Palestine Journal of Mathematics

Vol. 3(1) (2014) , 33–43

CERTAIN RESULTS ON A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

S. P. Goyal, Onkar Singh and Teodor Bulboacă

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 30C45, Secondary 30C80.

Keywords and phrases: Analytic functions, starlike functions, close-to-convex functions, Schwarz's lemma, differential subordination.

The author (S. P. G) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II. The second author (O. S.) is also thankful to CSIR for JRF under the same scheme.

Abstract. In this paper we introduce and study a subclass of close-to-convex functions defined in the open unit disk. We establish the inclusion relationship, coefficient estimates and some sufficient conditions for a normalized function to be in our classes of functions. Furthermore, we discuss Fekete-Szegő problem for a more generalized class. The results presented here would provide extensions of those given in some earlier works.

1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{U}, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the usual subclass of $\mathcal{A}$ whose members are close-to-convex and starlike in $\mathcal{U}$ respectively. We also denote by $\mathcal{S}^*(\alpha)$ the class of starlike functions of order α ($0 \le \alpha < 1$).

For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g, and write $f(z) \prec g(z)$, if there exists a *Schwarz function* w (i.e. w is analytic in $\mathcal{U}$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$), such that $f(z) = g(w(z))$ for all $z \in \mathcal{U}$.

In particular, if the function g is univalent in $\mathcal{U}$, then we have

$$f(z) \prec g(z) \Leftrightarrow f(0) = 0 \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

More recently, Kowalczyk and Leś-Bomba [4] studied a subclass $K_s(\alpha)$ of analytic function related to the starlike functions. Thus, let f be an analytic function in $\mathcal{U}$ defined by (1.1). We say that $f \in K_s(\alpha)$ ($0 \le \alpha < 1$) if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that

$$\mathrm{Re}\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \alpha,\ z \in \mathcal{U}.$$

Also, in terms of subordination, an analytic function $f \in \mathcal{A}$ belongs to the class $K_s(\alpha)$ $(0 \le \alpha < 1)$ if and only if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that

$$\frac{-z^2 f'(z)}{g(z)g(-z)} \prec \frac{1 + (1 - 2\alpha)z}{1 - z}.$$

Motivated by the work by Kowalczyk and Leś-Bomba [4], we introduce and study a new class $K_s(A, B; u, v)$ of analytic functions related to starlike functions, as follows:

Definition 1.1. If $f \in \mathcal{A}$, we say that $f \in K_s(A, B; u, v)$ if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \frac{1+Az}{1+Bz} \tag{1.2}$$

$$\left(-1 \leq B < A \leq 1;\ u, v \in \mathbb{C}^* := \mathbb{C} \setminus \{0\},\ |u| \leq 1 \text{ and } |v| \leq 1\right).$$

Also, we say that the function $f \in K_s(A, B; u, v)$ is generated by the function g.

Remarks 1.1. (i) For the special case $A = 1 - 2\alpha$ $(0 \leq \alpha < 1)$ and $B = -1$, we find that

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \frac{1+(1-2\alpha)z}{1-z}$$

implies

$$\operatorname{Re}\left(\frac{uvz^2 f'(z)}{g(uz)g(vz)}\right) > \alpha,\ z \in \mathcal{U}, \quad (0 \leq \alpha < 1), \tag{1.3}$$

and we denote this subclass of functions by $K_s(\alpha; u, v)$.

(ii) Obviously, $K_s := K_s(0; 1, -1)$, where K_s is the class of functions studied by Gao and Zhou [2]. Also, $K_s(\gamma) := K_s(\gamma; 1, -1)$, where $K_s(\gamma)$ is the class of functions due to Kowalczyk and Leś-Bomba [4].

(iii) By simple calculations it is easy to see that the inequality (1.2) is equivalent to

$$\left|\frac{uvz^2 f'(z)}{g(uz)g(vz)} - 1\right| < \left|\frac{Buvz^2 f'(z)}{g(uz)g(vz)} - A\right|,\ z \in \mathcal{U}. \tag{1.4}$$

In this present paper we investigate coefficient inequalities, inclusion relationship, and the Fekete-Szegő problem for functions belonging to the class $K_s(A, B; u, v)$.

In our proposed investigation of the class $K_s(A, B; u, v)$ we require the following lemmas.

The next lemma can be easily proved:

Lemma 1.2. *Let $u, v \in \mathbb{C}^*$, with $|u| \leq 1$, $|v| \leq 1$ and let*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*\left(\frac{1}{2}\right). \tag{1.5}$$

If we put

$$G(z) = \frac{g(uz)g(vz)}{uvz} = z + \sum_{n=2}^{\infty} C_n(u, v) z^n,\ z \in \mathcal{U}, \tag{1.6}$$

where

$$C_n(u, v) = \sum_{j=1}^{n} b_j b_{n-j+1} u^{j-1} v^{n-j} \quad (n = 2, 3, \dots), \tag{1.7}$$

with $b_1 = 1$, then $G \in \mathcal{S}^*$.

Remarks 1.2. (i) Since $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, from Lemma 1.2 we obtain that G given by (1.5) belongs to $\mathcal{S}^*$. Then, by (1.3) we see that the class $K_s(\alpha; u, v)$ is a subclass of the class $\mathcal{K}$ of close-to-convex functions.

(ii) If we put $u = 1$ and $v = -1$, from (1.7) we find that

$$C_n(u, v) = \begin{cases} 0, & \text{if} \quad n = 2k \\ B_{2k-1}, & \text{if} \quad n = 2k-1, \end{cases}$$

where

$$B_{2k-1} = 2b_{2k-1} - 2b_2 b_{2k-2} + \cdots + (-1)^k 2b_{k-1} b_{k+1} + (-1)^{k+1} b_k^2, \tag{1.8}$$

and we get the earlier given result by Gao and Zhou [2] for their class of functions.

Lemma 1.3. *Let the function*

$$H(z) = 1 + h_1 z + h_2 z^2 + \ldots, \ z \in \mathcal{U},$$

be analytic in the unit disk $\mathcal{U}$*. Then, the function* H *satisfies the condition*

$$\left| \frac{H(z) - 1}{A - BH(z)} \right| < \beta, \ z \in \mathcal{U}, \quad (-1 \le B < A \le 1)$$

for some β $(0 < \beta \le 1)$*, if and only if there exists an analytic function* φ *in the unit disk* $\mathcal{U}$*, such that* $|\varphi(z)| \le \beta$ *for* $z \in \mathcal{U}$*, and*

$$H(z) = \frac{1 - Az\varphi(z)}{1 - Bz\varphi(z)}, \ z \in \mathcal{U}.$$

Proof. We will employ the technique similar with those of Padamanabhan [7]. Assume that the function

$$H(z) = 1 + h_1 z + h_2 z^2 + \ldots, \ z \in \mathcal{U},$$

satisfies the condition

$$\left| \frac{H(z) - 1}{A - BH(z)} \right| < \beta, \ z \in \mathcal{U} \quad (-1 \le B < A \le 1).$$

Setting

$$h(z) = \frac{1 - H(z)}{A - BH(z)},$$

we see that the function h analytic in $\mathcal{U}$, satisfies the inequality $|h(z)| < \beta$ for $z \in \mathcal{U}$ and $h(0) = 0$. Now, by using the Schwarz's lemma, we get that the function h has the form $h(z) = z\varphi(z)$, where φ is analytic in $\mathcal{U}$ and satisfies $|\varphi(z)| \le \beta$ for $z \in \mathcal{U}$. Thus, we obtain

$$H(z) = \frac{1 - Ah(z)}{1 - Bh(z)} = \frac{1 - Az\varphi(z)}{1 - Bz\varphi(z)}.$$

On the other hand, if

$$H(z) = \frac{1 - Az\varphi(z)}{1 - Bz\varphi(z)}$$

and $|\varphi(z)| \le \beta$ for $z \in \mathcal{U}$, then H is analytic in the unit disk $z \in \mathcal{U}$. Furthermore, since $|z\varphi(z)| \le \beta\,|z| < \beta$ for $z \in \mathcal{U}$, we get

$$\left| \frac{H(z) - 1}{A - BH(z)} \right| = |z\varphi(z)| < \beta, \ z \in \mathcal{U},$$

which completes the proof of our lemma. □

Lemma 1.4. *[5] Let* $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$*. Then,*

$$\frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z}.$$

Let $\mathcal{P}$ denote the class of functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \ z \in \mathcal{U}, \tag{1.9}$$

which are analytic in the open unit disk $\mathcal{U}$.

Lemma 1.5. *[6] If $p \in \mathcal{P}$ has the form* (1.9) *and satisfies* $\operatorname{Re} p(z) > 0$*, $z \in \mathcal{U}$, then for any number $\mu \in \mathbb{C}$ we have*

$$\left|c_2 - \mu c_1^2\right| \leq 2 \max\{1; |2\mu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z}{1-z} \quad \text{and} \quad p(z) = \frac{1+z^2}{1-z^2}.$$

Lemma 1.6. *[3] A function $p \in \mathcal{P}$ satisfies* $\operatorname{Re} p(z) > 0$*, $z \in \mathcal{U}$, if and only if*

$$p(z) \neq \frac{x-1}{x+1},\ z \in \mathcal{U},$$

for all $|x| = 1$.

Lemma 1.7. *If $f \in \mathcal{A}$ has the form* (1.1)*, then $f \in K_s(\alpha; u, v)$ if and only if*

$$1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,\ z \in \mathcal{U},\ |x| = 1,$$

where

$$A_n = \frac{na_n + (1-2\alpha)C_n(u,v) + (na_n - C_n(u,v))\,x}{2(1-\alpha)} \tag{1.10}$$

and the coefficients $C_n(u,v)$ are given by (1.7).

Proof. . According to Lemma 1.6, we have that $f \in K_s(\alpha; u, v)$ if and only if

$$\frac{\frac{uvz^2 f'(z)}{g(uz)g(vz)} - \alpha}{1-\alpha} \neq \frac{x-1}{x+1},\ z \in \mathcal{U}, \tag{1.11}$$

for all $|x| = 1$.

For $z = 0$, the above relation holds, since

$$\left.\frac{\frac{uvz^2 f'(z)}{g(uz)g(vz)} - \alpha}{1-\alpha}\right|_{z=0} = 1 \neq \frac{x-1}{x+1},\ |x| = 1.$$

For $z \neq 0$, the relation (1.11) is equivalent to

$$(x+1)\left(uvz^2 f'(z) - \alpha g(uz)g(vz)\right) \neq (x-1)(1-\alpha)g(uz)g(vz),$$

for all $z \in \mathcal{U} \setminus \{0\}$ and $|x| = 1$. Thus, we have

$$2(1-\alpha)z + \sum_{n=2}^{\infty} \left[na_n + (1-2\alpha)C_n(u,v) + x(na_n - C_n(u,v)\right] z^n \neq 0,$$

for $z \in \mathcal{U} \setminus \{0\}$ and $|x| = 1$, equivalently

$$2(1-\alpha)z\left[1 + \frac{\sum_{n=2}^{\infty} \left[na_n + (1-2\alpha)C_n(u,v) + x(na_n - C_n(u,v)\right]}{2(1-\alpha)} z^{n-1}\right] \neq 0. \tag{1.12}$$

Dividing both sides of (1.12) by $2(1-\alpha)z$, we obtain

$$1 + \frac{\sum_{n=2}^{\infty} \left[na_n + (1-2\alpha)C_n(u,v) + x(na_n - C_n(u,v)\right]}{2(1-\alpha)} z^{n-1} \neq 0,$$

for $z \in \mathcal{U} \setminus \{0\}$ and $|x| = 1$, which completes our proof. □

2 Main Results

We will prove a theorem which provides us a sufficient condition for functions to belong into the class $K_s(A,B;u,v)$.

Theorem 2.1. *Let the functions f and g defined by* (1.1) *and* (1.5) *respectively, and for $n = 2,3,4,\ldots$ let define the coefficients $C_n(u,v)$ by* (1.7). *If*

$$(1+|B|)\sum_{n=2}^{\infty} n\,|a_n| + (1+|A|)\sum_{n=2}^{\infty} |C_n(u,v)| < A - B$$

$$\left(-1 \le B < A \le 1;\ u,v \in \mathbb{C}^*,\ |u| \le 1,\ |v| \le 1\right),$$

then $f \in K_s(A,B;u,v)$.

Proof. For the functions f given by (1.1) and g given by (1.5) set

$$\Delta = \left| zf'(z) - \frac{g(uz)g(vz)}{uvz} \right| - \left| Bzf'(z) - \frac{Ag(uz)g(vz)}{uvz} \right| =$$

$$\left| \sum_{n=2}^{\infty} na_n z^n - \sum_{n=2}^{\infty} C_n(u,v) z^n \right| - \left| (B-A)z + B\sum_{n=2}^{\infty} na_n z^n - A\sum_{n=2}^{\infty} C_n(u,v) z^n \right|.$$

From here, we have

$$\Delta \le -(A-B)|z| + (1+|B|)\sum_{n=2}^{\infty} n\,|a_n|\,|z|^n + (1+|A|)\sum_{n=2}^{\infty} |C_n(u,v)|\,|z|^n,$$

hence

$$\Delta \le \left(-(A-B) + (1+|B|)\sum_{n=2}^{\infty} n\,|a_n| + (1+|A|)\sum_{n=2}^{\infty} |C_n(u,v)| \right)|z|,\ z \in \mathcal{U}.$$

Using the assumption we obtain $\Delta < 0$, and thus we have

$$\left| zf'(z) - \frac{g(uz)g(vz)}{uvz} \right| < \left| Bzf'(z) - \frac{Ag(uz)g(vz)}{uvz} \right|,\ z \in \mathcal{U},$$

hence from (1.4) we conclude that $f \in K_s(A,B;u,v)$. □

Remark 2.2. Taking $u = 1$, $v = -1$, $A = 1 - 2\gamma$ $(0 \le \gamma < 1)$ and $B = -1$ in Theorem 2.1, we get the result obtained by Kowalczyk and Leś-Bomba [4].

The next theorem gives the estimate of the coefficients.

Theorem 2.3. *Let $-1 \le B < A \le 1$. Suppose that an analytic function f given by* (1.1) *and $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ given by* (1.5) *are such that the condition* (1.2) *holds. Then, for $n \ge 2$ we have*

$$|na_n - C_n(u,v)|^2 - |A-B|^2 \le \tag{2.1}$$

$$\sum_{k=2}^{n-1} \left(\left|B^2-1\right| k^2 |a_k|^2 + \left|A^2-1\right| |C_k(u,v)|^2 + 2k\,|a_k C_k(u,v)|\,|1-AB| \right),$$

where the coefficients $C_n(u,v)$ are defined by (1.7).

Proof. Since $f \in K_s(A,B;u,v)$ for some $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, the inequality (1.4) holds. From Lemma 1.3 we have

$$\frac{zf'(z)}{G(z)} = \frac{1 - Az\varphi(z)}{1 - Bz\varphi(z)},\ z \in \mathcal{U}, \tag{2.2}$$

where φ is an analytic functions in $\mathcal{U}$, $|\varphi(z)| \leq 1$ for $z \in \mathcal{U}$, and G is given by (1.6).

From (2.2), by using the definitions (1.1) and (1.6) for f and G respectively, we obtain that

$$\left[-B\left(z + \sum_{n=2}^{\infty} na_n z^n\right) + A\left(z + \sum_{n=2}^{\infty} C_n(u,v) z^n\right)\right] z\phi(z) =$$
$$\sum_{n=2}^{\infty} C_n(u,v) z^n - \sum_{n=2}^{\infty} na_n z^n,\ z \in \mathcal{U}. \tag{2.3}$$

Since the function $z\varphi(z)$ has the expansion

$$z\varphi(z) = \sum_{n=1}^{\infty} t_n z^n,\ z \in \mathcal{U},$$

from (2.3) we find that

$$\left((A-B)z - B\sum_{n=2}^{\infty} na_n z^n + A\sum_{n=2}^{\infty} C_n(u,v) z^n\right)\sum_{n=1}^{\infty} t_n z^n =$$
$$\sum_{n=2}^{\infty} C_n(u,v) z^n - \sum_{n=2}^{\infty} na_n z^n,\ z \in \mathcal{U}. \tag{2.4}$$

Now, equating the coefficient of z^n in (2.4), we get

$$C_n(u,v) - na_n = (A-B)t_{n-1} + \left(-2Ba_2 + AC_2(u,v)\right)t_{n-2} +$$
$$\left(-3Ba_3 + AC_3(u,v)\right)t_{n-3} + \cdots + \left(-(n-1)Ba_{n-1} + AC_{n-1}(u,v)\right)t_1.$$

and thus, the coefficient combination on the R.H.S. of (2.4) depends only upon the coefficients combinations

$$\left(-2Ba_2 + AC_2(u,v)\right), \ldots, \left(-(n-1)Ba_{n-1} + AC_{n-1}(u,v)\right).$$

Hence, for $n \geq 2$ we can write that

$$\left[(A-B)z + \sum_{k=2}^{n-1}\left(-Bka_k + AC_k(u,v)\right)z^k\right] z\varphi(z) =$$
$$\sum_{k=2}^{n}\left(C_k(u,v) - ka_k\right)z^k + \sum_{k=n+1}^{\infty} d_k z^k,\ z \in \mathcal{U},$$

and using the fact that $|z\varphi(z)| \leq |z| < 1$ for all $z \in \mathcal{U}$, this reduces to the inequality

$$\left|(A-B)\,z + \sum_{k=2}^{n-1}\left(-Bka_k + AC_k(u,v)\right)z^k\right| > \left|\sum_{k=2}^{n}\left(C_k(u,v) - ka_k\right)z^k + \sum_{k=n+1}^{\infty} d_k z^k\right|.$$

Squaring the above inequality and integrating along the circle $|z| = r$ $(0 < r < 1)$, we obtain

$$\int_0^{2\pi} \left| (A-B)re^{i\theta} + \sum_{k=2}^{n-1} \left(-Bka_k + AC_k(u,v)\right) r^k e^{ik\theta} \right|^2 d\theta >$$

$$\int_0^{2\pi} \left| \sum_{k=2}^{n} \left(C_k(u,v) - ka_k\right) r^k e^{ik\theta} + \sum_{k=n+1}^{\infty} d_k r^k e^{ik\theta} \right|^2 d\theta.$$

Using now the Parseval's inequality, we obtain

$$|A-B|^2 r^2 + \sum_{k=2}^{n-1} \left|-Bka_k + AC_k(u,v)\right|^2 r^{2k} >$$

$$\sum_{k=2}^{n} \left|ka_k - C_k(u,v)\right|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k}.$$

Letting $r \to 1$ in this inequality, we get

$$|A-B|^2 + \sum_{k=2}^{n-1} \left|-Bka_k + AC_k(u,v)\right|^2 \geq \sum_{k=2}^{n} \left|ka_k - C_k(u,v)\right|^2 + \sum_{k=n+1}^{\infty} |d_k|^2,$$

which implies

$$|A-B|^2 + \sum_{k=2}^{n-1} \left|-Bka_k + AC_k(u,v)\right|^2 \geq \sum_{k=2}^{n} \left|ka_k - C_k(u,v)\right|^2.$$

Hence we deduce that

$$|na_n - C_n(u,v)|^2 - |A-B|^2 \leq$$

$$\sum_{k=2}^{n-1} \left(\left|B^2 - 1\right| k^2 |a_k|^2 + \left|A^2 - 1\right| |C_k(u,v)|^2 + 2k \left|a_k C_k(u,v)\right| |1 - AB| \right),$$

and thus we obtain the inequality (2.1), which completes our proof. □

Remark 2.4. Taking $u = 1$, $v = -1$, $A = 1 - 2\gamma$ $(0 \leq \gamma < 1)$, and $B = -1$ in above theorem, we get the result obtained by Kowalczyk and Leś-Bomba [4].

Now we establish a result on inclusion relationship contained in the next theorem.

Theorem 2.5. *Let $u, v \in \mathbb{C}^*$, with $|u| \leq 1$, $|v| \leq 1$, and let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 < 1$. Then,*

$$K_s\left(A_1, B_1; u, v\right) \subset K_s\left(A_2, B_2; u, v\right).$$

Proof. Supposing that $f \in K_s(A_1, B_1; u, v)$, we have

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \frac{1 + A_1 z}{1 + B_1 z}.$$

Since $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 < 1$, by Lemma 1.4 we get

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \frac{1 + A_1 z}{1 + B_1 z} \prec \frac{1 + A_2 z}{1 + B_2 z},$$

hence $f \in K_s(A_2, B_2; u, v)$. □

Theorem 2.6. *If the function $f \in \mathcal{A}$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(ka_k+(1-2\alpha)C_k(u,v))(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|+\right.$$
$$\left.\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k-C_k(u,v)\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right)\leq 2(1-\alpha),$$

where $0\leq\alpha<1$, $\gamma,\delta\in\mathbb{R}$ and the coefficients $C_n(u,v)$ are given by (1.7)*, then $f\in K_s(\alpha;u,v)$.*

Proof. According to Lemma 1.7, to prove that $1+\sum\limits_{n=2}^{\infty}A_n z^{n-1}\neq 0$ for all $z\in\mathcal{U}$ and $|x|=1$, where A_n are given by (1.10), it is sufficient to show that

$$\left(1+\sum_{n=2}^{\infty}A_n z^{n-1}\right)(1-z)^{\gamma}(1+z)^{\delta}=$$
$$1+\sum_{n=2}^{\infty}\left[\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}A_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right]z^{n-1}\neq 0,$$

for all $z\in\mathcal{U}$ and $|x|=1$, where $A_0=0$, $A_1=1$ and $\gamma,\delta\in\mathbb{R}$. Thus, if the function f satisfies

$$\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}A_k(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\leq 1,\ |x|=1,$$

that is, if

$$\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(ka_k+(1-2\alpha)C_k(u,v))\right.\right.$$
$$\left.\left.+x\left(ka_k-C_k(u,v)\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\leq$$
$$\frac{1}{2(1-\alpha)}\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}(ka_k+(1-2\alpha)C_k(u,v))(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right.$$
$$\left.+|x|\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k-C_k(u,v)\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right)\leq 1,\ |x|=1,$$

then $f\in K_s(\alpha;u,v)$, and the proof is complete. □

Letting $\gamma=\delta=0$ in Theorem 2.6, we have:

Corollary 2.7. *If the function $f\in\mathcal{A}$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left(|na_n+(1-2\alpha)C_n(u,v)|+|na_n-C_n(u,v)|\right)\leq 2(1-\alpha)$$

for some α $(0\leq\alpha<1)$, where the coefficients $C_n(u,v)$ are given by (1.7)*, then $f\in K_s(\alpha;u,v)$.*

Taking $u=1$ and $v=-1$ in Theorem 2.6, we obtain:

Corollary 2.8. *If the function $f \in \mathcal{A}$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k+(1-2\alpha)B_{2k-1}\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|+\right.$$
$$\left.\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k-B_{2k-1}\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right)\leq 2(1-\alpha)$$

for some α $(0 \leq \alpha < 1)$ and $\gamma, \delta \in \mathbb{R}$, then $f \in K_s(\alpha) := K_s(\alpha; 1, -1)$, where B_{2k-1} $(k = 2, 3, 4 \dots)$ are given by (1.8) *and $B_1 = 0$.*

For $\alpha = 0$ the above Corollary reduces to the next special case:

Corollary 2.9. *If the function $f \in \mathcal{A}$ has the form* (1.1) *and satisfies the condition*

$$\sum_{n=2}^{\infty}\left(\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k+B_{2k-1}\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|+\right.$$
$$\left.\left|\sum_{l=1}^{n}\left\{\sum_{k=1}^{l}\left(ka_k-B_{2k-1}\right)(-1)^{l-k}\binom{\gamma}{l-k}\right\}\binom{\delta}{n-l}\right|\right)\leq 2,$$

where $\gamma, \delta \in \mathbb{R}$, then $f \in K_s(0)$, where B_{2k-1} $(k = 2, 3, 4 \dots)$ are given by (1.8) *and $B_1 = 0$.*

3 Fekete-Szegő Inequality

In this section we assume that the function φ is an analytic function with positive real part, that maps the unit disk $\mathcal{U}$ onto a starlike region which is symmetric with respect to real axis, and is normalized by $\varphi(0) = 1$ and $\varphi'(0) > 0$. In such case, the function φ has an expansion of the form $\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots$, $B_1 > 0$.

Definition 3.1. Let f be an analytic function in $\mathcal{U}$ defined by (1.1). We say that $f \in K_s(\varphi; u, v)$, if there exist $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$ such that

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} \prec \varphi(z)$$
$$\left(u, v \in \mathbb{C}^*,\ |u| \leq 1 \text{ and } |v| \leq 1\right),$$

where the function φ satisfies the requirements mentioned just above this definition.

Theorem 3.2 (Fekete-Szegő Inequality). *For a function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$ belonging to the class $K_s(\varphi; u, v)$, the following sharp estimate holds:*

$$\left|a_3-\mu a_2^2\right| \leq \frac{1}{3}\max\left\{B_1, \left|B_2-\frac{3\mu}{4}B_1^2\right|\right\}-\frac{uv}{3}+$$
$$B_1 c_1 b_2(u+v)\left(\frac{1}{6}-\frac{\mu}{4}\right)+(u+v)^2\left(\frac{b_3}{3}-\frac{\mu b_2^2}{4}\right). \tag{3.1}$$

Proof. Using the definition of the subordination between two analytic function, there exists a function w analytic in $\mathcal{U}$, normalized by $w(0) = 0$, satisfying $|w(z)| < 1$, $z \in \mathcal{U}$, and

$$\frac{uvz^2 f'(z)}{g(uz)g(vz)} = \varphi(w(z)),\ z \in \mathcal{U}.$$

If

$$p_1(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\ldots,\ z\in\mathcal{U}, \tag{3.2}$$

then p_1 is analytic and has positive real part in $\mathcal{U}$, with $p_1(0)=1$, and from (3.2) we obtain

$$w(z)=\frac{c_1}{2}z+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)z^2+\ldots,\ z\in\mathcal{U}. \tag{3.3}$$

Letting

$$p(z)=\frac{uvz^2f'(z)}{g(uz)g(vz)}=1+d_1z+d_2z^2+\ldots,\ z\in\mathcal{U}, \tag{3.4}$$

this gives

$$\begin{aligned} d_1&=2a_2-b_2(u+v),\\ d_2&=3a_3-2a_2b_2(u+v)-b_3(u^2+v^2)-b_2^2uv+b_2^2(u+v)^2. \end{aligned} \tag{3.5}$$

Since φ is univalent and $p(z)\prec\varphi(z)$, by using (3.3) we obtain

$$p(z)=\varphi(w(z))=1+\frac{B_1c_1}{2}z+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2\right]z^2+\ldots,\ z\in\mathcal{U}. \tag{3.6}$$

Now, from (3.4), (3.5), and (3.6), we obtain

$$\frac{B_1c_1}{2}=2a_2-b_2(u+v),$$
$$\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{1}{4}c_1^2B_2=$$
$$3a_3-2a_2b_2(u+v)-b_3(u^2+v^2)-b_2^2uv+b_2^2(u+v)^2,$$

and therefore, we conclude that

$$a_3-\mu a_2^2=\frac{1}{6}B_1\left(c_2-\nu c_1^2\right)-\frac{2uv}{3}\left(b_3-\frac{b_2^2}{2}\right)+$$
$$B_1c_1b_2(u+v)\left(\frac{1}{6}-\frac{\mu}{4}\right)+(u+v)^2\left(\frac{b_3}{3}-\frac{\mu b_2^2}{4}\right),$$

where

$$\nu=\frac{1}{2}\left(1-\frac{B_2}{B_1}+\frac{3\mu}{4}B_1\right).$$

The desired result follows upon using the Lemma 1.5 and using estimate that $\left|b_3-\frac{b_2^2}{2}\right|\le\frac{1}{2}$, for any analytic function $g(z)=z+b_2z^2+b_3z^3+\ldots$, $z\in\mathcal{U}$, which is starlike of order $\frac{1}{2}$ (see [1]). □

Remarks 3.1. (i) Putting $u=1$ and $v=-1$ in the above theorem we get the result obtained recently by Cho et al. [1].

(ii) Setting $\mu=0$ in Theorem 3.2 we get the sharp estimate for the third coefficient of function in $K_s(\varphi;u,v)$, that is

$$|a_3|\le\frac{B_1}{3}\max\left\{1,\left|\frac{B_2}{B_1}\right|\right\}-\frac{uv}{3}+\frac{B_1c_1b_2(u+v)}{6}+(u+v)^2\frac{b_3}{3}. \tag{3.7}$$

(iii) Putting $u = 1$ and $v = -1$ in (3.7) we get the sharp estimate for the third coefficient of function in the class $K_s(\varphi)$, due to Cho et al. [1].

(iv) If we let $\mu \to \infty$ in (3.1) we get the sharp estimate for $|a_2|$, i.e.

$$|a_2| \leq \sqrt{\frac{B_1^2}{4} - \frac{b_2(u+v)}{4}\,[B_1c_1 + (u+v)b_2]}. \tag{3.8}$$

(v) If we put $u = 1$ and $v = -1$ in (3.8) we get the result due to Cho et al. [1]. Also, for $u = 1$ and $v = -1$ and $\varphi(z) = \dfrac{1+z}{1-z}$, the results reduces to the corresponding one from [2, Theorem 2, p. 125].

References

[1] N. E. Cho, O. S. Kwon and V. Ravichandran, Coefficient, distortion and growth inequalities for certain close-to-convex functions, *J. Inequal. Appl.* (2011), 2011:100.

[2] C.-Y. Gao and S.-Q. Zhou, On a class of analytic functions related to the starlike functions, *Kyungpook Math. J.*, **45**(2005), 123–130.

[3] T. Hayami, S. Owa and H. M. Srivastava, Coefficient inequalities for certain classes of analytic and univalent functions, *J. Ineq. Pure Appl. Math.*, **8**(2007), Issue 4, Article 95, 1–10.

[4] J. Kowalczyk and E. Leś-Bomba, On a subclass of close-to-convex functions, *Appl. Math. Lett.*, **23**(2010), 1147–1151.

[5] M. S. Liu, On a subclass of p–valent close-to-convex functions of order β and type α, *J. Math. Study*, **30**(1997), 102–104.

[6] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, in *Proceedings of the Conference on Complex Analysis, Tianjin, 1992*, 157–169, Conf. Proc. Lecture Notes Anal. I, Int. Press, Cambridge, MA, 1994.

[7] K. S. Padmanabhan, On a certain classes of starlike functions in the unit disc, *J. Indian Math. Soc.*, **32**(1968), 89–103.

Author information

S. P. Goyal, Onkar Singh, Department of Mathematics, University of Rajasthan, Jaipur-302004, India.
E-mail: `somprg@gmail.com` `onkarbhati@gmail.com`

Teodor Bulboacă, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania.
E-mail: `bulboaca@math.ubbcluj.ro`

Received: January 2, 2013

Accepted: June 11, 2013