that the codes detecting and correcting such errors were studied by Tyagi and Das [1, 2, 3, 9]. These studies are confined to the situation when the disturbances affect periodically one digit only. But in a situation, when the periodically disturbances affect more than one digits (say $b$ number of consecutive positions), further work is required. So, there is a need to develop codes to combat such type of error. Such type of error may be termed as *periodic random error of length $b$*. Examples for channels occurring such errors are the data transmission via power lines, data channels in close distance to electronically controlled power supply units or inverters, car electric, compact discs and CD-ROM [5]; and also astrophotography [10]. An $s$-periodic random error of length $b$ may be defined as follows:

**Definition 1.1.** An $s$-periodic random error of length $b$ is an $n$-tuple whose non zero components are confined to some distinct sets of $b$ consecutive components (except the last set which may be of less than $b$ consecutive components) and sets are at a gap of $s$ positions.

For example, in a vector of length 17, 4-periodic random errors of length 3 are as follows:
101 0000 111 0000 010,
0 100 0000 011 0000 10,
00 010 0000 000 0000 1,
000 101 0000 101 0000, etc.

The bounds on the number of parity check symbols of a linear code are important from the point of efficiency of a code. The bounds provide us the knowledge of the capabilities and limitation of error correcting codes. This information indicates which problems may be virtually solved and which needs further work. This was initiated by Hamming [4] who was concerned with both code constructions and bounds. In view of this, the author gives an attempt to obtain bounds on the parity check digits for linear codes that are capable of detecting/correcting $s$-periodic random error of length $b$. The paper is organized as follows:

Section 1 i.e., the Introduction gives brief view of the importance of bounds on parity check digits of a code and the requirement for consideration of $s$-periodic random error of length $b$. Section 2 gives lower and upper bounds on the number of parity check digits of a linear code that detects such errors. It is followed by an example of such a code. Section 3 gives a bound on parity check digits of a linear code correcting such errors.

In what follows a linear code will be considered as a subspace of the space of all $n$-tuples over $GF(q)$. The distance between two vectors shall be considered in the Hamming sense. The parameters $s$ and $b$ will be known for known periodical disturbances implied by electrical machines.

## 2 Codes detecting $s$-periodic random error of length $b$

We consider linear codes over $GF(q)$ that are capable of detecting any $s$-periodic random error of length $b$. Clearly, the patterns to be detected should not be code words. In other words we consider codes that have no $s$-periodic random error of length $b$ as a code word. Firstly, we obtain a lower bound over the number of parity check digits required for such a code. The proof is based on the technique used in Theorem 4.13, Peterson and Weldon [6].

**Theorem 2.1.** *The number of parity check symbols in any $(n, k)$ linear code over $GF(q)$ that detects any s-periodic random error of length b $(n > b + s)$ must have at least*

$$(i) \qquad b\left\lceil \frac{n}{b+s} \right\rceil - 1, \qquad \text{if} \quad 0 < n \mod (b+s) < b.$$

$$(ii) \qquad b\left\lceil \frac{n}{b+s} \right\rceil, \qquad \text{if} \quad b \le n \mod (b+s) \le s+b.$$

**Proof.** The result will be proved on the basis that no detectable error vector can be a code word. Let $V$ be an $(n, k)$ linear code over GF($q$) and assume $n = \lambda(b + s) + t$, where $0 < t \le b + s$ and $\lambda$ be any positive integer.

*Case (i)* For $0 < t < b$, let $X$ be a set of all those vectors such that the non-zero components are confined to some distinct sets of $b$ consecutive positions starting from the $(t + 1)^{th}$ position, each set being at a gap of $s$ positions.

*Case (ii)* For $b \le t \le b + s$, then $t = b + \ell$, $0 \le \ell \le s$. Let $X$ be a set of all those vectors such that the non-zero components are confined to some distinct sets of $b$ consecutive positions starting from the $(\ell + 1)^{th}$ position, each set being at a gap of $s$ positions.

We claim that no two vectors of the set $X$ (in either cases) can belong to the same coset of the standard array; else a code word shall be expressible as a sum or difference of two error vectors.

Assume on the contrary that there is a pair, say $x_1, x_2$ in $X$ belonging to the same coset of the standard array. Their difference viz. $x_1 - x_2$ must be a code vector. But $x_1 - x_2$ is a vector all of whose non-zero components are confined to different sets of $b$ consecutive positions and the sets are at an interval of $s$ positions i.e., $x_1 - x_2$ is an $s$-periodic random error of length $b$, which is a contradiction. Thus all the vectors in $X$ must belong to distinct cosets of the standard array. The number of such vectors over $GF(q)$, including the vector of all zero, is clearly given by

$$(i) \qquad (q^b)^\lambda, \qquad \text{for case } (i)$$

$$(ii) \qquad (q^b)^{\lambda+1}, \qquad \text{for case } (ii).$$

The number of available cosets is $q^{n-k}$. Therefore, we must have

$$q^{n-k} \ge \begin{cases} (q^b)^\lambda, & \text{for case } (i) \\ (q^b)^{\lambda+1}, & \text{for case } (ii). \end{cases}$$

or,

$$n - k \ge \begin{cases} b\lambda, & \text{for case } (i) \\ b(\lambda+1), & \text{for case } (ii). \end{cases}$$

i.e.,

$$n - k \ge \begin{cases} b\left\lceil \frac{n}{b+s} \right\rceil - 1, & \text{for} \quad 0 < n \mod (b+s) < b. \\ b\left\lceil \frac{n}{b+s} \right\rceil, & \text{for} \quad b \le n \mod (b+s) \le b+s. \end{cases} \qquad (2.1)$$

□

**Remark 2.2.** For $b$ = 1, the bound (2.1) reduces to

$$n-k \geq \left\lceil \frac{n}{s+1} \right\rceil,$$

which coincides with the lower bound for the existence of a code detecting all $s$-periodic errors (refer Theorem 2.1, Das and Tyagi [2]).

In the following theorem, an upper bound on the number of check digits required for the construction of a linear code considered in Theorem 2.1 is provided. This bound assures the existence of a linear code that can detect all $s$-periodic random errors of length $b$. The proof is based on the well known technique used in Varshomov-Gilbert Sacks bound by constructing a parity check matrix for such a code (refer Sacks [8], also Theorem 4.7 Peterson and Weldon [6]).

**Theorem 2.3.** *For given positive integers s and b, there exists an $(n, k)$ linear code over $GF(q)$ that has no s-periodic random error of length b $(n > b + s)$ as a code word provided that*

$$n-k \geq \begin{cases} b\left(\left\lceil \frac{n}{b+s} \right\rceil - 1\right) + n \mod (b+s), & \text{if } \; 0 < n \mod (b+s) < b. \\ b\left\lceil \frac{n}{b+s} \right\rceil, & \text{if } \; b \leq n \mod (b+s) \leq b+s. \end{cases} \tag{2.2}$$

**Proof.** The existence of such a code will be shown by constructing an appropriate $(n-k) \times n$ parity-check matrix $H$. The requisite parity-check matrix $H$ shall be constructed as follows:

Select any non-zero $(n-k)$-tuples as the first $n-1$ columns $h_1$, $h_2$,..., $h_{n-1}$ appropriately, we lay down the condition to add $n^{th}$ column as follows.

*Case* $(i)$. If $n = \lambda(b+s)+t$ and $0 < t < b$ (as in Theorem 2.1).
Then $h_n$ should not be a linear combination of immediately preceding consecutive $b-1$ columns, together with previous sets of $b$ consecutive columns after a gap of $s$ positions each. In other words,

$$\begin{aligned} h_n \neq & \sum_{i=1}^{b-1} u_{n-i}h_{n-i} \\ & + \sum_{i=0}^{b-1} u_{n-(b-1)-(s+1)-i}h_{n-(b-1)-(s+1)-i} \\ & + \sum_{i=0}^{b-1} u_{n-2(b-1)-2(s+1)-i}h_{n-2(b-1)-2(s+1)-i} \\ & \cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots \\ & + \sum_{i=0}^{b-1} u_{n-(\lambda-1)(b-1)-(\lambda-1)(s+1)-i}h_{n-(\lambda-1)(b-1)-(\lambda-1)(s+1)-i} \\ & + \sum_{i=0}^{t-1} u_{n-\lambda(b-1)-\lambda(s+1)-i}h_{n-\lambda(b-1)-(\lambda-1)(s+1)-i}, \end{aligned} \tag{2.3}$$

where $u_i$'s belong to $GF(q)$

*Case* $(ii)$. If $n = \lambda(b+s)+t$ and $b < t < b+s$ (as in Theorem 2.1). Let $t = b+\ell$ and $0 \leq \ell \leq s$.
Then $h_n$ should not be a linear combination of immediately preceding consecutive $b-1$ columns, together with previous sets of $b$ consecutive columns after a gap of $s$ positions each. In other

words,

$$
\begin{aligned}
h_n \neq & \sum_{i=1}^{b-1} u_{n-i} h_{n-i} \\
& + \sum_{i=0}^{b-1} u_{n-(b-1)-(s+1)-i} h_{n-(b-1)-(s+1)-i} \\
& + \sum_{i=0}^{b-1} u_{n-2(b-1)-2(s+1)-i} h_{n-2(b-1)-2(s+1)-i} \\
& \dots\dots\dots\dots \\
& \dots\dots\dots\dots \\
& + \sum_{i=0}^{b-1} u_{n-(\lambda-1)(b-1)-(\lambda-1)(s+1)-i} h_{n-(\lambda-1)(b-1)-(\lambda-1)(s+1)-i} \\
& + \sum_{i=0}^{b-1} u_{n-\lambda(b-1)-\lambda(s+1)-i} h_{n-\lambda(b-1)-(\lambda-1)(s+1)-i},
\end{aligned} \tag{2.4}
$$

where $u_i$'s belong to $GF(q)$.

The conditions (3) and (4) ensure that no $s$-periodic random error of length $b$ will be a code word which thereby means that the code shall be able to detect $s$-periodic random error of length $b$.

The number of ways in which the coefficients $u_i$ can be selected, including the vector of all zeros, is

$$
\begin{array}{lll}
(i) & q^{b-1}.(q^b)^{\lambda-1}.q^t = q^{\lambda b-1+t}, & \text{for case } (i) \\
(ii) & q^{b-1}.(q^b)^{\lambda} = q^{\lambda b-1+b}, & \text{for case } (ii).
\end{array}
$$

At worst, all these linear combinations might yield a distinct sum.
Therefore column $h_n$ can be added to $H$ provided that

$$
q^{n-k} > \begin{cases} q^{\lambda b-1+t}, & \text{for case } (i) \\ q^{\lambda b-1+b}, & \text{for case } (ii). \end{cases}
$$

or,

$$
n-k \geq \begin{cases} b\lambda + t, & \text{for case } (i) \\ b(\lambda+1), & \text{for case } (ii). \end{cases}
$$

i.e.,

$$
n-k \geq \begin{cases} b\left(\left\lceil \dfrac{n}{b+s} \right\rceil - 1\right) + n \mod (s+b), & \text{if } \quad 0 < n \mod (b+s) < b. \\ b\left\lceil \dfrac{n}{b+s} \right\rceil, & \text{if } \quad b \leq n \mod (b+s) \leq b+s. \end{cases}
$$

□

**Remark 2.4.** For $b$ = 1, the bound (2.2) reduces to

$$
n-k \geq \left\lceil \frac{n}{s+1} \right\rceil,
$$

which coincides with the upper bound for the existence of a code detecting all $s$-periodic errors (refer Theorem 2.2, Das and Tyagi [2]).

**Example 2.5.** Consider a $(11, 6)$ binary code with the $5 \times 11$ matrix $H$ which is constructed by the synthesis procedure given in the proof of Theorem 2.3 by taking $s = 3, b = 2, n = 11$.

$$H = \begin{bmatrix} 101010000001 \\ 010100000001 \\ 000000101001 \\ 000000010101 \\ 000000000011 \end{bmatrix}$$

The null space of this matrix can be used to correct all 3-periodic random errors of length 2. It may be verified from error pattern-syndromes table 2.4 that the syndromes of all 3-periodic random errors of length 2 are non zero, showing thereby that the code that is the null space of this matrix can detect all 3-periodic random errors of length 2.

**Table 2.4**
Error pattern - syndromes Table

| Error patterns | Syndromes | Error patterns | Syndromes |
|---|---|---|---|
| 00 000 10 000 0 | 00100 | 0 01 000 11 000 | 10110 |
| 00 000 01 000 0 | 00010 | 0 11 000 00 000 | 11000 |
| 00 000 00 000 1 | 11111 | 0 11 000 10 000 | 11010 |
| 00 000 11 000 0 | 00110 | 0 11 000 01 000 | 11100 |
| 00 000 10 000 1 | 11011 | 0 11 000 11 000 | 11110 |
| 00 000 01 000 1 | 11101 | 00 00 000 01 00 | 00010 |
| 00 000 11 000 1 | 11001 | 00 00 000 11 00 | 00011 |
| 10 000 00 000 0 | 10000 | 00 10 000 01 00 | 10010 |
| 10 000 10 000 0 | 10100 | 00 10 000 11 00 | 10110 |
| 10 000 01 000 0 | 10010 | 00 01 000 00 00 | 01000 |
| 10 000 00 000 1 | 01111 | 00 01 000 10 00 | 01100 |
| 10 000 11 000 0 | 10110 | 00 01 000 01 00 | 01010 |
| 10 000 10 000 1 | 01011 | 00 01 000 11 00 | 01011 |
| 10 000 01 000 1 | 01101 | 00 11 000 00 00 | 11000 |
| 10 000 11 000 1 | 01001 | 00 11 000 10 00 | 11100 |
| 01 000 00 000 0 | 01000 | 00 11 000 01 00 | 11010 |
| 01 000 10 000 0 | 01100 | 00 11 000 11 00 | 11011 |
| 01 000 01 000 0 | 01010 | 000 00 000 01 0 | 00001 |
| 01 000 00 000 1 | 10111 | 000 00 000 11 0 | 00011 |
| 01 000 11 000 0 | 01110 | 000 10 000 01 0 | 01001 |
| 01 000 11 000 1 | 10001 | 000 10 000 11 0 | 01011 |
| 01 000 10 000 1 | 10011 | 000 01 000 00 0 | 10000 |
| 01 000 01 000 1 | 10101 | 000 01 000 10 0 | 10010 |
| 11 000 00 000 0 | 11000 | 000 01 000 01 0 | 10001 |
| 11 000 10 000 0 | 11100 | 000 01 000 11 0 | 10011 |
| 11 000 01 000 0 | 11010 | 000 11 000 00 0 | 11000 |
| 11 000 00 000 1 | 00111 | 000 11 000 10 0 | 11010 |
| 11 000 10 000 1 | 00011 | 000 11 000 01 0 | 11001 |
| 11 000 01 000 1 | 00101 | 000 11 000 11 0 | 11011 |
| 11 000 11 000 0 | 11110 | 0000 10 000 01 | 01111 |
| 11 000 11 000 1 | 00001 | 0000 10 000 11 | 01110 |
| 0 00 000 01 000 | 00100 | 0000 01 000 10 | 00110 |
| 0 00 000 11 000 | 00110 | 0000 01 000 11 | 11010 |
| 0 10 000 01 000 | 01100 | 0000 11 000 00 | 10100 |
| 0 10 000 11 000 | 01110 | 0000 11 000 10 | 10101 |
| 0 01 000 00 000 | 10000 | 0000 11 000 01 | 01011 |
| 0 01 000 10 000 | 10010 | 0000 11 000 11 | 01010 |
| 0 01 000 01 000 | 10100 | | |

## 3 Codes correcting $s$-periodic random error of length $b$

The following theorem gives a bound on the number of parity-check digits for a linear code that corrects $s$-periodic random error of length $b$. The proof is based on the technique used to establish Reiger's bound [7] (also refer Theorem 4.15, Peterson and Weldon [6]) for correction of such errors.

**Theorem 3.1.** *The number of parity check symbols in an $(n, k)$ linear code over $GF(q)$ that corrects all $t$-periodic random errors of length b, $t = 2s + 1$ must have at least*

$$(i) \qquad b\left\lceil \frac{n}{s+b} \right\rceil - 1, \qquad \textit{if} \quad 0 < n \mod (s+b) < b.$$

$$(ii) \qquad b\left\lceil \frac{n}{s+b} \right\rceil, \qquad \textit{if} \quad b \le n \mod (s+b) \le s+b.$$

**Proof.** Any vector that has the form of an $s$-periodic random error of length $b$ can be expressible as a sum or difference of two vectors, each of which is a $t$-periodic random error of length $b$. These component vectors must belong to different cosets of the standard array, because both such errors are correctable errors. Accordingly, such a vector viz. $s$-periodic random error of length $b$ can not be a code vector. In view of Theorem 2.1, the number of parity check digits, such a code must have, is at least

$$(i) \qquad b\left\lceil \frac{n}{s+b} \right\rceil - 1, \qquad \text{if} \quad 0 < n \mod (s+b) < b.$$

$$(ii) \qquad b\left\lceil \frac{n}{s+b} \right\rceil, \qquad \text{if} \quad b \le n \mod (s+b) \le s+b. \tag{3.1}$$

□

**Remark 3.2.** For $b$ = 1, the bound (3.1) reduces to

$$n - k \ge \left\lceil \frac{n}{s+1} \right\rceil,$$

which coincides with the bound for a code correcting all $t$-periodic errors (refer Theorem 3.1, Das and Tyagi [2]).

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