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On Chains of Intermediate Rings Resulting from the Juxtaposition of Minimal Ring Extensions

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Abstract. Let $R \subset S$ and $S \subset T$ be minimal ring extensions of (commutative) rings. If (i) each of these extensions is integral or (ii) each of these extensions is integrally closed or (iii) $R \subset S$ is integral while $S \subset T$ is integrally closed, then each chain of rings between R and T is finite. Examples are given of minimal extensions $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is any of the three possible kinds of integral minimal extensions, and there exists (resp., does not exist) an infinite chain of intermediate rings between R and T.

1 Introduction

All rings considered below are commutative with identity; all subrings and inclusions of rings are (unital) ring extensions. Recall (cf. [9]) that a ring extension $R \subset S$ is a *minimal ring extension* if there does not exist a ring properly contained between R and S. (As usual, $\subset$ denotes proper inclusion.) A minimal ring extension $R \subset S$ is either integrally closed (in the sense that R is integrally closed in S) or integral. If $R \subset S$ is a minimal ring extension, it follows from [9, Théorème 2.2 (i) and Lemme 1.3] that there exists a unique maximal ideal M of R (called the *crucial maximal ideal* of $R \subset S$) such that the canonical injective ring homomorphism $R_M \to S_M$ $(:= S_{R\setminus M})$ can be viewed as a minimal ring extension while the canonical ring homomorphism $R_P \to S_P$ is an isomorphism for all prime ideals P of R except M. If $R \subset S$ is an integral minimal ring extension with crucial maximal ideal M, there are three possibilities: $R \subset S$ is said to be respectively *inert*, *ramified*, or *decomposed* if S/MS ($= S/M$) is isomorphic, as an algebra over the field $K := R/M$, to a minimal field extension of K, $K[X]/(X^2)$, or $K \times K$.

Let $R \subset S$ and $S \subset T$ each be minimal ring extensions. It is natural to ask whether $R \subset T$ inherits any interesting properties from the two given extensions. Of course, $R \subset T$ is not a minimal ring extension. However, it follows from [6, Theorems 4.2 (a) and 6.3 (a)] that if $R \subset S$ and $S \subset T$ are either both integral or both integrally closed, then $R \subset T$ satisfies the FCP property. (Recall that a ring extension $A \subseteq B$ is said to satisfy FCP if each chain of rings contained between A and B is finite.) In fact, if both $R \subset S$ and $S \subset T$ are integrally closed, one can say more, namely, $R \subset T$ satisfies the FIP property. (Recall that a ring extension $A \subseteq B$ is said to satisfy FIP if there are only finitely many rings contained between A and B. It is clear that FIP $\Rightarrow$ FCP, but the converse is false.) Our main interest here is in the behavior of $R \subset T$ when exactly one of $R \subset S$ and $S \subset T$ is integral (while the other is necessarily integrally closed). Proposition 2.2 and Theorem 2.3 determines this behavior relative to the FCP property. It turns out that if $R \subset S$ is integral and $S \subset T$ is integrally closed, then $R \subset T$ satisfies FIP (and hence FCP): see Theorem 2.2 (c). However, if $R \subset S$ is integrally closed with crucial maximal ideal M and $S \subset T$ is integral with crucial maximal ideal N, then $R \subset T$ satisfies FCP $\Leftrightarrow$ $R \subset T$ satisfies FIP $\Leftrightarrow$ $N \cap R \not\subseteq M$: see Theorem 2.3.

Sections 3 and 4 are devoted to examples showing that the above results cannot be extended, in the following sense. Let $\mathcal{P}$ be any of the three properties "decomposed," "inert," and "ramified". Then there exists a tower of rings $R \subset S \subset T$ such that $R \subset S$ is an integrally closed minimal ring extension, $S \subset T$ is an integral minimal ring extension that satisfies the property $\mathcal{P}$, and $R \subset T$ either does or does not (as one wishes to prescribe) satisfy FCP.

If D is an integral domain, it will be convenient to let D' denote the integral closure of D (in its quotient field). Given rings $A \subseteq B$, we let $[A, B]$ denote the set of intermediate rings, that is, the set of rings C such that $A \subseteq C \subseteq B$. Any unexplained material is standard, as in [10].

2 Some sufficient conditions for FCP and a criterion

We begin by collecting some (essentially known) results that give sufficient conditions for the juxtaposition of minimal ring extensions $R \subset S$ and $S \subset T$ to produce an extension $R \subset T$ that satisfies FCP.

Proposition 2.1. (a) Let $R_1 \subset R_2, \ldots, R_n \subset R_{n+1}$ be n $(< \infty)$ integral minimal ring extensions. Then $R_1 \subset R_{n+1}$ satisfies FCP.

(b) Let $R_1 \subset R_2, \ldots, R_n \subset R_{n+1}$ be n $(< \infty)$ integrally closed minimal ring extensions. Then $R_1 \subset R_{n+1}$ satisfies FIP (and FCP).

(c) Let $R_1 \subset R_2, \ldots, R_m \subset R_{m+1}$ be m (with $1 \leq m < \infty$) integral minimal ring extensions and let $R_{m+1} \subset R_{m+2}, \ldots, R_{m+n} \subset R_{m+n+1}$ be n (with $1 \leq n < \infty$) integrally closed minimal ring extensions. Then $R_1 \subset R_{m+n+1}$ satisfies FCP. Moreover if, in addition, $m = 1$, then $R_1 \subset R_{m+n+1}$ satisfies FIP (and FCP).

Proof. (a) (resp., (b)) Since $R_1 \subset R_2 \subset \ldots \subset R_n \subset R_{n+1}$ is a finite maximal chain of rings going from R_1 to R_{n+1} and the extension $R_1 \subset R_{n+1}$ is integral (resp., integrally closed) the assertion follows at once from [6, Theorem 4.2(a)] (resp., [6, Theorem 6.3(a)]).

(c) The hypotheses ensure that R_{m+1} is the integral closure of R_1 in R_{m+n+1}. By (a), $R_1 \subset R_{m+1}$ satisfies FCP. Moreover, by (b) or [1, Theorem 4.1], $R_{m+1} \subset R_{m+n+1}$ satisfies FIP (and hence FCP). Therefore, by [6, Theorem 3.13] $R_1 \subset R_{m+n+1}$ satisfies FCP. The stronger assertion in case $m = 1$ also follows from [6, Theorem 3.13] since, in this case, the (minimal) ring extension $R_1 \subset R_{m+1}$ satisfies FIP. □

In view of Proposition 2.1, the only remaining context to consider is $R \subset S \subset T$ where $R \subset S$ is an integrally closed minimal ring extension and $S \subset T$ is an integral minimal ring extension. For this context, Theorem 2.3 gives a necessary and sufficient condition for $R \subset T$ to satisfy FCP. Section 3 (resp., 4) is devoted to examples where this condition is not (resp., is) satisfied, regardless of which kind of integral minimal ring extension $S \subset T$ is. We leave to the reader the task of collecting examples that satisfy the hypotheses of the various parts of Proposition 2.1.

To facilitate the proof of Theorem 2.3 (as well as that of Example 3.1), we isolate the following lemma, which is of some independent interest.

Lemma 2.2. Let D be an integral domain which is not a field. View D as a subring of $D \times D$ via the diagonal map $D \to D \times D$, $d \mapsto (d, d)$. Then $D \subset D \times D$ does not satisfy FCP.

Proof. Pick a nonzero nonunit element $d \in D$. It is straight-forward to verify that $\{D[(d^n, 0)] \mid n = 1, 2, 3, \ldots\}$ is a strictly descending chain, and so the conclusion follows. □

Theorem 2.3. Let $R \subset S$ be an integrally closed minimal ring extension with crucial maximal ideal M and let $S \subset T$ be an integral minimal ring extension with crucial maximal ideal N. Then the following conditions are equivalent:

(1) $R \subset T$ satisfies FIP;
(2) $R \subset T$ satisfies FCP;
(3) $N \cap R \not\subseteq M$.

Proof. (3) ⇒ (1): Assume that $N \cap R \not\subseteq M$. Then the Crosswise Exchange Lemma [6, Lemma 2.7] provides a ring S^* such that $R \subset S^*$ is an integral minimal ring extension and $S^* \subset T$ is an integrally closed minimal ring extension. As S^* is necessarily the integral closure of R in T, with both $R \subset S^*$ and $S^* \subset T$ satisfying FIP, it follows from [6, Theorem 3.13] that $R \subset T$ satisfies FIP.

It was noted earlier that (1) ⇒ (2) trivially. Thus, to complete the proof, it suffices to establish the contrapositive of the assertion that (2) ⇒ (3). Assume that $N \cap R \subseteq M$; our task is to prove that $R \subset T$ does not satisfy FCP.

Note that $R \setminus M \subseteq S \setminus N$. Therefore, as $S_N \subset T_N$, it cannot be the case that $S_M = T_M$ (for otherwise, one could then localize at $S \setminus N$ to get $S_N = T_N$, a contradiction). Thus, we have the tower of proper ring extensions $R_M \subset S_M \subset T_M$. Since $R_M \subset S_M$ is an integrally closed minimal ring extension (cf. [9, Lemme 1.3]), it is clear that (R_M, S_M) is a normal pair. Since R_M is quasi-local, it follows from the characterization of normal pairs with quasi-local base rings [6, Theorem 6.8] (one may appeal instead to [3, Theorem 1] if S_M is an integral domain) that S_M is also quasi-local, say with maximal ideal Q, such that Q is also a prime ideal of R_M, $S_M = (R_M)_Q$ and R_M/Q is a valuation domain of Krull dimension 1. (Note also that $Q = QS_M = Q(R_M)_Q$ and that S_M/Q is the quotient field of R_M/Q.) As the minimal ring extension $R_M \subset S_M$ is integrally closed, with crucial maximal ideal MR_M, it follows from [9, Théorème 2.2] that no prime ideal of S_M can lie over MR_M. Therefore $Q \neq MR_M$; necessarily, $Q \subset MR_M$. Note that NS_M is the crucial maximal ideal of the integral minimal ring extension $S_M \subset T_M$ and that this ideal must be both the (unique) maximal ideal of S_M and the conductor $(S_M : T_M)$. It follows that $Q = NS_M$ is an ideal of T_M. In particular, Q is a common ideal of R_M, S_M, and T_M. Consider the associated factor rings, $D := R_M/Q$, $K := S_M/Q$, and $L := T_M/Q$. We have the tower $D \subset K \subset L$, where D is a valuation domain of Krull dimension 1, hence an integral domain that is distinct from its quotient field K, and $K \subset L$ is an integral minimal ring extension. We claim that $K \subset L$ is the same kind of integral minimal ring extension (namely, inert, decomposed or ramified) as was $S \subset T$.

It follows from a standard homomorphism theorem (cf. [7, Proposition 4.7]) that $K \subset L$ is the same kind of integral minimal ring extension (namely, inert, decomposed or ramified) as is $S_M \subset T_M$. Thus, to prove the above claim, it suffices to show that $S_M \subset T_M$ is the same kind of integral minimal ring extension as is $S \subset T$. It suffices to prove that $T_M/Q \cong T/N$ as algebras over the field $S_M/Q \cong S_M/NS_M \cong (S/N)_{R\setminus M} \cong S/N$. (Actually, the last-mentioned isomorphism should be explicated, as its details will be germane later as well. Since S/N is a field, there is a (unital, necessarily injective) ring homomorphism $\theta : S/N \to (S/N)_{R\setminus M}$, given by $u \mapsto u/1$ for all $u \in S/N$. To show that θ is an isomorphism, it is enough to prove that if $z \in R \setminus M$, then there exists $t \in T$ such that $(t + N)/1 = (1 + N)/z$ in $(S/N)_{R\setminus M}$. As $z \in S \setminus N$ and S/N is a field, there exists $v \in S$ such that $(z + N)^{-1} = v + N$; that is, $zv - 1 \in N$. Then $t := v$ has the desired behavior, and so θ is an isomorphism.) It remains to prove that $T_M/Q \cong T/N$. We have that $T_M/Q = T_M/NTM \cong (T/N)_{R\setminus M}$. It suffices to show that the map $\varphi : T/N \to (T/N)_{R\setminus M}$, given by $u \mapsto u/1$ for all $u \in T/N$, is an isomorphism. To see that φ is injective, note that if $t \in T$ is such that $(t + N)/1 = 0/1$ in $(T/N)_{R\setminus M}$, then there exists $\zeta \in R \setminus M$ $(\subseteq S \setminus N)$ such that $\zeta t \in N$; as $\zeta + N$ is a unit of the field S/N (and hence a unit of T/N), the equation $(\zeta + N)(t + N) = 0$ in T/N leads to $t \in N$, so that $\ker(\theta) = 0$. To show that φ is surjective, it is straightforward to adapt the above proof that θ is surjective. This completes the proof of the above claim.

We return to the consideration of the extension $D = R_M/Q \subset L = T_M/Q$. Note that $D \subset L$ satisfies FCP if and only if $R_M \subset T_M$ satisfies FCP. On the other hand, it is well known (and easy to see) that if $R \subset T$ satisfies FCP, then so does $R_M \subset T_M$. The rest of the proof (that $R \subset T$ does not satisfy FCP) consists of showing that $D \subset L$ does not satisfy FCP. This will be done in each of the three cases, determined by whether $S \subset T$ is inert, decomposed or ramified.

Consider first the case where $S \subset T$ is inert. Then $K \subset L$ is inert; that is, L is a minimal field extension of K. As $D \subset K \subset L$, with L a field and K the quotient field of D, [4, Theorem 2.1] yields an infinite chain of intermediate rings between D and L, so that $D \subset L$ does not satisfy FCP.

Next, consider the case where $S \subset T$ is decomposed. Then $L \cong K \times K$ as K-algebras, and we can identify the inclusion map $K \hookrightarrow L$ with the diagonal map $K \to K \times K$. Similarly, view $D \subset D \times D$ via the diagonal map $D \to D \times D$. We then have the tower $D \subset D \times D \subset K \times K = L$. By Lemma 2.1, $D \subset D \times D$ fails to satisfy FCP. Then, *a fortiori*, $D \subset L$ also fails to satisfy FCP.

It remains only to consider the case where $S \subset T$ is ramified. Then (up to K-algebra isomorphism) we can identify $L = K[X]/(X^2) = K \oplus Kx$, where $x := X + (X^2)$. Since D is not a field, we can pick a nonzero nonunit element $m \in D$. In the spirit of the proof of Lemma 2.1, we will show that the descending chain $\{D[m^n x] \mid n = 1, 2, 3, \ldots\}$ is strictly descending. If this assertion fails, then $m^t x \in D[m^{t+1}x]$ for some positive integer t. Thus, $m^t x = d_0 + d_1 m^{t+1} x$ for some uniquely determined coefficients $d_0, d_1 \in D$ (as 1 and x are linearly independent over K). Hence, $d_0 = 0$ and $m^t = d_1 m^{t+1}$. As $m^t \neq 0$ and D is an integral domain, we can divide by m^t (in K) to get $1 = d_1 m$ (in D), contradicting the fact that m was chosen as a nonunit of D. Thus, $\{D[m^n x]\}$ is indeed strictly descending, whence $D \subset L$ fails to satisfy FCP, to complete the proof. □

The proof of Theorem 2.3 established the following result which is of independent interest. Examples of towers $R \subset S \subset T$ that satisfy the hypotheses of Proposition 2.4 will be given in Section 4.

Proposition 2.4. Let $R \subset S$ be an integrally closed minimal ring extension with crucial maximal ideal M and let $S \subset T$ be an integral minimal ring extension with crucial maximal ideal N such that $N \cap R \subseteq M$. Then $S_M \subset T_M$ is the same kind of integral minimal ring extension (that is, inert, ramified, or decomposed) as $S \subset T$.

Remark 2.5. (a) Proposition 2.1 (a) is best possible, in the following sense. Its "FCP" conclusion cannot be strengthened to "FIP". For instance, if X and Y are algebraically independent indeterminates over the finite field $\mathbb{F}_p$ for some prime number p, then the p-dimensional field extensions $\mathbb{F}_p(X^p, Y^p) \subset \mathbb{F}_p(X, Y^p)$ and $\mathbb{F}_p(X, Y^p) \subset \mathbb{F}_p(X, Y)$ are each necessarily minimal ring extensions, but it is well known that $\mathbb{F}_p(X^p, Y^p) \subset \mathbb{F}_p(X, Y)$ does not satisfy FIP.

(b) The criterion in Theorem 2.3 cannot be applied to the context where the minimal ring extension $R \subset S$ is integral and the minimal ring extension $S \subset T$ is integrally closed. To see this, take $R \subset S$ to be the decomposed (integral minimal ring) extension $D \subset D'$ in [11, Example 4.3] and take T to be the localization of D' at one of its (two) maximal ideals. Since D' is a Prüfer domain of Krull dimension 1, standard facts ensure that $S \subset T$ is an integrally closed minimal ring extension [10, Theorems 17.5 (1), 22.1 and 26.1 (1), (2)]. By Proposition 2.1 (c), $R \subset T$ satisfies FCP. However, the present data do not satisfy the "$N \cap R \not\subseteq M$" criterion from Theorem 2.3. Indeed, let Q_1 and Q_2 denote the maximal ideals of S, with $T := D_{Q_1}$. Then the crucial maximal ideal of $S \subset T$ is $N := Q_2$ (cf. [9, Théorème 2.2]). It was shown in [11] that D is quasi-local, with unique maximal ideal $M := Q_1 \cap Q_2$; necessarily, M is the crucial maximal ideal of $R \subset S$; and it is plain that $N \cap R \subseteq$ (in fact, $=$) M.

3 Examples where $R \subset T$ does not satisfy FCP

In this section, we collect examples of towers of integral domains $R \subset S \subset T$ such that $R \subset S$ is an integrally closed minimal ring extension, $S \subset T$ is any of the possible kinds of integral minimal ring extensions (that is, decomposed, inert or ramified), and (contrary to expectations that may have been raised by Proposition 2.1) $R \subset T$ does not satisfy FCP. Of course, such $R \subset T$ also fails to satisfy FIP.

Example 3.1. There exist minimal ring extensions of integral domains $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is decomposed, and $R \subset T$ does not satisfy FCP.

Proof. Take V to be a valuation domain of Krull dimension 1 with quotient field K. Let X be an indeterminate over K. With α, β distinct elements of K, consider the maximal ideals $M_1 := (X - \alpha)K[X]$ and $M_2 := (X - \beta)K[X]$ of $K[X]$, and put $Q := M_1 \cap M_2$. Take $R := V + Q$, $S := K + Q$, and $T := K[X]$. Since $V \subset K$ is an integrally closed minimal ring extension, so is $R \subset S$ (cf. [7, Proposition 4.7]). Moreover, $S \subset T$ is a decomposed (hence integral) minimal ring extension since $S/Q \cong K$ and T/Q is K-algebra isomorphic to $T/M_1 \times T/M_2 \cong K \times K$. Finally, it remains to show that $R \subset T$ does not satisfy FCP; equivalently, that $R/Q (\cong V) \subset T/Q (\cong K \times K)$ does not satisfy FCP. In view of the tower $V \subset V \times V \subset K \times K$ (where V is viewed as a subring of $V \times V$ via the diagonal map), it is enough to prove that $V \subset V \times V$ does not satisfy FCP. This, in turn, follows immediately from Lemma 2.2, to complete the proof. (One can appeal to Theorem 2.3 to give an alternate proof that $R \subset T$ does not satisfy FCP. Indeed, the crucial maximal ideal of $R \subset S$ is $M := m + Q$, where m denotes the maximal ideal of V; the crucial maximal ideal of $S \subset T$ is $N := Q$; and $N \cap R = Q \subseteq$ (in fact, $\subset$) M.) □

Example 3.2. There exist minimal ring extensions of integral domains $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is inert, and $R \subset T$ does not satisfy FCP.

Proof. Take R to be a valuation domain of Krull dimension 1 such that its quotient field S is not algebraically closed. Next, take T to be a minimal field extension of S. Then $R \subset S$ is an integrally closed minimal ring extension and $S \subset T$ is an inert minimal ring extension. However, since $R \subset S \subset T$, [4, Theorem 2.1] yields an infinite chain of intermediate rings between R and T, so that $R \subset T$ does not satisfy FCP. (One can appeal to Theorem 2.3 to give an alternate proof that $R \subset T$ does not satisfy FCP. Indeed, the crucial maximal ideal of $R \subset S$ is $M := m$, where m denotes the unique maximal ideal of R; the crucial maximal ideal of $S \subset T$ is $N := \{0\}$; and $N \cap R = \{0\} \subseteq$ (in fact, $\subset$) M.) □

Example 3.3. There exist minimal ring extensions of integral domains $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is ramified, and $R \subset T$ does not satisfy FCP.

Proof. Data constructed in [2, Example 6.4] can be shown to exhibit the asserted behavior. Recall that that construction involved a DVR, V, with quotient field k and an analytic indeterminate X over k, with $R := V + X^2k[[X]], S := k + X^2k[[X]]$, and $T := k[[X]]$. In view of what was explicitly shown in [2, Example 6.4], it remains only to observe that $R \subset S$ is an integrally closed extension and that $S \subset T$ is ramified. (One can appeal to Theorem 2.3 to give an alternate proof that $R \subset T$ does not satisfy FCP. Indeed, the crucial maximal ideal of $R \subset S$ is

$M := m + X^2k[[X]]$, where m denotes the maximal ideal of V; the crucial maximal ideal of $S \subset T$ is $N := X^2k[[X]]$; and $N \cap R = X^2k[[X]] \subseteq$ (in fact, $\subset$) M.) $\square$

Remark 3.4. (a) In presenting examples of suitable chains $R \subset S \subset T$ in Examples 3.1-3.3, we have taken care to use integral domains. For readers who would be satisfied with towers of rings that are not necessarily domains but exhibit the asserted behavior, the constructions can be simplified. For instance, for the zero-divisor analogue of Example 3.1, one could take $R \subset S \subset T$ to be $V \subset K \subset K \times K$.

(b) The kind of simplification that was noted in (a) cannot be carried much further. In particular, there does not exist a tower $R \subset S \subset T$ such that R is a integral domain of Krull dimension 1, $R \subset S$ and $S \subset T$ are minimal ring extensions, T is contained in the quotient field of R, and $R \subset T$ does not satisfy FCP. To see this, observe that $R \subset S \subset T$ is a finite maximal chain of rings going from R to T and apply [1, Corollary 4.4].

(c) Examples 3.1-3.3 each show that the conclusion of Proposition 2.1 (c) may fail if the hypotheses are changed by placing an integral minimal extension "on top of" an integrally closed minimal ring extension. Similarly, the conclusion of Proposition 2.1 (c) can fail for $n \geq 3$ minimal ring extensions if one of the "higher" minimal ring extensions is integral. For instance, let R_2, R_3 and R_4 be the rings that were denoted by R, S, and T respectively, in Example 3.3. Assume further that the valuation domain V in that example is of the form $V = F_1[[Y]]$ for some field F_1 and analytic indeterminate Y, and that F_2 is a subfield of F_1 such that $[F_1 : F_2] = 2$. Put $R_1 := F_2 + YV$. Then the minimal ring extension $R_1 \subset R_2$ is integral (in fact, inert). Also, we have seen that $R_2 \subset R_3$ is an integrally closed minimal ring extension and that $R_3 \subset R_4$ is a ramified minimal ring extension. Finally, since $R_2 \subset R_4$ fails to satisfy FCP, neither does $R_1 \subset R_4$.

4 Examples where $R \subset T$ satisfies FCP

In contrast to the examples in Section 3, the examples in this section will return to the flavor exhibited in Proposition 2.1. Specifically, we give examples of towers $R \subset S \subset T$ such that $R \subset S$ is an integrally closed minimal ring extension, $S \subset T$ is any of the possible kinds of integral minimal ring extensions, and $R \subset T$ satisfies FCP. Except for the case where $S \subset T$ is decomposed, all the rings in the examples in this section will be integral domains. The following result will enable us to handle that decomposed context (with applicability to the other integral contexts as well).

Proposition 4.1. Let $D \subset E$ be integral domains. Let (V, n) be a valuation domain of Krull dimension 1 with quotient field K. Put $R := V \times D$, $S := V \times E$ and $T := K \times E$. Then:

(a) $R \subset S$ is a minimal ring extension if and only if $D \subset E$ is a minimal ring extension.

(b) $R \subset S$ is an integral extension if and only if $D \subset E$ is an integral extension.

(c) Let $D \subset E$ be an integral minimal ring extension (equivalently, let $R \subset S$ be an integral minimal ring extension). If m is the crucial maximal ideal of $D \subset E$, then $M := V \times m$ is the the crucial maximal ideal of $R \subset S$.

(d) $D \subset E$ is a decomposed (resp., inert; resp., ramified) extension of and only if $R \subset S$ is decomposed (resp., inert; resp., ramified).

(e) $S \subset T$ is an integrally closed minimal ring extension, and its crucial maximal ideal is $N := n \times E$.

(f) $N \cap R = n \times D \not\subseteq M$.

Proof. (a) Note that a ring extension $A \subseteq B$ is a minimal ring extension if and only if the cardinality of $[A, B]$ is 2. Therefore, the assertion follows from that fact that $[R, S] = \{V\} \times [D, E]$ and $[D, E]$ have the same cardinality.

(b) Let A denote the integral closure of D in E. The assertion follows from the fact that the integral closure of R in S is $V \times A$.

(c) The parenthetical equivalence follows from (a) and (b). Since $(D : E) = m$, the crucial maximal ideal of $R \subset S$ is $(R : S) = V \times (D : E) = V \times m = M$.

(d) By (a)-(c), we can suppose that $D \subset E$ and $R \subset S$ are each integral minimal ring extensions, with crucial maximal ideals m and $M = V \times m$, respectively. Note that $k := D/m$ can be identified with $V/V \times D/m = (V \times D)/(V \times m) = R/M$. Therefore, by the characterizations of "decomposed," "inert" and "ramified" in the Introduction, it suffices to prove that $S/M \cong E/m$ as k-algebras. In fact, $S/M = (V \times E)/(V \times m) \cong V/V \times E/m \cong E/m$, as required.

(e) Since V is integrally closed in K, $S = V \times E$ is integrally closed in $K \times E = T$; that is, $S \subset T$ is an integrally closed extension. To see that this is a minimal ring extension, observe that $[S,T] = [V,K] \times \{E\}$ and argue as in the proof of (a), noting that $V \subset K$ is a minimal ring extension. Finally, by [9, Théorème 2.2], the crucial maximal ideal of $S \subset T$ is the only prime ideal of S that is not lain over from T, namely, $n \times E = N$.

(f) This is clear from (e) and (c). □

We can now produce towers of rings that answer the main questions for this section.

Corollary 4.2. Let $\mathcal{P}$ be any of the three properties "decomposed," "inert," and "ramified". Then there exist minimal ring extensions $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ satisfies property $\mathcal{P}$, and $R \subset T$ satisfies FIP (and hence FCP).

Proof. Pick $D \subset E$ to be any (integral minimal) ring extension that satisfies property $\mathcal{P}$. Let (V,n), K, R, S, T, m, M and N be as in Proposition 4.1. By parts (d) and (c) of Proposition 4.1, $R \subset S$ inherits the property $\mathcal{P}$ from $D \subset E$ and the crucial maximal ideal of $R \subset S$ is $M = V \times m$; by Proposition 4.1 (e), $S \subset T$ is an integrally closed minimal ring extension, with crucial maximal ideal $N = n \times E$. Note that $R \subset T$ satisfies FIP, by Proposition 2.1 (c). Also, since $N \cap R \not\subseteq M$ by Proposition 4.1 (f), we may apply the Crosswise Exchange Lemma [6, Lemma 2.7]. The upshot is the existence of a ring $S^* \in [R,T]$ such that $R \subset S^*$ inherits the property of being an integrally closed minimal ring extension from $S \subset T$ and $S^* \subset T$ inherits the property of being an integral minimal ring extension with the property $\mathcal{P}$ from $R \subset S$. Thus, the tower $R \subset S^* \subset T$ exhibits the asserted behavior. □

Remark 4.3. One can give an alternate proof of Corollary 4.2 that avoids an explicit use of the Crosswise Exchange Lemma, as follows. As before, take $D \subset E$ to be an integral minimal ring extension that satisfies $\mathcal{P}$. Then, instead of considering the tower $V \times D \subset V \times E \subset K \times E$, consider the tower $V \times D \subset K \times D \subset K \times E$. By reasoning as in the proof of Proposition 4.1, we see that $V \times D \subset K \times D$ is an integrally closed minimal ring extension with crucial maximal ideal $n \times D$; $K \times D \subset K \times E$ is an integral minimal ring extension that satisfies the property $\mathcal{P}$ and has crucial maximal ideal $K \times m$, where $m = (D : E)$; and

$$(K \times m) \cap (V \times D) = V \times m \not\subseteq n \times D.$$

Therefore, by Theorem 2.3, $R \subset T$ satisfies FIP, thus completing the alternate proof of Corollary 4.2. We wish to point out that this second proof is not really simpler than the above proof, since the second proof appealed to Theorem 2.3 and the proof of Theorem 2.3 made explicit use of the Crosswise Exchange Lemma.

In the spirit of Section 3, we next augment Corollary 4.2, at least in case $\mathcal{P}$ is "ramified" or "inert," by giving domain-theoretic examples of the behavior in Corollary 4.2.

Example 4.4. There exist minimal ring extensions of integral domains $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is ramified, and $R \subset T$ satisfies FIP (and hence FCP).

Proof. Let X, Y be algebraically independent indeterminates over a field K, and let R be the ring of fractions obtained when one localizes $K[X^2, X^3, Y]$ at (the complement of) the union of the prime ideals (X^2, X^3) and (Y). Put $S := R_{(X^2,X^3)}$ and $T := S[X]$. We will show that R, S and T have the asserted behavior.

We begin by verifying the assertion concerning $R \subset S$. As this extension is a flat epimorphism, it is not integral, and so one need only prove that it is a minimal ring extension. Therefore, by [7, Proposition 4.6], it suffices to show that $R_{(Y)} \subseteq S_{(Y)}$ is a minimal ring extension and that $R_{(X^2,X^3)} = S_{(X^2,X^3)}$. The latter equality is clear from the definition of S. It now follows by globalization that $R_{(Y)} \subset S_{(Y)}$. Hence, it will be enough to prove that $R_{(Y)}$ is a valuation domain of Krull dimension 1. In fact, this ring is a DVR, since it is a local Noetherian domain of Krull dimension 1 whose unique nonzero prime ideal is principal.

We turn next to the assertions concerning $S \subset T$. This is a proper integral extension (since $X \in T \setminus S$). Therefore, by the characterization of ramified extensions in terms of condition (3) in [8, Proposition 2.12], we need only observe that $q := X$ satisfies $q^2 \in S, q^3 \in S$, and $q(X^2, X^3) \subseteq S$.

It remains to explain why $R \subset T$ satisfies FCP. This is an immediate consequence of [1, Proposition 5.3], since $R \subset S \subset T$ is a finite maximal chain of rings inside the quotient field of the Noetherian integral domain R. One can appeal to Theorem 2.3 to give an alternate proof that $R \subset T$ satisfies FIP. Indeed, the crucial maximal ideal of $R \subset S$ is $M := (Y)$; the crucial maximal ideal of $S \subset T$ is $N := (X^2, X^3)$; and $N \cap R = (X^2, X^3) \not\subseteq M$. □

Example 4.5. There exist minimal ring extensions of integral domains $R \subset S$ and $S \subset T$ such that $R \subset S$ is integrally closed, $S \subset T$ is inert, and $R \subset T$ satisfies FIP (and hence FCP).

Proof. We begin by specializing the setting in [5, Theorem 3.4]. Take D to be a Dedekind (hence Prüfer) domain with exactly two maximal ideals, M_1 and M_2 (and, necessarily a Y-shaped spectrum). Arrange also that the field $L := D/M_1$ has a subfield F such that $F \subset L$ is a minimal field extension. Consider the pullback $R := D \times_L F$. Put $S := R_{M_1}$ and $T := S'$. We will show that the tower $R \subset S \subset T$ exhibits the asserted behavior.

Let K denote the quotient field of R. By parts (a) and (c) of [5, Theorem 3.4], $R \subset R' = D$, D is a minimal ring extension of R, and S is the only overring of R (that is, the only R-subalgebra of K) which is incomparable with R'. Also, by [5, Theorem 3.4 (b)], Spec(R) is canonically homeomorphic to, hence order-isomorphic to, Spec(D) (in the Zariski topology). Consequently, R inherits from D the property of having Krull dimension 1. Therefore, by [5, Theorem 3.1], R has exactly 6 overrings. It follows that the set of overrings of R must be $\{R, D, S, D_{M_1}, D_{M_2}, K\}$. Note that $R \neq S$ since R is not quasi-local and S is quasi-local. In fact, the above list of overrings reveals that $R \subset S$ is a minimal ring extension. Moreover, $R \subset S$ is integrally closed (by [9, Théorème 2.2], since S is an R-flat overring of R that is distinct from R). The crucial maximal ideal of $R \subset S$ cannot be M_1 since $R_{M_1} = S_{M_1}$ canonically, and so, by the process of elimination, the crucial maximal ideal of $R \subset S$ must be $M := M_2 \cap R$.

We claim that $S' = D_{M_1}$. Indeed, the above list of overrings reveals that $S' = D_{M_i}$ for some $i \in \{1, 2\}$. However, $M \in$ Spec(R) cannot be lain over from R_{M_1}, since $R \subset S$ is an integrally closed minimal ring extension [9, Théorème 2.2]. Thus, $M_2 \cap R$ cannot be lain over from any overring of R_{M_1} (such as S'). It follows that $S' \neq D_{M_2}$ and so, by the process of elimination, $(T =)\ S' = D_{M_1}$, as claimed.

Consider the integral extension $S = R_{M_1} \subseteq T = S' = D_{M_1}$. The above list of overrings reveals that $S \neq T$ and, in fact, that $S \subset T$ is a minimal ring extension. We claim that this extension is inert. To see this, note first, from the definition of R, that the crucial maximal ideal of $R \subset D$ is $(R : D) = M_1$. As M_1 is a maximal ideal of (both R and) D, the extension $R \subset D$ is inert. It is well known (cf. [7, Proposition 4.6]) that $R_{M_1} \subset D_{M_1}$ inherits the "inert" property from $R \subset D$, which proves the above claim.

The crucial maximal ideal of $S \subset T$ must be the unique maximal ideal of S, namely, $N := M_1 R_{M_1}$. Observe that $N \cap R = M_1 \not\subseteq M_2 \cap R = M$. Therefore, by Theorem 2.3, $R \subset T$ satisfies FIP, to complete the proof. □

The work in this section leaves the following open question. Does there exist a tower of domains $R \subset S \subset T$ such that $R \subset S$ is an integrally closed minimal ring extension, $S \subset T$ is a decomposed (integral minimal ring) extension, and $R \subset T$ satisfies FCP?

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