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THE GROWTH RATE OF χ^2 DEFINED BY MODULUS FUNCTION

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Abstract In this paper we introduce the growth rate of χ^2 defined by modulus function and study general properties of these spaces and also establish some inclusion results and duals among them..

1 Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [3]. Later on, they were investigated by Hardy [4], Moricz [8], Moricz and Rhoades [9], Basarir and Solankan [1], Tripathy [15], Turkmenoglu [16], Tripathy [40], Tripathy and Dutta([45],[56]), Tripathy and Sarma ([48],[50],[52]) and many others.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - L|^{t_{mn}} = 1 \, for \, some \, L \in \mathbb{C} \right\},$$
$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$
$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$
$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [18,19] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and calculated the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, Zelter [20] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] and Tripathy [15] have recently introduced the notion of statistical convergence and statistically Cauchy for double sequences independently and proved a relation between statistical convergent and strongly Cesàro summable double sequences. Mursaleen [22] and Mursaleen and Edely [23] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. Altay and Basar [24] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [25] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Subramanian and Misra [26] have studied the space $\chi^2_M(p, q, u)$ of double sequences and proved some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [28], Maddox [29], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [7] as an extension of the definition of strongly Cesàro summable sequences. Connor [30] further extended this definition and introduced the notion of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [31] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [32]-[33], and [34] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1,1}^{m,n} x_{ij} (m, n \in \mathbb{N})$.

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz[11] used the idea of Orlicz function to construct the space $\left(L^M\right)$. Lindenstrauss and Tzafriri [6] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \,(1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [12], Mursaleen et al., Bektas and Altin [2], Tripathy et al. [43], Rao and Subramanian [13], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [5].

Recalling [11] and [5], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function M is replaced by subadditivity of M, then this function is called the modulus function, defined by Nakano [10] and further discussed by Ruckle [14] and Maddox [7], Tripathy and Chandra [53] and many others.

An modulus function M is said to satisfy the Δ_2- condition for small u or at 0 if for each $k \in \mathbb{N}$, there exist $R_k > 0$ and $u_k > 0$ such that $M(ku) \leq R_k M(u)$ for all $u \in (0, u_k]$. Moreover, an modulus function M is said to satisfy the Δ_2- condition if and only if

$$lim_{u \to 0+} sup \frac{M(2u)}{M(u)} < \infty$$

Two Modulus functions M_1 and M_2 are said to be equivalent if there are positive constants α, β and b such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \text{ for all } u \in [0, b] .$$

An modulus function M can always be represented in the following integral form

$$M(u) = \int_0^u \eta(t)\, dt,$$

where η, the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$ for $t > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$ whenever $\frac{M(u)}{u} \uparrow \infty$ as $u \uparrow \infty$.

Consider the kernel η associated with the modulus function$M$ and let

$$\mu(s) = sup\{t : \eta(t) \leq s\}.$$

Then μ possesses the same properties as the function η. Suppose now

$$\Phi = \int_0^x \mu(s)\, ds.$$

Then, Φ is an modulus function. The functions M and Φ are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let M and Φ be mutually complementary modulus functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's \quad inequality) \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \, (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

If X is a sequence space, we procure the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \quad for \quad each \quad x \in X\right\}$;

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is \quad convegent, \quad for \quad each \quad x \in X\right\}$;

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn} \geq 1 \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \quad for \quad each \quad x \in X\right\}$;

(v)$let \quad X \quad be \quad an \quad FK-space \quad \supset \phi; \quad then \quad X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \quad for \quad each \quad x \in X\right\}$;

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha-\quad(or \quad Köthe-Toeplitz)$ dual of $X, \beta-(or \quad generalized - Köthe-Toeplitz) \quad dual \quad of X, \gamma- \quad dual \quad of \quad X, \quad \delta \quad - \quad dual \quad of X \quad respectively. X^{\alpha}$ is found in Gupta and Kamptan [17]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [27] as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay in [37] and in the case $0 < p < 1$ by Altay and Başar in [38]. The spaces $c(\Delta), c_0(\Delta), \ell_\infty(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{m,n} - x_{m,n+1}) - (x_{m+1,n} - x_{m+1,n+1}) = x_{m,n} - x_{m,n+1} - x_{m+1,n} + x_{m+1,n+1}$ for all $m, n \in \mathbb{N}$.

2 Definitions and Preliminaries

2.1 Definition

A modulus function was introduced by Nakano [10]. We recall that a modulus f is a function from $[0,\infty) \to [0,\infty)$, such that
(1) $f(x) = 0$ if and only if $x = 0$
(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x-y|)$, it follows from here that f is continuous on $[0,\infty)$.

2.2 Definition

Let $A = \left(a^{mn}_{k,\ell}\right)$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the $k,\ell-$ th term to Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell} x_{mn}$$

such transformation is said to be nonnegative if $a^{mn}_{k\ell}$ is nonnegative.

The notion of regularity for two dimensional matrix transformations was presented by Silverman [35] and Toeplitz [36]. Following Silverman and Toeplitz, Robison and Hamilton presented the following four dimensional analogue of regularity for double sequences in which they both added an adiditional assumption of boundedness. This assumption was made because a double sequence which is $P-$ convergent is not necessarily bounded.

2.3 Definition

A sequence t is called a growth gai sequence of modulus, for a set X of sequences if $x_{mn} = o(t_{mn}) \Leftrightarrow f\left((m+n)!\left|\frac{x_{mn}}{t_{mn}}\right|\right)^{1/m+n} \to 0 \ \ as \ \ m,n \to \infty.$

2.4 Definition

A sequence t is called a growth analytic sequence of modulus, for a set X of sequences if $x_{mn} = O(t_{mn}) \Leftrightarrow f\left(\left|\frac{x_{mn}}{t_{mn}}\right|\right)^{1/m+n} < \infty \ \ \forall m,n.$

3 Main Results

3.1 Theorem

If χ^2 has a growth sequence of modulus then $\chi^2_{f\pi}$ has a growth sequence of modulus

Proof: Let $\chi^2_{f\pi}$ be a growth sequence of modulus. Then $f\left((m+n)!\left|\frac{x_{mn}}{t_{mn}}\right|\right)^{1/m+n} \to 0 \ \ as \ \ m,n \to \infty$. Let $x \in \chi^2_{f\pi}$. Then $\left\{\frac{x_{mn}}{\pi_{mn}}\right\} \in \chi^2_f$. We have $f\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \leq f\left((m+n)!\,|x_{mn}|\right)^{1/m+n} \leq |\pi_{mn}t_{mn}| \to 0 \ \ as \ \ m,n \to \infty$, which means that
$f\left((m+n)!\,|x_{mn}|\right)^{1/m+n} \leq |\pi_{mn}t_{mn}| \to 0 \ \ as \ \ m,n \to \infty$. Thus $\{\pi_{mn}t_{mn}\}$ is a growth sequence for $\chi^2_{f\pi}$. In other words, $\chi^2_{f\pi}$ has the growth sequence πt.

3.2 Theorem

Let χ^2_f be a BK metric space. Then the rate space $\chi^2_{f\pi}$ has a growth sequence of modulus.

Proof: Let $x \in \chi^2_{f\pi}$. Then $\left\{\frac{x_{mn}}{\pi_{mn}}\right\} \in \chi^2_f$.

Put $P_{mn}(x) = \frac{x_{mn}}{\pi_{mn}} \forall x \in \chi^2_{f\pi}$. Then P_{mn} is a continuous functional on $\chi^2_{f\pi}$. Hence $|P_{mn}| \to 0$ as $m, n \to \infty$.
Also for every positive integer m, n, we have $f\left((m+n)! \left|\frac{x_{mn}}{\pi_{mn}}\right|\right)^{1/m+n} = |P_{mn}(x)| \leq$
$|P_{mn}(x)\pi_{mn}| f\left((m+n)! |x_{mn}|\right)^{1/m+n} \to 0 \ as \ m, n \to \infty$. Hence $x_{mn} = o\left(P_{mn}\pi_{mn}\right)$. Thus $\{P_{mn}\pi_{mn}\}$ is a growth sequence for $\chi^2_{f\pi}$.

3.3 Theorem

$\left(\chi^2_\pi\right)^\alpha = \Lambda^2_{1/\pi}$

Proof: Let $x \in \Lambda^2_{1/\pi}$. Then there exists $M > 0$ with $|\pi_{mn} x_{mn}| \leq M^{m+n} \quad \forall m, n \geq 1$. Choose $\epsilon > 0$ such that $\epsilon M < 1$.
If $y \in \chi^2_\pi$, we have $\left((m+n)! \left|\frac{y_{mn}}{\pi_{mn}}\right|\right) \leq \epsilon^{m+n} \quad \forall m, n \geq m_0 n_0$ depending on ϵ.
Therefore $\sum |x_{mn} y_{mn}| \leq \sum \frac{(M\epsilon)^{m+n}}{(m+n)!} < \infty$, Hence

$$\Lambda^2_{1/\pi} \subset \left(\chi^2_\pi\right)^\alpha \tag{3.1}$$

On the other hand, let $x \in \left(\chi^2_\pi\right)^\alpha$. Assume that $x \notin \Lambda^2_{1/\pi}$. Then there exists an increasing sequence $\{p_{mn} q_{mn}\}$ of positive integers such that
$|\pi_{p_{mn} q_{mn}} x_{p_{mn} q_{mn}}| > (m+n)! (m+n)^{2(p_{mn}+q_{mn})} \forall m, n > m_0 n_0$. Take Take $y = \{y_{mn}\}$ by

$$y_{mn} = \begin{cases} \frac{\pi_{mn}}{(m+n)!(m+n)^{2(p_{mn}+q_{mn})}}, & \text{for } (p,q) = (p_m, q_n) \\ 0, & \text{for } (p, q \neq p_m q_n) \end{cases} \tag{3.2}$$

Then $\{y_{mn}\} \in \chi^2_\pi$, but $\sum |x_{mn} y_{mn}| = \infty$, a contradiction. This contradiction shows that

$$\left(\chi^2_\pi\right)^\alpha \subset \Lambda^2_{1/\pi} \tag{3.3}$$

From (3.1) and (3.3) it follows that $\left(\chi^2_\pi\right)^\alpha = \Lambda^2_{1/\pi}$.

3.4 Theorem

$\left[\Lambda^2_{f\pi}\right]^\beta = \left[\Lambda^2_{f\pi}\right]^\alpha = \left[\Lambda^2_{f\pi}\right]^\gamma = \eta^2_{M\pi}$,
where $\eta^2_M = \bigcap_{N \in N - \{1\}} \left\{x = x_{mn} : \sum_{m,n} \left(f\left(\frac{\left|\frac{x_{mn}}{\pi_{mn} t_{mn}}\right| N^{m+n}}{\rho}\right)\right) < \infty\right\}$.

Proof (1) First we show that $\eta^2_{f\pi} \subset \left[\Lambda^2_{f\pi}\right]^\beta$.
Let $x \in \eta^2_{f\pi}$ and $y \in \Lambda^2_{f\pi}$. Then we can find a positive integer N such that $\left(|y_{mn}|^{1/m+n}\right) < max\left(1, sup_{m,n \geq 1}\left(|y_{mn}|^{1/m+n}\right)\right) < N$, for all m, n.

Hence we may write

$\left|\sum_{m,n} x_{mn} y_{mn}\right| \leq \sum_{m,n} |x_{mn} y_{mn}| \leq \sum_{mn} \left(f\left(\frac{|x_{mn} y_{mn}|}{\rho}\right)\right)$
$\leq \sum_{m,n} \left(f\left(\frac{\left|\frac{x_{mn}}{\pi_{mn} t_{mn}}\right| N^{m+n}}{\rho}\right)\right)$.
Since $x \in \eta^2_{f\pi}$. the series on the right side of the above inequality is convergent, whence $x \in \left[\Lambda^2_{f\pi}\right]^\beta$. Hence $\eta^2_{f\pi} \subset \left[\Lambda^2_{f\pi}\right]^\beta$.

Now we show that $\left[\Lambda^2_{f\pi}\right]^\beta \subset \eta^2_{f\pi}$.

For this, let $x \in \left[\Lambda^2_{f\pi}\right]^\beta$, and suppose that $x \notin \Lambda^2_{f\pi}$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n} \left(f\left(\frac{\left|\frac{x_{mn}}{\pi_{mn} t_{mn}}\right| N^{m+n}}{\rho}\right)\right) = \infty$.

If we define $y_{mn} = \left(N^{m+n}/\pi_{mn} t_{mn}\right) Sgn\left(x_{mn}\right) \ m, n = 1, 2, \cdots$, then $y \in \Lambda^2_{f\pi}$.

But, since

$\left|\sum_{m,n} x_{mn} y_{mn}\right| = \sum_{mn}\left(f\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right) = \sum_{m,n}\left(f\left(\frac{|x_{mn}|\left(N^{m+n}/\pi_{mn}t_{mn}\right)}{\rho}\right)\right) = \infty$, we get $x \notin \left[\Lambda^2_{f\pi}\right]^\beta$, which contradicts to the assumption $x \in \left[\Lambda^2_{f\pi}\right]^\beta$. Therefore $x \in \eta^2_{f\pi}$. Therefore $\left[\Lambda^2_{f\pi}\right]^\beta = \eta^2_{f\pi}$.

(ii)and (iii) can be shown in a similar way of (i). Therefore we omit it.

3.5 Theorem

Let f be an modulus function which satisfies the Δ_2-condition and if $\chi^2_{f\pi}$ is a growth sequence then $\chi^2_\pi \subset \chi^2_{f\pi}$

Proof Let

$$x \in \chi^2_\pi. \tag{3.4}$$

Then $\left(\left((m+n)!\,|x_{mn}/\pi_{mn}t_{mn}|\right)^{1/m+n}\right) \leq \epsilon$ for sufficiently large m, n and every $\epsilon > 0$.
But then by taking $\rho \geq 1/2$,

$$\left(f\left(\frac{\left((m+n)!|x_{mn}/\pi_{mn}t_{mn}|\right)^{1/m+n}}{\rho}\right)\right) \leq \left(f\left(\frac{\epsilon}{\rho}\right)\right) \text{ (because } M \text{ is non-decreasing)}$$

$$\leq \left(f\left(2\epsilon\right)\right)$$

$$\Rightarrow \left(f\left(\frac{\left(|x_{mn}/\pi_{mn}t_{mn}|\right)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \leq Kf\left(\epsilon\right) \text{ (by the } \Delta_2- \text{ condition, for some } k > 0)$$

$$\leq \epsilon \text{ (by defining } f\left(\epsilon\right) < \epsilon/K)$$

$$\left(f\left(\frac{\left(|x_{mn}/\pi_{mn}t_{mn}|\right)^{1/m+n}}{\rho}\right)\right)^{p_{mn}} \to 0\, as\, m, n \to \infty. \tag{3.5}$$

Hence

$$x \in \chi^2_{f\pi}. \tag{3.6}$$

From (3.4) and (3.6) we get $\chi^2_\pi \subset \chi^2_{f\pi}$.

3.6 Theorem

If $\chi^2_{f\pi}$ is a growth sequence then $\eta^2_{f\pi} \subset \left[\chi^2_{f\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$

ProofCase1: First we show that $\eta^2_{f\pi} \subset \left[\chi^2_{f\pi}\right]^\beta$.

We know that $\chi^2_{f\pi} \subset \Lambda^2_{f\pi}$.

$\left[\Lambda^2_{f\pi}\right]^\beta \subset \left[\chi_{f\pi}\right]^\beta$. But $\left[\Lambda^2_{f\pi}\right]^\beta = \eta^2_{f\pi}$, by Theorem 3.4.

Therefore

$$\eta^2_{f\pi} \subset \chi^2_{f\pi}. \tag{3.7}$$

case2: Now we show that $\left[\chi^2_{f\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$.

Let $y = \{y_{mn}\}$ be an arbitrary point in $\left(\chi^2_{f\pi}\right)^\beta$. If y is not in Λ^2_π, then for each natural number q, we can find an index $m_q n_q$ such that

$$\left(f\left(\frac{\left((m_q+n_q)!\left|y_{m_q n_q}/\pi_{m_q n_q}t_{m_q n_q}\right|\right)^{1/m_q+n_q}}{\rho}\right)\right) > q, (1, 2, 3, \cdots)$$

Define $x = \{x_{mn}\}$ by $\left(f\left(\frac{(x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = \frac{1}{(m+n)!q^{m+n}}$ for $(m, n) = (m_q, n_q)$ for some $q \in \mathbb{N}$; and $\left(f\left(\frac{x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$ otherwise.

Then x is in $\chi^2_{f\pi}$, but for infinitely mn,

$$\left(f\left(\frac{|y_{mn}x_{mn}|}{\rho}\right)\right)^{p_{mn}} > 1. \tag{3.8}$$

Consider the sequence $z = \{z_{mn}\}$, where $\left(f\left(\frac{2!z_{11}/\pi_{11}t_{11}}{\rho}\right)\right) = \left(f\left(\frac{2!x_{11}/\pi_{11}t_{11}}{\rho}\right)\right) - s$ with $s = \sum\left(f\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)$; and
$\left(f\left(\frac{(m+n)!z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = \left(f\left(\frac{(m+n)!x_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right)(m,n = 1,2,3,\cdots)$

Then z is a point of $\chi^2_{f\pi}$. Also $\sum\left(f\left(\frac{(m+n)!z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$. Hence z is in $\chi^2_{f\pi}$.

But, by the equation (3.8), $\sum\left(M\left(\frac{z_{mn}y_{mn}}{\rho}\right)\right)$ does not converge.

$\Rightarrow \sum x_{mn}y_{mn}$ diverges.

Thus the sequence y would not be in $\left(\chi^2_{f\pi}\right)^\beta$. This contradiction proves that

$$\left(\chi^2_{f\pi}\right)^\beta \subset \Lambda^2_\pi. \tag{3.9}$$

If we now choose $f = id$, where id is the identity and $(1+n)!y_{1n}/\pi_{1n}t_{1n} = (1+n)!x_{1n}/\pi_{1n}t_{1n} = 1$ and $(m+n)!y_{mn}/\pi_{mn}t_{mn} = (m+n)!x_{mn}/\pi_{mn}t_{mn} = 0\,(m>1)$ for all n, then obviously $x \in \chi^2_{f\pi}$ and $y \in \Lambda^2_\pi$, but $\sum_{m,n=1}^\infty x_{mn}y_{mn} = \infty$, hence

$$y \notin \left(\chi^2_{f\pi}\right)^\beta. \tag{3.10}$$

From (3.9) and (3.10) we are granted

$$\left(\chi^2_{f\pi}\right)^\beta \overset{\subset}{\neq} \Lambda^2_\pi. \tag{3.11}$$

Hence (3.7)and (3.11)we are granted $\eta^2_{f\pi} \subset \left[\chi^2_{f\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$.

3.7 Proposition

$\chi^2_{f\pi} \subset \Gamma^2_{f\pi}$

Proof: Let $x \in \chi^2_{f\pi}$.

Then we have $\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \to 0\, as\, m,n \to \infty$.

Here, we get $\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|^{1/m+n} \to 0\, as\, m,n \to \infty$. Thus we have $x \in \Gamma^2_{f\pi}$ and so $\chi^2_{f\pi} \subset \Gamma^2_{f\pi}$.

3.8 Theorem

If $\Gamma^2_{f\pi}$ is a growth sequence then $\eta^2_{f\pi} \subset \left[\Gamma^2_{f\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$

ProofCase1: First we show that $\eta^2_{f\pi} \subset \left[\Gamma^2_{f\pi}\right]^\beta$.

We know that $\Gamma^2_{f\pi} \subset \Lambda^2_{f\pi}$.

$\left[\Lambda^2_{f\pi}\right]^\beta \subset \left[\Gamma_{f\pi}\right]^\beta$. But $\left[\Lambda^2_{f\pi}\right]^\beta = \eta^2_{f\pi}$, by Theorem 3.4.

Therefore

$$\eta^2_{f\pi} \subset \Gamma^2_{f\pi}. \tag{3.12}$$

case2: Now we show that $\left[\Gamma^2_{f\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$.

Let $y = \{y_{mn}\}$ be an arbitrary point in $\left(\Gamma^2_{f\pi}\right)^\beta$. If y is not in Λ^2_π, then for each natural number q, we can find an index $m_q n_q$ such that

$$\left(f\left(\frac{\left(\left|y_{m_q n_q}/\pi_{m_q n_q}t_{m_q n_q}\right|\right)^{1/m_q+n_q}}{\rho}\right)\right) > q, (1,2,3,\cdots)$$

Define $x = \{x_{mn}\}$ by $\left(f\left(\frac{(x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = \frac{1}{q^{m+n}}$ for $(m,n) = (m_q, n_q)$ for some $q \in \mathbb{N}$; and $\left(f\left(\frac{x_{mn}\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$ otherwise.

Then x is in $\Gamma^2_{f\pi}$, but for infinitely mn,

$$\left(f\left(\frac{|y_{mn}x_{mn}|}{\rho}\right)\right)^{p_{mn}} > 1. \tag{3.13}$$

Consider the sequence $z = \{z_{mn}\}$, where $\left(f\left(\frac{z_{11}/\pi_{11}t_{11}}{\rho}\right)\right) = \left(f\left(\frac{x_{11}/\pi_{11}t_{11}}{\rho}\right)\right) - s$ with $s = \sum\left(f\left(\frac{(m+n)!x_{mn}}{\rho}\right)\right)$; and
$\left(f\left(\frac{z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = \left(f\left(\frac{x_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right)(m, n = 1, 2, 3, \cdots)$

Then z is a point of $\Gamma^2_{f\pi}$. Also $\sum\left(f\left(\frac{z_{mn}/\pi_{mn}t_{mn}}{\rho}\right)\right) = 0$. Hence z is in $\Gamma^2_{f\pi}$.

But, by the equation (3.13), $\sum\left(f\left(\frac{z_{mn}y_{mn}}{\rho}\right)\right)$ does not converge.

$\Rightarrow \sum x_{mn}y_{mn}$ diverges.

Thus the sequence y would not be in $\left(\Gamma^2_{fM\pi}\right)^\beta$. This contradiction proves that

$$\left(\Gamma^2_{M\pi}\right)^\beta \subset \Lambda^2_\pi. \tag{3.14}$$

If we now choose $M = id$, where id is the identity and $y_{1n}/\pi_{1n}t_{1n} = x_{1n}/\pi_{1n}t_{1n} = 1$ and $y_{mn}/\pi_{mn}t_{mn} = x_{mn}/\pi_{mn}t_{mn} = 0\,(m > 1)$ for all n, then obviously $x \in \Gamma^2_{M\pi}$ and $y \in \Lambda^2_\pi$, but $\sum_{m,n=1}^\infty x_{mn}y_{mn} = \infty$, hence

$$y \notin \left(\Gamma^2_{M\pi}\right)^\beta. \tag{3.15}$$

From (3.14) and (3.15) we are granted

$$\left(\Gamma^2_{M\pi}\right)^\beta \overset{\subset}{\neq} \Lambda^2_\pi. \tag{3.16}$$

Hence (3.12)and (3.16)we are granted $\eta^2_{M\pi} \subset \left[\Gamma^2_{M\pi}\right]^\beta \overset{\subset}{\neq} \Lambda^2_\pi$.

3.9 Proposition

The $\beta-$ dual space of $\chi^2_{f\pi}$ is $\Lambda^2_{f\pi}$

Proof: First, we observe that $\chi^2_{f\pi} \subset \Gamma^2_{f\pi}$, by Proposition 3.7. Theorefore $\left(\Gamma^2_{f\pi}\right)^\beta \subset \left(\chi^2_{f\pi}\right)^\beta$. But $\left(\Gamma^2_{f\pi}\right)^\beta \overset{\subset}{\neq} \Lambda^2_{f\pi}$, by Proposition 3.8. Hence

$$\Lambda^2_{f\pi} \subset \left(\chi^2_{f\pi}\right)^\beta \tag{3.17}$$

Next we show that $\left(\chi^2_{f\pi}\right)^\beta \subset \Lambda^2_{f\pi}$. Let $y = (y_{mn}) \in \left(\chi^2_{f\pi}\right)^\beta$. Consider $f(x) = \sum_{m=1}^\infty \sum_{n=1}^\infty x_{mn}y_{mn}$ with $x = (x_{mn}) \in \chi^2_{f\pi}$
$x = [(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})]$

$$= \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{\pi_{mn}t_{mn}}{(m+n)!}, & \frac{-\pi_{mn}t_{mn}}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \end{pmatrix} - \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{\pi_{mn}t_{mn}}{(m+n)!}, & \frac{-\pi_{mn}t_{mn}}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \end{pmatrix}$$

$$\left\{\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n}\right\} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{\pi_{mn}t_{mn}}{(m+n)!}, & \frac{-\pi_{mn}t_{mn}}{(m+n)!}, & ... & 0 \\ \\ 0, & 0, & ...\frac{-\pi_{mn}t_{mn}}{(m+n)!}, & \frac{\pi_{mn}t_{mn}}{(m+n)!}, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \\ \end{pmatrix}$$

. Hence converges to zero.

Therefore $[(\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1})] \in \chi^2_{f\pi}$.

Hence $d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) = 1$. But

$|y_{mn}| \le \|f\|\, d\left((\Im_{mn} - \Im_{mn+1}) - (\Im_{m+1n} - \Im_{m+1n+1}), 0\right) \le \|f\| \cdot 1 < \infty$ for each m, n. Thus (y_{mn}) is a double growth rate of an bounded sequence and hence an growth rate of an analytic sequence. In other words $y \in \Lambda^2_{f\pi}$. But $y = (y_{mn})$ is arbitrary in $\left(\chi^2_{f\pi}\right)^{\beta}$. Therefore

$$\left(\chi^2_{f\pi}\right)^{\beta} \subset \Lambda^2_{f\pi} \tag{3.18}$$

From (3.17) and (3.18) we get $\left(\chi^2_{f\pi}\right)^{\beta} = \Lambda^2_{f\pi}$.

3.10 Proposition

$\chi^2_{f\pi}$ has AK

Proof: Let $x = (x_{mn}) \in \chi^2_{f\pi}$ and take the $[m,n]^{th}$ sectional sequence of x. We have $d\left(x, x^{[r,s]}\right) = sup_{mn}\left\{\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} : m \ge r, n \ge s\right\} \to 0\, as\ [r,s] \to \infty$. Therefore $x^{[r,s]} \to x$ in $\chi^2_{f\pi}$ as $r, s \to \infty$. Thus $\chi^2_{f\pi}$ has AK.

3.11 Proposition

$\chi^2_{f\pi}$ is solid

Proof: Let $|x_{mn}| \le |y_{mn}|$ and let $y = (y_{mn}) \in \chi^2_{f\pi}$. We have $\left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \le \left((m+n)!\left|\frac{y_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n}$. But $\left((m+n)!\left|\frac{y_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \in \chi^2_{f\pi}$, because $y \in \chi^2_{f\pi}$. That is $\left((m+n)!\left|\frac{y_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \to 0 \Rightarrow \left((m+n)!\left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n} \to 0\, as\, m, n \to \infty$. Therefore $x = (x_{mn}) \in \chi^2_{f\pi}$.

3.12 Proposition

$\Lambda-$ dual of $\chi^2_{f\pi}$ is $\Lambda^2_{f\pi}$

Proof: Let $y \in \Lambda-$ dual of $\chi^2_{f\pi}$. Then $|x_{mn}y_{mn}| \le M^{m+n}$ for some constant $M > 0$ and for each $x \in \chi^2_{f\pi}$. Therefore $|y_{mn}| \le M^{m+n}$ for each m, n by taking

$$x = \Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{\pi_{mn}t_{mn}}{(m+n)!}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \\ \end{pmatrix}.$$

This shows that $y \in \Lambda^2_{f\pi}$. Then

$$\left(\chi^2_{f\pi}\right)^{\Lambda} \subset \Lambda^2_{f\pi} \tag{3.19}$$

On the other hand, let $y \in \Lambda^2_{f\pi}$. Let $\epsilon > 0$ be given. Then $|y_{mn}| < M^{m+n}$ for each m, n and for some constant $M > 0$. But $x \in \chi^2_{f\pi}$. Hence $\left((m+n)! \left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right) < \epsilon^{m+n}$ for each m, n and for each $\epsilon > 0$. i.e $|x_{mn}| < \frac{\epsilon^{m+n}\pi_{mn}t_{mn}}{(m+n)!}$. Hence

$$|x_{mn}y_{mn}| = |x_{mn}|\,|y_{mn}| < \frac{\epsilon^{m+n}\pi_{mn}t_{mn}}{(m+n)!}M^{m+n} = \frac{(\epsilon M)^{m+n}\pi_{mn}t_{mn}}{(m+n)!}$$

$$\Rightarrow y \in \left(\chi^2_{f\pi}\right)^{\Lambda}$$

$$\Lambda^2_{f\pi} \subset \left(\chi^2_{f\pi}\right)^{\Lambda} \tag{3.20}$$

From (3.19) and (3.20) we get $\left(\chi^2_{f\pi}\right)^{\Lambda} = \Lambda^2_{f\pi}$.

3.13 Proposition

Let $\left(\chi^2_{f\pi}\right)^{*}$ denote the dual space of $\chi^2_{f\pi}$. Then we have $\left(\chi^2_{f\pi}\right)^{*} = \Lambda^2_{f\pi}$.

Proof: We recall that

$$x = \Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...\frac{\pi_{mn}t_{mn}}{(m+n)!}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \end{pmatrix}.$$

with $\frac{\pi_{mn}t_{mn}}{(m+n)!}$ in the $(m,n)^{th}$ position and zero other wise, with

$$x = \Im_{mn}, \left\{\left((m+n)! \left|\frac{x_{mn}}{\pi_{mn}t_{mn}}\right|\right)^{1/m+n}\right\} =$$

$$\begin{pmatrix} 0^{1/2}, & 0, & ...0, & 0, & ... & 0^{1/1+n} \\ . \\ . \\ . \\ 0^{1/m+1}, & 0, & ...\left(\frac{(m+n)!\pi_{mn}t_{mn}}{(m+n)!\pi_{mn}t_{mn}}\right)^{1/m+n}, & 0, & ... & 0^{1/m+n+1} \\ 0^{1/m+2}, & 0, & ...0, & 0, & ... & 0^{1/m+n+2} \\ \end{pmatrix}.$$

$$= \begin{pmatrix} 0, & 0, & ...0, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ . \\ . \\ . \\ 0, & 0, & ...1^{1/m+n}, & 0, & ... & 0 \\ 0, & 0, & ...0, & 0, & ... & 0 \\ \end{pmatrix}.$$

which is a double growth rate of χ sequence. Hence $\Im_{mn} \in \chi^2_{f\pi}$. Let us take $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with $x \in \chi^2_{f\pi}$ and $f \in \left(\chi^2_{f\pi}\right)^{*}$. Take $x = (x_{mn}) = \Im_{mn} \in \chi^2_{f\pi}$. Then

$$|y_{mn}| \leq \|f\| \, d\left(\Im_{mn}, 0\right) < \infty \text{ for each } m, n$$

Thus (y_{mn}) is a growth rate of bounded sequence and hence double growth rate of an analytic sequence. In other words $y \in \Lambda^2_{f\pi}$. Therefore $\left(\chi^2_{f\pi}\right)^* = \Lambda^2_{f\pi}$.

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