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Wave Equations with Nonlinear Sources and Damping: Weak vs. Regular Solutions

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Abstract. We provide an up to date and unified exposition of the well-posedness theory for nonlinear wave equations on bounded domains, with the focus on the interaction between highly nonlinear sources and damping present in the system. This work summarizes and compares well-posedness results for finite-energy ($H^1 \times L^2$) and regular ($H^2 \times H^1$) solutions.

1 Introduction

This exposition is devoted to a review of recent results, including the latest developments, on well-posedness analysis of weak ($H^1 \times L^2$) and regular ($H^2 \times H^1$) solutions for nonlinear wave equations on bounded domains with either Dirichlet or Neumann-type boundary conditions. In contrast to systems on $\mathbb{R}^n$, here one has to account for the interference of the boundary conditions and the boundary traces. The equations are characterized by the interplay of two competing nonlinearities:

(i) nonlinear sources, especially of order above the super-critical exponent (above exponent 5 for the interior source, and above 3 for the boundary one in dimensions $\dim(\Omega) = 3$), which are neither monotone, nor dissipative, nor corresponds to a locally Lipschitz perturbation on the associated finite energy space; instead such sources build up the energy and drive the solution to a blow up in finite time;

(ii) a nonlinear damping on the interior/boundary whose role becomes two-fold: it is intended not only to stabilize the system, as in the classical control theory for dissipative dynamics, but also to prevent the finite time blow-up of solution and guarantee its existence and uniqueness.

The idea of using damping to positively influence the lifespan and properties of solutions was introduced more than 20 years ago in the context of control theory for boundary or point controls, which by themselves, without any damping, do not lead to well-posed dynamics [LT90]. General "energy-building" sources, even if locally Lipschitz (thus allowing for local in time well-posedness) on the energy space, in general lead to blow-up of solutions in a finite time unless they are counteracted by sufficiently nonlinear damping [BL08a, Boc09, GT94, PS75, Vit02a].

The interaction of source and damping can take place in the interior of the physical domain and/or in the boundary conditions. While there are certain similarities between the treatment of interior and boundary sources, the latter case is more subtle and brings in new challenges to the problem, as detailed below.

A lot of work has been done on this topic, including some soon-to-appear [BRT] results on local and global existence for regular ($H^2(\Omega) \times H^1(\Omega)$) solutions. The goal of this paper is to provide a concise exposition of the recent developments, while also offering a comparison between the challenges and strategies involved in the analysis done for the weak solutions vs. the corresponding study of regular solutions.

2 Weak solutions

We begin with a complete overview of the well-posedness results for weak solutions: local and global existence, as well as blow-up in finite time.

2.1 Notation

Let us first briefly introduce some notation. Henceforth $C_w([0,T];Y)$ denote the space of weakly continuous functions from $[0,T]$ into a Banach space Y.

Also define

$$Q_T := \Omega \times (0,T) \quad \text{and} \quad \Sigma_T := \Gamma \times (0,T)$$

and the associated integrals

$$\int_0^T \int_\Omega := \int_{Q_T} \quad \text{and} \quad \int_0^T \int_\Gamma := \int_{\Sigma_T} .$$

The relation $a(s) \lesssim b(s)$ will indicates that $a(s) \leq Cb(s)$ for some constant $C > 0$ independent of s. If $b(s) \lesssim a(s)$ also holds then we will write $a(s) \sim b(s)$.

2.2 The model and the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set a with sufficiently smooth boundary Γ (e.g. of class C^2, locally on one side of the domain Γ). We are interested in the well-posedness analysis for the following system

$$\begin{cases} u_{tt} - \Delta u + g_0(u_t) &= f(u) \quad \text{in} \quad \Omega \times [0,\infty), \\ \partial_\nu u + u + g(u_t) &= h(u) \quad \text{on} \quad \Gamma \times [0,\infty), \\ \{u(0), u_t(0)\} &= \{u_0, u_1\} \in H^1(\Omega) \times L_2(\Omega) . \end{cases} \tag{2.1}$$

The scalar maps $g_0(s)$, $g(s)$ model respectively the interior and boundary damping feedbacks; they are monotone increasing and continuous functions such that $g(0) = g_0(0) = 0$. In the absence of any source, or with sources terms that correspond to globally Lipschitz operators on the state spaces (e.g. linearly bounded ones), the maps g and g_0 can be relatively unrestricted and described via subgradients of appropriately defined convex functionals. This framework has been extensively developed as an application of the monotone operators theory, see [Bar93] for instance.

In our source-laden case the benchmark model considers damping maps with polynomial-like growth bounds at infinity

$$g_0(s)s \sim |s|^{m+1} \quad \text{and} \quad g(s)s \sim |s|^{q+1} \quad \text{for some } m,q > 0 \text{ and all large } |s| > 1 .$$

Remark 2.1 (Absence of damping). In case of up-to-critical interior sources and without boundary sources the presence of (any) damping in the system is not necessary for local existence. However, whenever a nonlinear boundary source is involved, regardless of its exponent, the presence of the boundary damping g becomes instrumental to proving local well-posedness.

While the upper bounds on the dissipation maps make them tractable in the corresponding Lebesgue spaces and facilitate the analysis of local well-posedness of solutions, the lower bounds at infinity are necessary for establishing global existence of weak solutions.

The Nemytski operators associated with differentiable scalar maps f and h represent source terms. The dynamics of the equation is fundamentally affected by the behavior of the nonlinear terms $f(u)$ and $h(u)$; henceforth, their growth orders will be denoted respectively by p and k, via the estimates

$$|f(s)| \lesssim |s|^p, \qquad |h(s)| \lesssim |s|^k .$$

With minor adjustments to the argument one can without loss of generality consider f and h of the form

$$f(s) = \alpha|s|^{p-1}s, \quad h(s) = \beta|s|^{k-1}s .$$

The sign of these sources (appearing on the right-hand side of the equation (2.1)) correspond to *focusing* or "energy-building" terms. Such nonlinear perturbations do not conserve or dissipate the finite energy, but rather have the capacity to increase it, possibly driving solutions to blow-up in finite time. Then well-known energy law for the system (2.1) can be recast in terms of the energy functional

$$E(t) := \frac{1}{2}\|\nabla u\|^2_{L_2(\Omega)} + \frac{1}{2}\|u_t\|^2_{L_2(\Omega)} - \frac{\alpha}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)} - \frac{\beta}{k+1}\|u\|^{k+1}_{L^{k+1}(\Gamma)}. \tag{2.2}$$

When $\alpha = \beta = 0$ the energy is non-increasing along trajectories, thus the standard regularity level for weak solutions is

$$(u, u|_\Gamma, u_t) \in H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times L^2(\Omega) .$$

Based on the strength of Sobolev embeddings

$$H^1(\Omega) \subset L^{\frac{2n}{n-2}}(\Omega) \quad \text{and} \quad H^{\frac{1}{2}}(\Gamma) \subset L^{\frac{2n-2}{n-2}}(\Gamma) \tag{2.3}$$

one immediately infers that higher-order nonlinearities present a regularity problem. From this point of view we divide the ranges of source exponents into the following categories:

Sub-critical/Critical in this context will mean sub-criticality with respect to Sobolev embeddings. In particular it is the scenario when the embedding results for finite-energy regularity assure that the nonlinear terms themselves stay in the finite-energy space:

$$u^p \in L^2(\Omega) \quad \text{and} \quad u^k|_\Gamma \in L^2(\Gamma) .$$

Specifically (in dimensions above 2)

$$p \leq \frac{n}{n-2} \quad \text{and} \quad k \leq \frac{n-1}{n-2}$$

The critical case refers to the equality when the embeddings hold, but are no longer compact.

Super-critical range contains nonlinearities that are above the embedding levels for the finite-energy, *but for which the potential energy of solutions, namely the norms* $\|u\|_{L^{p+1}(\Omega)}$ *and* $\|u\|_{L^{k+1}(\Gamma)}$, *are defined*. In this case the nonlinearities u^{p+1} and u^{k+1}

are still integrable on Ω and Γ respectively. This condition is preserved up to and including the equality

$$p+1 = \frac{2n}{n-2} \quad \text{and} \quad k+1 = \frac{2n-2}{n-2}$$

in dimensions $n > 2$. This is the highest level at which the potential energy component of the energy functional (2.2) is defined for finite energy solutions.

Above super-critical include exponents for p and k for up to one degree above the preceding exponent level, namely

$$p \leq \frac{2n}{n-2} \quad \text{and} \quad k \leq \frac{2n-2}{n-2}$$

the nonlinear functions $f(u)$ and $h(u)$ are still in $L^1(\Omega)$, and $L^1(\Gamma)$, respectively for finite energy solutions, but they are no longer tractable in the framework of the potential well theory.

Nevertheless, above the super-critical level if one considers higher powers for the damping terms (i.e. $m \to \infty$ and $q \to \infty$) one expects the dynamics to gain some stability and at least assure local existence of finite energy solutions assuming only

$$p \leq \frac{2n}{n-2} \quad \text{and} \quad k \leq \frac{2n-2}{n-2}$$

(i.e. $p \leq 6$ and $k \leq 4$ in 3D). Thus it is clear that the presence of damping becomes essential in ascertaining existence, uniqueness and the life-span of solutions by preventing finite-time blow-up. This fact had been observed even for interior sources below the super-critical level, as described below, in the summary of the results.

Characterizing how the relation between damping orders m, q and source exponents p, k affects the well-posedness is one of the fundamental questions.

Having introduced the exponents, now we are ready to state the notion of weak solution for the system (2.1):

Definition 2.2 (Weak solution). A weak solution of (2.1), defined on some interval $[0, T]$, is a function $u \in C_w([0,T]; H^1(\Omega))$ with $u_t \in C_w([0,T]; L_2(\Omega))$ such that:

(i) $u_t \in L_{m+1}((0,T)\times\Omega)$, $\quad u_t|_{(0,T)\times\Gamma} \in L_{q+1}(0,T;\Gamma)$;

(ii) For all $\phi \in C([0,T]; H^1(\Omega)) \cap C^1([0,T]; L_2(\Omega)) \cap L_{m+1}(0,T;\Omega)$, and $\phi|_\Gamma \in L_{q+1}(0,T;\Gamma)$ the function u verifies:

$$\begin{aligned} &\int_{Q_T} (-u_t\phi_t + \nabla u \cdot \nabla\phi)\, dQ + \int_{\Sigma_T} \phi u \, d\Sigma + \int_{Q_T} g_0(u_t)\phi \, dQ \\ &= -\int_\Omega u_t\phi d\Omega\Big|_0^T + \int_{\Sigma_T} h(u)\phi \, d\Sigma - \int_{\Sigma_T} g(u_t)\phi \, d\Sigma + \int_{Q_T} f(u)\phi \, dQ \end{aligned} \tag{2.4}$$

(iii) In addition, for $\psi_0 \in H^{-1}(\Omega)$, $\psi_1 \in L_2(\Omega)$

$$\lim_{t\to 0}\langle u(t) - u_0,\, \psi_0\rangle = 0 \quad \text{and} \quad \lim_{t\to 0}(u_t(t) - u_1,\, \psi_1)_\Omega = 0\,.$$

2.3 Well-posedness results for sources up to super-critical level

Local existence and uniqueness:

In 3 dimensions, if the interior source f has exponent $p \leq 3$, then f defines a locally Lipschitz operator from $H^1(\Omega) \to L_2(\Omega)$, and existence and uniqueness of local-in-time solution follows from monotone operator theory [Bar93]. However, if the exponent p goes above the critical embedding 3, then one has to take advantage of the damping term, as was shown in [Fei95, BLR05], where existence of solutions (without uniqueness) was obtained for sources up to the super-critical level ($p < 5$) under the following relation between the exponents of the source and the damping:

$$p < \frac{6m}{m+1}\,.$$

In the case of boundary sources in Neumann-type boundary condition, the issue is more difficult, due to the fact that the linearized problem fails the uniform Lopatinskii (Kreiss-Sakamoto) condition. If the boundary data is $L^2(\Gamma)$, the maximal amount of regularity that one obtains in the general case is $H^{2/3}(\Omega) \times H^{-1/3}(\Omega)$ with improvement to $H^{3/4}(\Omega) \times H^{-1/4}(\Omega)$ for parallelepiped domains [LT90, Tat98]. Thus even for locally Lipschitz boundary source h (i.e. $k \leq 2$ in 3D), the loss of a $1/3$ interior derivative interferes with the Lipschitz behavior on the finite-energy space. As a consequence, local existence of solution even in this case requires the "help" of boundary damping g, unlike in the interior scenario. The idea is to use the damping as regularizer, in order to offset the singularities brought in by the inhomogeneous Neumann data. Existence (without uniqueness) of solutions was obtained [Vit02a] for the case of boundary sources up to the super-critical level (i.e. $k = 3$ in 3D) under a corresponding (due to the Sobolev embedding $H^1(\Omega) \to L^4(\Gamma)$) relationship between the exponents of the source and the damping

$$k < \frac{4q}{q+1}.$$

The methods employed for the above results (for both interior and boundary sources) heavily relied on the at most super-critical sources, and only recently it was proved that the exclusion of sources above the super-critical level was inherent to the methods, and not the problem. In a series of papers [BL08a, BR09, Boc09, BL10, BL08b] the authors introduced new techniques that employ monotonicity methods combined with suitable truncations-approximations of nonlinear terms, rather than on compactness arguments which had limited the results in the previous attempts. This strategy made it possible to extend the range of Sobolev exponents for which the analysis is applicable ($5 \leq p < 6$ for the interior exponent and $3 \leq k < 4$ for the boundary one in 3 dimensions), and extend previously available results on this problem for the case of interior super-critical sources and boundary sources [Fei95], [GT94], [BLR05], [Rad05], [STV03], [LT93], [Vit02a], [CDCM04].

Global existence

Once the issue of local existence and uniqueness is settled, the next step is to investigate what happens to the solution as time goes to infinity. Local finite energy solutions, without further hypotheses on the nonlinear sources, may blow up in norm in finite time and thus not be global. However, the damping has the tendency to extend the lifespan of solution. This fact was first realized in the case of interior source-damping interaction in [GT94] wherein it was proved that if the damping dominates the source (i.e. the exponent m of the damping is greater that the exponent p of the source), then the solution exists forever, while in the opposite case (i.e. $m < p$), the solution blows-up in the energy norm in finite time, at least for negative initial energy.

2.4 Results for sources above the super-critical range

The recent key contributions to sources exceeding the super-critical threshold, i.e. beyond the range where the potential energy is defined, can be summarized as follows:

1.) Full Hadamard well-posedness of local (in time) flows: solutions exist locally, are *unique*, and *depend continuously* on the initial data in the finite-energy topology. This is the first result on local Hadamard well-posedness of wave flows generated by super-critical boundary and interior sources and damping terms [Boc09, BL10, BL08b].

2.) The introduction of new techniques that rely on monotonicity methods combined with suitable truncations-approximations, rather than compactness arguments that limited the results in the previous literature [Vit02a].

3.) Verification of sharpness of the results for the exponent ranges that illustrate the border between well-posedness and blow-up for finite energy solutions (starting with negative initial energy). This step completes the picture for Hadamard well-posedness of weak solutions [BL08a, Boc09]:

   (a) *Local existence* was obtained for the range of interior source exponents $1 \leq p \leq 3$ or $p \leq 6m/(m+1)$ if $p > 3$ (Figure 1-A), and boundary source exponents $1 \leq k \leq 4q/(q+1)$ (Figure 1-B). The dark shaded regions indicate the improvement over previous results by including sources above the super-critical range: when interior exponent p exceeds 5 and boundary exponent k exceeds 3.

   (b) *Global existence* holds in region I (Fig. 1-A,B), i.e. when $m \geq p$ and $q \geq k$. The complementary region II indicates the ranges of source-damping exponents for which solutions may *blow-up* in finite time.

   (c) *Uniqueness* and *continuous dependence on initial data* has been established for all of the above solutions.

The above theorems were subsequently complemented by the results in [BRT11], where the authors focused on energy decay rates for potential well solutions associated with system (2.1). The findings include:

(i) Global existence of potential well solutions - without exploiting interior and boundary overdamping ($m \geq p$ and $q \geq k$).

(ii) Blow-up in finite time for initial data of *non-negative* energy thus complementing the one in [BL08a] for initial data of *negative energy*.

(iii) Explicit uniform algebraic decay rates of the finite energy for super-critical sources ($p \leq 5, q \leq 3$), provided local existence criteria are satisfied and solutions originate in a potential well.

3 Regular solutions

Now we move our attention to the well-posedness of smooth, in particular $H^2(\Omega) \times H^1(\Omega)$, solutions for (2.1). The questions of interest are: (i) Can the longstanding restriction imposed on the range of exponents for the source-damping interactions (necessary

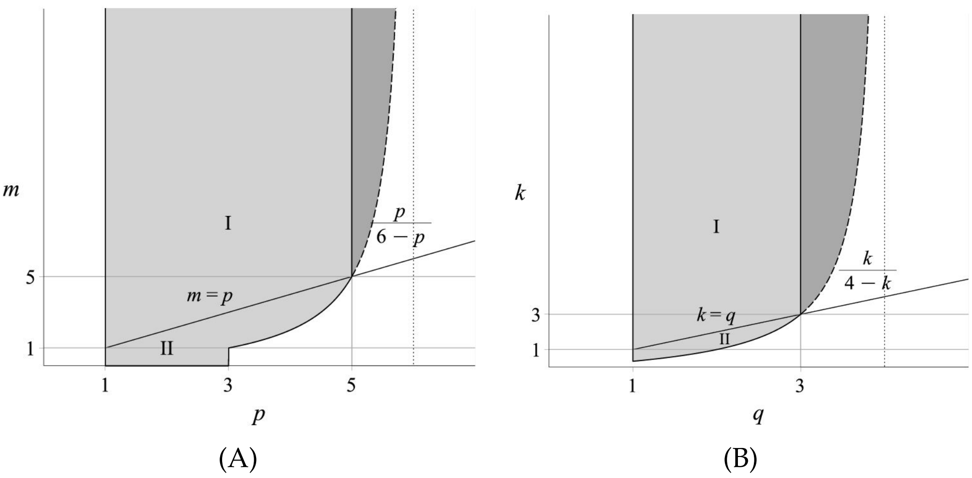

Figure 1. Ranges of source and damping exponents for weak solutions.

for existence of weak solutions) be removed?, and (ii) What is the new/optimal range of exponents for both interior and boundary sources and damping terms to provide local and global existence of regular solutions?

The case of interior source with exponent up to the super-critical level ($p < 5$) (and no boundary source-damping interaction) has been addressed in [BLR05] for degenerate damping and more recently in [Rad13] for the case of non-degenerate dissipation. It was shown in both papers that the range of the source parameter p is larger than the range typically assumed in local existence results for weak solutions; e.g. interdependence of source and damping $p \leq \frac{6m}{m+1}$ for $n = 3$, necessary for weak solutions has been dropped. The method employed in [Rad13] consisted of a combination of a suitable approximation for the source term, a-priori estimates on higher derivatives for small initial data, and compactness arguments in order to pass with the limit in the source terms.

Only recently in [BRT] the authors bypassed the compactness arguments, by working with Galerkin approximations and taking advantage of the regular initial data as well as theory of Orlicz function spaces. The paper proves local existence of regular solutions for interior sources of any exponent p and any damping when $\dim(\Omega) = 3, 4$. Moreover, the argument employed the more general framework of Orlicz spaces that admits non-polynomially bounded damping, e.g. growing exponentially or logarithmically at infinity. To our knowledge, this is the first paper that deals with non-polynomial damping in the context of sources *above the super-critical level*. Below we present a summary of these results.

3.1 PDE Model

In this context the results so far focus on interior damping and homogeneous Dirichlet data:

$$\begin{cases} u_{tt} + g_0(u_t) = \Delta u + f(u) & \text{in} \quad \Omega \times [0, \infty) \\ u = 0 & \text{on} \quad \Gamma \times [0, \infty) \\ u(0) = u_0 \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad u_t(0) = u_1 \in H^1_0(\Omega) \end{cases} \tag{3.1}$$

Damping g_0

The feedback map $g_0 : \mathbb{R} \to \mathbb{R}$ is continuous monotone function, $g_0(0) = 0$. The framework below considers functions g_0 less general than the ones that could be analyzed in subgradient framework (e.g. as was done in the context of subcritical or critical–with respect to Sobolev embeddings–sources), but of more general growth than polynomial. To this end two additional structural assumptions will be imposed: g_0 is odd (this property can be in fact relaxed to $|g_0(s)| \sim |g_0(-s)|$), non-decreasing, and non-saturating: $\lim_{s\to\infty} g_0(s) = \infty$ (however, as will be indicated, some local well-posedness results also hold with $g_0 \equiv 0$).

Orlicz spaces associated with damping

For local well-posedness, we take advantage of the damping's monotonicity properties to work within Orlicz space framework. Let g_0^{-1} denote the inverse of g_0 if defined, or otherwise the general *right-inverse* of g_0 (see e.g. [KR61, pp. 11–12]), so g_0 does not necessarily have to be *strictly* monotone increasing. Then we can define the following $\mathcal{N}$*-functions*:

$$\Phi(t) = \int_0^t g_0(s)ds \quad \text{and} \quad \Psi(t) = \int_0^t g_0^{-1}(s)ds \quad \text{for} \quad t \geq 0\,. \tag{3.2}$$

For $T > 0$ and a measurable function $u : Q_T \to \mathbb{R}$ define the *modulars*

$$\rho(u;\Phi) := \int_{Q_T} \Phi(|u|)dQ \quad \text{and} \quad \rho(u;\Psi) := \int_{Q_T} \Psi(|u|)dQ\,. \tag{3.3}$$

It can be shown that Φ and Ψ are *complementary* $\mathcal{N}$-functions, in particular they are convex conjugates of each other. The functions for which the corresponding modulars ρ are finite form the *Orlicz classes* $K^\Phi(Q_T)$ and $K^\Psi(Q_T)$ respectively. The linear hulls (under pointwise addition and scalar multiplications) of these classes are the *Orlicz spaces* $L^\Phi(Q_T)$ and $L^\Phi(Q_T)$, which are Banach spaces of measurable functions with the norms

$$\|u\|_\Phi = \sup_{\rho(v;\Psi)\leq 1} \left| \int_{Q_T} uv \right| \quad \text{and} \quad \|u\|_\Psi = \sup_{\rho(v;\Phi)\leq 1} \left| \int_{Q_T} uv \right|.$$

The last assumption imposed on the damping g_0 is the following: with the $\mathcal{N}$-functions Φ and Ψ defined above, assume that for any set $\mathcal{V}$ of measurable functions on Q_T the boundedness of scalars

$$\left\{ \int_{Q_T} g_0(v)v : v \in \mathcal{V} \right\}$$

implies that the family $\{g_0(v) : v \in \mathcal{V}\}$ is bounded in $L^\Psi(Q_T)$. This is one of the key assumptions, necessary in the proof of local existence. From the energy inequality associated with the equation, one can obtain an upper bound on the term $\int_{Q_T} g_0(v)v$, which combined with the above mentioned assumption, provides a bound on $g_0(u_t)$ in the Orlicz space $L^\Psi(Q_T)$ wherein one can appeal to appropriate weak compactness results. This is clearly a generalization of the case of polynomially bounded dissipation, where the assumption is automatically satisfied: if $g_0(s) = |s|^{m-1}s$, $m > 1$, then the Orlicz spaces $L^\Psi(Q_T)$ and $L^\Phi(Q_T)$ are topologically isomorphic to the Lebesgue spaces $L^{\frac{m+1}{m}}(Q_T)$ and $L^m(Q_T)$ respectively. Moreover, the assumption is also satisfied for the cases of exponentially and logarithmically growing damping terms, as shown in [BRT], and thus permits the presence of such more general dissipative feedbacks in the system.

Interior source f

As before the map $f \in C^1(\mathbb{R})$ models nonlinear amplitude-modulated forces, is "energy-building" and thus counteracts the effect of the damping $g_0(u_t)$. We use the same notation (as the one in the case of weak solutions) for the polynomial upper bound for f

$$|f'(s)| \leq C|s|^{p-1}\,.$$

where $p \geq 1$ for $\dim(\Omega) = n \leq 4$, and $1 \leq p < \frac{n-2}{n-4}$ for $\dim(\Omega) = n \geq 5$.

3.2 Regular solutions

Throughout this section let A denote the Laplace operator on functions with vanishing traces:

$$Au = -\Delta u \quad \text{with} \quad \mathcal{D}(A) := \Big\{ u \in H^2(\Omega) \mid u = 0 \quad \text{on} \quad \Gamma \Big\} = H^2(\Omega) \cap H^1_0(\Omega)\,. \tag{3.4}$$

It is well-known that A is positive self-adjoint on $L_2(\Omega)$ and maximal accretive $L_2(\Omega) \to L_2(\Omega)$. So we can define fractional powers of A and identify via the topological isomorphism $\mathcal{D}(A^{1/2}) \approx H^1_0(\Omega)$.

The regular solutions will not (at least in dimensions $3, 4$) rely on the correlation between source and damping, and moreover will be considered for non-polynomially bounded dissipation. For this reason, the definition is slightly more detailed than merely asserting extra regularity on weak solutions.

Definition 3.1 (Regular Solution). Regular solutions will be defined for a range of interior source exponents: $1 \leq p < \infty$ if $n \leq 4$, or $1 \leq p \leq 2^{**}$ if $n > 4$, where $2^{**} = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embedding $H^2(\Omega) \to L^{2^{**}}(\Omega)$. By a **regular solution** of (3.1) defined on some interval $[0, T]$, we mean function u such that

$$(u, u_t) \;\in\; C_w\Big([0,T]; H^2(\Omega) \times H^1(\Omega)\Big) \cap L^\infty\Big(0,T; H^2(\Omega) \times H^1(\Omega)\Big)\,.$$

Moreover,

(i) Let r be sufficiently large to ensure the embedding $\mathcal{D}(A^r) \subset L^\infty(\Omega)$. For any test-function $\phi \in H^1(0,T;\mathcal{D}(A^r))$ the following identity must hold

$$\int_{Q_T} (-u_t\phi_t + \nabla u \cdot \nabla\phi)\, dQ + \int_{Q_T} g_0(u_t)\phi\, dQ = -(u_t, \phi)_\Omega \Big|_0^T + \int_{Q_T} f(u)\phi\, dQ \tag{3.5}$$

(ii) In addition, for $\psi_0 \in H^{-1}(\Omega)$, $\psi_1 \in L_2(\Omega)$

$$\lim_{t\to 0}\langle u(t) - u_0,\, \psi_0\rangle = 0 \quad \text{and} \quad \lim_{t\to 0}(u_t(t) - u_1,\, \psi_1)_\Omega = 0\,.$$

3.3 Results

The recent results in [BRT] demonstrate:

1.) Existence of local regular solutions for *all* interior sources with exponent $p \geq 1$ (and $1 \leq p < \frac{n-2}{n-4}$ for $\dim(\Omega) = n \geq 5$) without correlation with the damping. For example, if $|g_0(s)| \sim s^m$ then Figure 2-B shows that we obtain existence of local regular solutions for all values of $m > 0$. Moreover, the conclusion holds for damping of exponential or logarithmic growth (in modulus $|g_0(s)|$ as $s \to \pm\infty$) at infinity, or without damping at all ($g_0 \equiv 0$). The solution is unique if in addition,

$$u_t \in E^{\Phi}(Q_T),\ g(u_t) \in L^{\Psi}(Q_T), \text{ and } E^{\Phi}(Q_T) \text{ coincides with } L^m(Q_T), \text{ for some } m \geq 1, \tag{3.6}$$

where $E^{\Phi}(Q_T)$ is the closure of $L^{\infty}(Q_T)$ functions with respect to the Orlicz norm of $L^{\Phi}(Q_T)$.

2.) The condition $p < \frac{n-2}{n-4}$ in $n \geq 5$ for *local existence* can be relaxed to:

$$m > \frac{p(n-4)+4}{(2n-4)-p(n-4)} \quad \text{or} \quad p < \frac{2m(n-2)-4}{(n-4)(m+1)},$$

where $m > 0$ is such that

$$|s|^m \lesssim |g_0(s)| \quad \text{for all} \quad |s| \geq 1,$$

and the assumptions on f are adjusted to

(i) $|f''(s)| \lesssim |s|^{p-2}$ for all $s \in \mathbb{R}$.

(ii) $f(0) = 0$ (required, whereas it was not necessary in the presiding item).

Furthermore, if 3.6 are satisfied, and if in addition g_0 satisfiesĚ

$$|s-r|^{m+1} \lesssim (g_0(s) - g_0(r))(s-r)$$

then this solution is unique.

3.) Existence and uniqueness of global regular solutions whenever the damping grows at least as fast as the source, for example, when $m \geq p$ if g_0 has a lower polynomial bound of order m, analogously to the global existence statement for weak solutions.

The results for weak and regular solutions are summarized in Figure 2.

4 Open problems

In conclusion it is worth mentioning that the above foray into regular solutions reflects at some level the presently open problems concerning damping-source interaction for finite-energy solutions:

- Presently the analysis of finite-energy solutions for the source exponent $p = \frac{2n}{n-2}$ (i.e. merely ensuring that $f(u)$ is integrable) is open. Similarly one could consider the maximal boundary exponent $k = \frac{2(n-1)}{n-2}$.
- Likewise it is not known whether the restriction (e.g. in 3D)

$$p \leq \frac{6m}{m+1}$$

could be relaxed for initial data $(u, u_t) \in (H^1(\Omega) \cap L^{p+1}(\Omega)) \times L^{m+1}(\Omega)$. An analogous question holds for the boundary source and damping exponents k and q respectively.

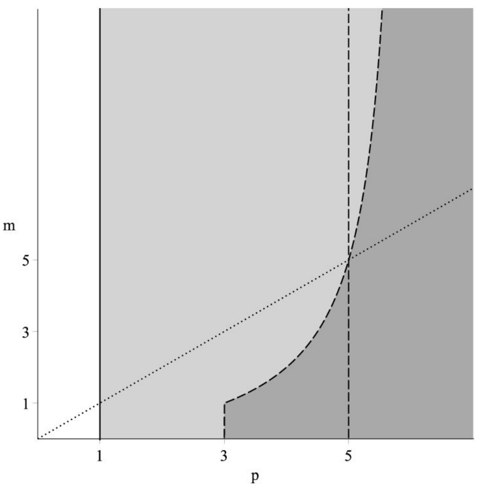

Figure 2. Comparison of well-posedness results for weak and regular solutions of (3.1) in dimension $n = 3$ when the damping has a polynomial like growth $|g_0(s)| \sim |s|^m$. The darker region corresponds to the extension in the range of exponents for regular solutions, with global existence at or above the indicated line $p = m$.

- Another interesting direction would be to investigate possible interaction between a strong boundary source and interior damping. Existence theorems for subcritical sources in the context of the potential well theory can be found in [Vit02b]; some blow-up results are also available [FLZ12]. However, to our knowledge it is still an open question whether a supercritical boundary source may allow for local existence of weak or regular solutions in the presence of interior damping only.

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