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ON RECIPROCITY FORMULA OF CHARACTER DEDEKIND SUMS

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Abstract In this paper, we give a new proof for the reciprocity formula of character Dedekind sums with the help of character analogue of the Euler–MacLaurin summation formula. Moreover, we obtain some relations for the integrals having generalized Bernoulli function as integrands.

1 Introduction

For positive integer c and integer b the classical Dedekind sum $s(b,c)$, arising in the theory of Dedekind η–function, were introduced by R. Dedekind in 1892 by

$$s(b,c)=\sum_{m(\text{mod } c)}\left(\left(\frac{m}{c}\right)\right)\left(\left(\frac{bm}{c}\right)\right),$$

where the *sawtooth* function is defined by

$$((x))=\begin{cases}x-[x]-1/2, & \text{if } x\in\mathbb{R}\backslash\mathbb{Z},\\ 0, & \text{if } x\in\mathbb{Z}\end{cases}$$

with $[x]$ floor function. The most important property of Dedekind sums is the reciprocity law

$$s(b,c)+s(c,b)=-\frac{1}{4}+\frac{1}{12}\left(\frac{b}{c}+\frac{c}{b}+\frac{1}{bc}\right) \tag{1.1}$$

for $(b,c)=1$. The well–known reference for Dedekind sums is Rademacher and Grosswald [9]. Several generalizations of Dedekind sums have been defined and the corresponding reciprocity formulas have been obtained (for instance, see [1, 2, 6, 7, 9, 10]).

A character analogue of the classical Dedekind sum, called character Dedekind sum, appears in the transformation formula of a generalized Eisenstein series associated to a non-principal primitive character χ of modulus k defined by Berndt [2]. This sum is defined by

$$s\left(b,c:\chi\right)=\sum_{n=1}^{ck-1}\chi\left(n\right)\overline{B}_{1,\chi}\left(\frac{bn}{c}\right)\overline{B}_{1}\left(\frac{n}{ck}\right)$$

and possesses the reciprocity formula

$$s\left(c,b:\chi\right)+s\left(b,c:\overline{\chi}\right)=B_{1,\chi}B_{1,\overline{\chi}}, \tag{1.2}$$

whenever b and c are coprime positive integers, and either c or $b\equiv 0(\text{mod } k)$ ([2, Theorem 4]). Here $\overline{B}_p\left(x\right)$ and $\overline{B}_{p,\chi}\left(x\right)$ are the Bernoulli and the generalized Bernoulli functions (see Section 2), respectively, with $B_{p,\chi}=\overline{B}_{p,\chi}\left(0\right)$. For the proofs of (1.1) and (1.2) via Poisson summation formula and periodic Poisson summation formula, see [4].

The sum $s\left(b,c:\chi\right)$ is generalized by Cenkci et al [7] as

$$s_p\left(b,c:\chi\right)=\sum_{n=1}^{ck-1}\chi\left(n\right)\overline{B}_{p,\chi}\left(\frac{bn}{c}\right)\overline{B}_{1}\left(\frac{n}{ck}\right)$$

and the following reciprocity formula is established.

Theorem 1.1. *([7]) Let p be odd and let b, c be coprime positive integers. Let χ be a non-principal primitive character of modulus k, where k is a prime number if $(k, bc) = 1$, otherwise k is an arbitrary integer. Then*

$$\begin{aligned}&(p+1)\left(bc^{p}s_{p}(b,c:\chi)+cb^{p}s_{p}(c,b:\overline{\chi})\right)\\&=\sum_{j=0}^{p+1}\binom{p+1}{j}b^{j}c^{p+1-j}B_{j,\overline{\chi}}B_{p+1-j,\chi}+\frac{p}{k}\chi(c)\overline{\chi}(-b)\left(k^{p+1}-1\right)B_{p+1}.\end{aligned}$$

The reciprocity formulas of Dedekind sums are proved by employing various techniques and theories such as transformation formulas, residue theory, Riemann–Stieltjes integral, Franel integral, Poisson summation formula and arithmetic methods.

In this paper, we give a new proof of Theorem 1.1 for $p > 1$ by applying the character analogue of the Euler–MacLaurin summation formula to the generalized Bernoulli function, motivated by [5] and [8].

We summarize this study as follows: Section 2 is the preliminary section where we give definitions and known results needed. Section 3 is devoted to prove Theorem 1.1 for $(b, c) = q$. We will accomplish this by applying the character analogue of the Euler–MacLaurin summation formula to the generalized Bernoulli function. In Section 4, we present several relations for the integral having generalized Bernoulli functions as integrands.

2 Preliminaries

The Bernoulli polynomials $B_n(x)$ are defined by means of the generating function

$$\frac{te^{xt}}{e^{t}-1}=\sum_{n=0}^{\infty}B_{n}(x)\frac{t^{n}}{n!},\quad (|t|<2\pi)$$

and $B_n = B_n(0)$ are the Bernoulli numbers with $B_0 = 1$, $B_1 = -1/2$ and $B_{2n+1} = 0$ for $n \geq 1$. The Bernoulli functions $\overline{B}_n(x)$ are defined by

$$\overline{B}_{n}(x)=B_{n}(\{x\})\ \text{ for } n>1 \text{ and } \overline{B}_{1}(x)=((x))$$

where $\{x\}$ denotes the fractional part of a real number x.

Let χ be a primitive character of modulus k. The generalized Bernoulli polynomials $B_{n,\chi}(x)$ are defined by means of the generating function [3]

$$\sum_{a=0}^{k-1}\frac{\overline{\chi}(a)te^{(a+x)t}}{e^{kt}-1}=\sum_{n=0}^{\infty}B_{n,\chi}(x)\frac{t^{n}}{n!},\quad (|t|<2\pi/k)$$

and $B_{n,\chi} = B_{n,\chi}(0)$ are the generalized Bernoulli numbers. In particular, if χ_0 is the principal character, then $B_{n,\chi_0}(x) = B_n(x)$ for $n \geq 0$ and $B_{0,\chi}(x) = 0$ for $\chi \neq \chi_0$. The generalized Bernoulli functions $\overline{B}_{n,\chi}(x)$, are functions with period k, may be defined by ([3, Theorem 3.1])

$$\overline{B}_{m,\chi}(x)=k^{m-1}\sum_{n=1}^{k-1}\overline{\chi}(n)\overline{B}_{m}\left(\frac{n+x}{k}\right),\ m\geq 1,$$

for all real x. We will need the following properties in the sequel ([3]):

$$\overline{B}_{m,\chi}(-x)=(-1)^{m}\overline{\chi}(-1)\overline{B}_{m,\chi}(x), \tag{2.1}$$

$$B_{m,\chi}=0 \text{ when } (-1)^{m}\chi(-1)=-1, \tag{2.2}$$

$$\frac{d}{dx}\overline{B}_{m,\chi}(x)=m\overline{B}_{m-1,\chi}(x),\ m\geq 2. \tag{2.3}$$

As mentioned in introductory section, the aim of this study is to give an alternative proof of Theorem 1.1 by using character analogue of the Euler–MacLaurin summation formula. For this purpose we make use of the following theorem and lemma.

Theorem 2.1. *([3, Theorem 4.1]) (Character analogue of the Euler–MacLaurin summation formula)*

Let $f \in C^{(l+1)}[\alpha,\beta]$, $-\infty<\alpha<\beta<\infty$. Then,

$$\sum_{\alpha\leq n\leq\beta}{}' \chi(n) f(n) = \chi(-1)\sum_{j=0}^{l}\frac{(-1)^{j+1}}{(j+1)!}\left(\overline{B}_{j+1,\overline{\chi}}(\beta) f^{(j)}(\beta) - \overline{B}_{j+1,\overline{\chi}}(\alpha) f^{(j)}(\alpha)\right)$$
$$+\chi(-1)\frac{(-1)^{l}}{(l+1)!}\int_{\alpha}^{\beta}\overline{B}_{l+1,\overline{\chi}}(u) f^{(l+1)}(u)du$$

where the dash indicates that if $n=\alpha$ or $n=\beta$, only $\frac{1}{2}\chi(\alpha)f(\alpha)$ or $\frac{1}{2}\chi(\beta)f(\beta)$, respectively, is counted.

Lemma 2.2. *([7, Lemma 5.5]) Let $(b,c)=1$ with $c>0$. Let χ be a non-principal primitive character of modulus k, where k is a prime number if $(k,bc)=1$ and p is even, otherwise k is an arbitrary integer. Then*

$$\sum_{n=1}^{ck-1}\chi(n)\overline{B}_{p,\chi}\left(\frac{bn}{c}\right) = c^{1-p}\chi(c)\overline{\chi}(-b)\left(k^{p}-1\right)\overline{B}_{p}(0).$$

3 Proof of Theorem 1.1

In the sequel we assume that $\alpha=0$, $\beta=ck$ in Theorem 2.1 and $p>1$. With the aid of $\overline{B}_1(x)=x-1/2$, $0<x<1$, and Lemma 2.2, $s_p(d,c:\chi)$ can be written as

$$\begin{aligned} & ck\cdot s_p(b,c:\chi) \\ &= \sum_{n=1}^{ck-1}\chi(n)\, n\overline{B}_{p,\chi}\left(\frac{bn}{c}\right) - \frac{ck}{2}\sum_{n=1}^{ck-1}\chi(n)\overline{B}_{p,\chi}\left(\frac{bn}{c}\right) && (3.1)\\ &= \sum_{n=1}^{ck-1}\chi(n)\, n\overline{B}_{p,\chi}\left(\frac{bn}{c}\right) - \frac{ck}{2}q^{p}c^{1-p}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(-\frac{b}{q}\right)(k^{p}-1)B_{p},\ \text{for } (b,c)=q. && (3.2)\end{aligned}$$

So, it is convenient to consider the function $f(x)=x\overline{B}_{p,\chi}(xb/c)$ in Theorem 2.1. The property (2.3) entails that $f\in C^{(p-1)}[\alpha,\beta]$ and

$$\frac{d^{j}}{dx^{j}}f(x) = \frac{p!}{(p-j)!}\left(\frac{b}{c}\right)^{j} x\overline{B}_{p-j,\chi}\left(\frac{b}{c}x\right) + j\frac{p!}{(p+1-j)!}\left(\frac{b}{c}\right)^{j-1}\overline{B}_{p+1-j,\chi}\left(\frac{b}{c}x\right) \quad (3.3)$$

for $0\leq j\leq p-1$. Therefore, from Theorem 2.1 and the fact $\overline{B}_{m,\chi}(k)=\overline{B}_{m,\chi}(0)=B_{m,\chi}$ we have

$$\begin{aligned}\sum_{n=1}^{ck-1}\chi(n)\, n\overline{B}_{p,\chi}\left(n\frac{b}{c}\right) &= \chi(-1)\frac{ck}{p+1}\sum_{j=0}^{l}(-1)^{j+1}\binom{p+1}{j+1}\left(\frac{b}{c}\right)^{j} B_{j+1,\overline{\chi}}B_{p-j,\chi}\\ &+\chi(-1)\binom{p}{l+1}\left(-\frac{b}{c}\right)^{l} bc\int_{0}^{k} x\overline{B}_{l+1,\overline{\chi}}(cx)\,\overline{B}_{p-(l+1),\chi}(bx)\,dx\\ &+\chi(-1)\frac{l+1}{p-l}\binom{p}{l+1}\left(-\frac{b}{c}\right)^{l} c\int_{0}^{k}\overline{B}_{l+1,\overline{\chi}}(cx)\,\overline{B}_{p-l,\chi}(bx)\,dx \quad (3.4)\end{aligned}$$

for $1\leq l+1\leq p-1$. To evaluate the integral occurs in last row, we apply Theorem 2.1 to the generalized Bernoulli function $\overline{B}_{p+1,\chi}(bx/c)$, along with $\overline{B}_{m,\chi}(k)=B_{m,\chi}$, and use Lemma 2.2. Then it can be seen that

$$\int_{0}^{k}\overline{B}_{l+1,\overline{\chi}}(cx)\,\overline{B}_{p-l,\chi}(bx)\,dx = q^{p+1}\frac{(-1)^{l}}{\binom{p+1}{l+1}}\frac{c^{l-p}}{b^{l+1}}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(\frac{b}{q}\right)\left(k^{p+1}-1\right)B_{p+1} \quad (3.5)$$

for $(b,c)=q$ (see also [5, p. 759]).

Let $p>1$ be odd and $(b,c)=q$. Using the fact that $B_p=0$ for odd $p>1$, it follows from (3.2), (3.4) and (3.5) that

$$\begin{aligned}
ck\cdot s_p\left(b,c:\chi\right) &= \sum_{n=1}^{ck-1}\chi\left(n\right)n\overline{B}_{p,\chi}\left(\frac{bn}{c}\right)\\
&=\chi\left(-1\right)\frac{ck}{p+1}\sum_{j=1}^{l+1}\left(-1\right)^j\binom{p+1}{j}\left(\frac{b}{c}\right)^{j-1}B_{j,\overline{\chi}}B_{p+1-j,\chi}\\
&+\chi\left(-1\right)\binom{p}{l+1}\left(-\frac{b}{c}\right)^l bc\int_0^k x\overline{B}_{l+1,\overline{\chi}}\left(cx\right)\overline{B}_{p-(l+1),\chi}\left(bx\right)dx\\
&+\chi\left(-1\right)q^{p+1}\frac{c^{1-p}}{b}\frac{l+1}{p+1}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(\frac{b}{q}\right)\left(k^{p+1}-1\right)B_{p+1}. \qquad (3.6)
\end{aligned}$$

First setting $l+1=p-1$ in (3.6) and multiplying by bc^{p-1} we have

$$\begin{aligned}
bc^p s_p\left(b,c:\chi\right) &= \frac{\chi\left(-1\right)}{p+1}\sum_{j=1}^{p-1}\left(-1\right)^j\binom{p+1}{j}b^jc^{p+1-j}B_{j,\overline{\chi}}B_{p+1-j,\chi}\\
&-\chi\left(-1\right)\frac{p}{k}b^pc^2\int_0^k x\overline{B}_{p-1,\overline{\chi}}\left(cx\right)\overline{B}_{1,\chi}\left(bx\right)dx\\
&+\chi(-1)q^{p+1}\frac{p-1}{p+1}\frac{1}{k}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(\frac{b}{q}\right)(k^{p+1}-1)B_{p+1}. \qquad (3.7)
\end{aligned}$$

Now setting $l=0$ and interchanging b and c in (3.6), then multiplying by cb^{p-1} we have

$$\begin{aligned}
cb^p s_p\left(c,b:\overline{\chi}\right) &= -\overline{\chi}\left(-1\right)cb^pB_{p,\overline{\chi}}B_{1,\chi}\\
&+\overline{\chi}\left(-1\right)\frac{p}{k}b^pc^2\int_0^k x\overline{B}_{p-1,\overline{\chi}}\left(cx\right)\overline{B}_{1,\chi}\left(bx\right)dx\\
&+\overline{\chi}(-1)q^{p+1}\frac{1}{p+1}\frac{1}{k}\overline{\chi}\left(\frac{b}{q}\right)\chi\left(\frac{c}{q}\right)(k^{p+1}-1)B_{p+1}. \qquad (3.8)
\end{aligned}$$

Combining (3.7) and (3.8), with the help of (2.1), we arrive at the following reciprocity formula

$$\begin{aligned}
&\left(p+1\right)\left(bc^ps_p(b,c:\chi)+cb^ps_p(c,b:\overline{\chi})\right)\\
&=\sum_{j=0}^{p+1}\binom{p+1}{j}b^jc^{p+1-j}B_{j,\overline{\chi}}B_{p+1-j,\chi}+q^{p+1}\frac{p}{k}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(-\frac{b}{q}\right)\left(k^{p+1}-1\right)B_{p+1} \qquad (3.9)
\end{aligned}$$

for $(b,c)=q$ under the assumptions of Theorem 1.1.

Therefore, Theorem 1.1 is a direct consequence of (3.9).

4 Further consequences

We consider the function $f(x)=\overline{B}_p\left(xb/kc\right)$ in Theorem 2.1 with $\alpha=0$, $\beta=ck$ and $p>2$. Using the property

$$\frac{d}{dx}\overline{B}_p\left(x\right)=p\overline{B}_{p-1}\left(x\right),\ p>2$$

we find that

$$\sum_{n=1}^{ck}\chi\left(n\right)\overline{B}_p\left(\frac{bn}{ck}\right)=\chi\left(-1\right)\left(-1\right)^l\binom{p}{l+1}\left(\frac{b}{ck}\right)^{l+1}ck\int_0^1\overline{B}_{l+1,\overline{\chi}}\left(ckx\right)\overline{B}_{p-(l+1)}\left(bx\right)dx$$

for $0\le l+1\le p-2$. On the other hand, using Raabe formula

$$\sum_{m=0}^{c-1}\overline{B}_p\left(\frac{m+x}{c}\right)=c^{1-p}\overline{B}_p\left(x\right),$$

we deduce for $b=b_1q$, $c=c_1q$, $(b_1,c_1)=1$ and $(b,k)=1$ that

$$\begin{aligned}\sum_{n=1}^{ck}\chi(n)\overline{B}_p\left(\frac{bn}{ck}\right) &= \sum_{h=1}^{k}\sum_{j=0}^{c-1}\chi(h)\overline{B}_p\left(\frac{bh}{ck}+\frac{b_1}{c_1}j\right)\\ &= q\left(c_1\right)^{1-p}\sum_{h=1}^{k}\chi(h)\overline{B}_p\left(\frac{b_1h}{k}\right)\\ &= q\left(c_1\right)^{1-p}\overline{\chi}(b_1)\sum_{h=1}^{k}\chi(b_1h)\overline{B}_p\left(\frac{b_1h}{k}\right)\\ &= \frac{q^p}{(ck)^{p-1}}\overline{\chi}\left(\frac{b}{q}\right)B_{p,\overline{\chi}}.\end{aligned}$$

Then we obtain

$$\int_0^1 \overline{B}_{m,\overline{\chi}}\left(ckx\right)\overline{B}_r\left(bx\right)dx=\chi\left(-1\right)\left(-1\right)^{m-1}\frac{m!r!}{(m+r)!}\frac{q^{m+r}}{b^m\left(ck\right)^r}\overline{\chi}\left(\frac{b}{q}\right)B_{m+r,\overline{\chi}} \tag{4.1}$$

for $(b,c)=q$ and $(b,k)=1$ (where $l+1=m\geq 1$ and $p-m=r>1$).

Notice that (4.1) reduces to Berndt's result [3, Proposition 6.7] for $b=c=1$.

Finally we present a few further consequences of (3.4) for even $p\geq 2$. In this case we use that

$$s_p\left(b,c:\chi\right)=0, \tag{4.2}$$

which is obvious from the definition of $s_p\left(b,c:\chi\right)$. First consider $b=ck$. Taking into consideration that

$$\int_0^k \overline{B}_{l+1,\overline{\chi}}\left(x\right)\overline{B}_{p-l,\chi}\left(kx\right)dx=0,$$

which follows from (3.5), we conclude from (2.2), (3.1), (4.2) and (3.4) that

$$\int_0^k x\overline{B}_{l+1,\overline{\chi}}\left(x\right)\overline{B}_{p-l-1,\chi}\left(kx\right)dx=0.$$

Now let $(b,c)=q$. Then, it is seen from (3.2) and (4.2) that

$$\sum_{n=1}^{ck-1}\chi\left(n\right)n\overline{B}_{p,\chi}\left(\frac{bn}{c}\right)=q\frac{ck}{2}\left(\frac{c}{q}\right)^{1-p}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(-\frac{b}{q}\right)(k^p-1)B_p. \tag{4.3}$$

Combining (3.4), (3.5) and (4.3), and using (2.2) we find that

$$\int_0^k x\overline{B}_{l+1,\overline{\chi}}\left(cx\right)\overline{B}_{p-l-1,\chi}\left(bx\right)dx=\frac{k}{2}q^p\frac{\left(-1\right)^l}{\binom{p}{l+1}}\frac{c^{l+1-p}}{b^{l+1}}\chi\left(\frac{c}{q}\right)\overline{\chi}\left(\frac{b}{q}\right)(k^p-1)B_p \tag{4.4}$$

for k as in Lemma 2.2. In particular

$$\int_0^k x\overline{B}_{l+1,\overline{\chi}}\left(x\right)\overline{B}_{p-l-1,\chi}\left(x\right)dx=\left(-1\right)^l\frac{k}{2}\binom{p}{l+1}^{-1}(k^p-1)B_p.$$

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