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FIP and FCP products of ring morphisms

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Abstract. We characterize some types of FIP and FCP ring extensions $R \subset S$, where S is not an integral domain and R may not be an integral domain, contrary to a general trend. In each of the sections, S is a product of finitely many rings that are related to R in various ways. Ring extensions of the form $R^n \hookrightarrow R^p$ associated to some matrices are also considered. Our tools are minimal ring morphisms and seminormalization, while Artinian conditions on rings are ubiquitous.

1 Introduction and Notation

All rings R considered are commutative, nonzero and unital; all morphisms of rings are unital. Let $R \subseteq S$ be a (ring) extension. The set of all R-subalgebras of S is denoted by $[R, S]$. The extension $R \subseteq S$ is said to have FIP (for the "finitely many intermediate algebras property") if $[R, S]$ is finite. A *chain* of R-subalgebras of S is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that the extension $R \subseteq S$ has FCP (for the "finite chain property") if each chain of R-subalgebras of S is finite. It is clear that each extension that satisfies FIP must also satisfy FCP. Our main tool are the minimal (ring) extensions, a concept introduced by Ferrand-Olivier [10]. Recall that an extension $R \subset S$ is called *minimal* if $[R, S] = \{R, S\}$. The key connection between the above ideas is that if $R \subseteq S$ has FCP, then any maximal (necessarily finite) chain $R = R_0 \subset R_1 \subset \cdots \subset R_{n-1} \subset R_n = S$, of R-subalgebras of S, with *length* $n < \infty$, results from juxtaposing n minimal extensions $R_i \subset R_{i+1}$, $0 \leq i \leq n-1$. Following [14], the *length of* $[R, S]$, denoted by $\ell[R, S]$, is the supremum of the lengths of chains of R-subalgebras of S. In particular, if $\ell[R, S] = r$, for some integer r, there exists a maximal chain $R = R_0 \subset R_1 \subset \cdots \subset R_{r-1} \subset R_r = S$ of R-subalgebras of S with length r. Against the general trend, we characterized arbitrary FCP and FIP extensions in [8], a joint paper by D. E. Dobbs and ourselves whereas most of papers on the subject are concerned with extensions of integral domains. It is worth noticing here that FCP extensions of integral domains (ignoring fields) are generally nothing but extensions of overrings as a quick look at [6, Theorems 4.1,4.4] shows because FCP extensions are composite of minimal extensions.

In this paper, we will continue to consider the FCP or FIP properties of extensions for special types of extensions between not necessarily integral domains, like $K \to K^n$ where K is a field. It is known that these latter extensions have FIP and actually they motivated us to study generalizations. Our study shows phenomena that do not arise in the integral domain case and provides us a lot of new examples, that may be sometimes surprising. They are most of time integral and seminormal within the meaning of Swan. Problems arise when they are not seminormal, leading to the computation of seminormalizations. The Gilmer's seminal work on FIP and FCP is settled for overrings of an integral domain R, with quotient field K. In particular, [12, Theorem 2.14] shows that $R \subseteq S$ has FCP for each overring S of R only if R/C is an Artinian ring, where $C = (R : \overline{R})$ is the conductor of R in its integral closure. This necessary Artinian condition is not surprisingly present in all our results.

Product morphisms $R \to \prod_{i=1}^{n} R_i$ that are extensions are the theme of our work. We warn the reader that we have developed a similar theory for idealizations of modules, with necessarily finitely many submodules [19]. We will observe that results may depend on the value of n, and a lot of them are only valid for $n = 2$.

In Section 2, we look at diagonal extensions $R \subseteq \prod_{i=1}^{n} R_i$, for some finitely many FCP or FIP extensions $R \subseteq R_i$. When $R \subseteq R_i$ has FCP for each i, Theorem 2.11 asserts that $R \subseteq \prod_{i=1}^n R_i$ has FCP if and only if R is an Artinian ring. The FIP condition is much more complicated. For instance, R has finitely many ideals if $R \subseteq \prod_{i=1}^n R_i$ has FIP (Proposition 2.2). Moreover, $R \subseteq R^2$ has FIP if and only if R has finitely many ideals (Corollary 2.5).

Section 3 is concerned with extensions of the form $R/\cap_{j=1}^n I_j \subseteq \prod_{j=1}^n (R/I_j)$, where $I_1, \dots, I_n$ are proper ideals of a ring R, not necessarily distinct and such that $\cap_{j=1}^n I_j = 0$. Then, $R \subseteq \prod_{j=1}^n R/I_j$ has FCP if and only if R/C is Artinian, where its conductor C can be computed as follows. Setting $J_j := \cap_{k=1, k\neq j}^n I_k$ for each $j \in \{1, \dots, n\}$, we get that $C := \sum_{j=1}^n J_j$ (Proposition 3.1). We are able to generalize a Ferrand-Olivier's result. It states that if R is a ring and $\{I_1, \dots, I_n\}$, $n > 2$, is a family of ideals of R such that $\cap_{j=1}^n I_j = 0$, then $R \subseteq \prod_{j=1}^n (R/I_j)$ is a minimal extension if and only if there exist $j_0, k_0 \in \{1, \dots, n\}$, $j_0 \neq k_0$ such that $I_{j_0} + I_{k_0} \in \mathrm{Max}(R)$ and $I_j + I_k = R$ for any $(j,k) \neq (j_0, k_0)$, $j \neq k$. If this condition holds, then $\{I_1, \dots, I_n\}$ satisfies a weak Chinese Remainder Theorem (Theorem 3.13).

Section 4 is devoted to diagonal extensions $R \subseteq R^n$ and heavily uses results of Section 3. We get in Theorem 4.2 that $R \subseteq R^n$ has FIP if and only if R has finitely many ideals and $n \leq 2$ as soon as there exists a maximal ideal M of R such that R_M is not a field and R/M is an infinite field. We are then able to give a general characterization of FIP extensions $R \subseteq \prod_{i=1}^n R_i$ studied in Section 2. We show that R^n may have different structures of R^p-algebras if $p < n$ are two positive integers, leading to different occurrences of FIP extensions $R^p \hookrightarrow R^n$.

Let R be a ring. As usual, $\mathrm{Spec}(R)$ (resp. $\mathrm{Max}(R)$) denotes the set of all prime ideals (resp. maximal ideals) of R. If I is an ideal of R, we set $\mathrm{V}_R(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$. If $R \subseteq S$ is a ring extension and $P \in \mathrm{Spec}(R)$, then S_P is the localization $S_{R\setminus P}$ and $(R : S)$ is the conductor of $R \subseteq S$. When there is no possible confusion, we denote the integral closure of R in S by $\overline{R}$. Recall that if E is an R-module, its *support* $\mathrm{Supp}_R(E)$ is the set of prime ideals P of R such that $E_P \neq 0$ and $\mathrm{MSupp}_R(E) := \mathrm{Supp}_R(E) \cap \mathrm{Max}(R)$. If E is an R-module, $\mathrm{L}_R(E)$ is its length. We will shorten finitely generated module into f.g. module. Recall that a *special principal ideal ring* (SPIR) is a principal ideal ring R with a unique nonzero prime ideal $M = Rt$, such that M is nilpotent of index $p > 0$. Hence a SPIR is not a field. Each nonzero element of a SPIR is of the form ut^k for some unit u and some *unique* integer $k < p$. Finally, as usual, $\subset$ denotes proper inclusion and $|X|$ denotes the cardinality of a set X.

There are four types of minimal extension, but we only need ramified minimal extensions.

Theorem 1.1. *[10, Théorème 2.2], [18, Theorem 3.3] Let $R \subset T$ be a ring extension and $M := (R : T)$. Then $R \subset T$ is a minimal* **ramified** *extension if and only if $M \in \mathrm{Max}(R)$ and there exists $M' \in \mathrm{Max}(T)$ such that $M'^2 \subseteq M \subset M'$, $[T/M : R/M] = 2$ (resp. $\mathrm{L}_R(M'/M) = 1$), and the natural map $R/M \to T/M'$ is an isomorphism.*

If these conditions hold, then $R_P = T_P$ for each $P \in \mathrm{Spec}(R) \setminus \{M\}$.

We also need some results about seminormality and t-closedness that we recall here.

Definition 1.2. An integral extension $f : R \hookrightarrow S$ is termed:

(1) *infra-integral* if all its residual extensions are isomorphisms [17].

(2) *subintegral* if f is infra-integral and ${}^a f$ is bijective [20].

A minimal morphism is ramified if and only if it is subintegral. Let $\{R_1, \dots, R_n\}$ be finitely many infra-integral extensions of a ring R. It is easy to show that $R \to \prod_{i=1}^n R_i$ is infra-integral. But this result is no longer valid for subintegrality.

A ring extension $R \subseteq S$ is called *t-closed* if $b \in S$, $r \in R$, $b^2 - rb, b^3 - rb^2 \in R \Rightarrow b \in R$ [17]. Now, $R \subseteq S$ is called *seminormal* if $b \in S$, $b^2, b^3 \in R \Rightarrow b \in R$ [20]. If $R \subset S$ is seminormal, $(R : S)$ is a radical ideal of S. The *t-closure* ${}_S^tR$ (resp. *seminormalization* ${}_S^+R$) of R in S is the smallest $B \in [R,S]$ such that $B \subseteq S$ is t-closed (resp. seminormal). Moreover, ${}_S^tR$ (resp. ${}_S^+R$) is the greatest $B \in [R,S]$ such that $R \subseteq B$ is infra-integral (resp. subintegral). The chain $R \subseteq {}_S^+R \subseteq {}_S^tR \subseteq S$ is called the *canonical decomposition* of $R \subseteq S$.

T-closures and seminormalizations both commute with localization at arbitrary multiplicatively closed subsets ([16, Proposition 3.6], [20, Proposition 2.9]).

According to J. A. Huckaba and I. J. Papick [13], an extension $R \subseteq S$ is termed a Δ_0*-extension* provided each R-submodule of S containing R is an element of $[R,S]$. We recall here for later use an unpublished result of the Gilbert's dissertation.

Proposition 1.3. *[11, Proposition 4.12] Let $R \subseteq S$ be a ring extension with conductor I and such that $S = R + Rt$ for some $t \in S$. Then the R-modules R/I and S/R are isomorphic. Moreover, each of the R-modules between R and S is a ring (and so there is a bijection from $[R, S]$ to the set of ideals of R/I).*

We end this introduction with a new result that introduces and gives the flavor of the next section.

Proposition 1.4. *Let R be a commutative ring and $n \geq 2$ a positive integer.*

(1) $(R : R^n) = 0$ and $R \subseteq R^n$ is infra-integral. Moreover, $R \subseteq R^n$ is seminormal if and only if R is reduced.

(2) $R \subseteq R^n$ has FCP if and only if R is an Artinian ring.

Proof. (1) Obviously, $R \subseteq R^n$ has a zero conductor and is infra-integral. Assume that R is reduced. Then, [20, Lemma 3.1] gives that $R \subseteq R^n$ is seminormal. Conversely, if $R \subseteq R^n$ is seminormal, then $0 = (R : R^n)$ is a radical ideal of R, so that R is reduced.

(2) Assume that $R \subseteq R^n$ has FCP and that there is an infinite chain $\{I_j\}_{j\in J}$ of ideals of R. For each $j \in J$, set $S_j := R + (0 \times I_j)$. Then, $\{S_j\}_{j\in J}$ is an infinite chain of R-subalgebras of R^n, which is absurd. Hence, any chain of ideals of R is finite and R is Artinian.

Conversely, R^n is f.g. over R. Thus $R \subseteq R^n$ has FCP in view of [8, Theorem 4.2], if R is Artinian. □

The following results will be useful.

Proposition 1.5. *Let (R, M) be a local Artinian ring such that R/M is infinite and $R \subseteq S$ a ring extension with conductor $C := (R : S)$.*

(1) If $R \subset S$ has FIP and is subintegral, then $[R, S]$ is linearly ordered.

(2) If $R \subseteq S$ is finite, seminormal and infra-integral, then $R \subseteq S$ has FIP.

(3) If $R \subset S$ is finite and infra-integral, then $R \subset S$ has FIP if and only if $R \subseteq {}^+_S R$ has FIP.

Proof. (1) There is no harm to assume that $C = 0$ because the map $[R, S] \to [R/C, S/C]$ defined by $T \mapsto T/C$ is bijective. If R is not a field, then the proof of [8, Proposition 5.15] shows that $[R, S]$ is linearly ordered.

Now, assume that R is a field, so that $0 = (R : S)$ and R is infinite. Since $R \subset S$ is an FIP subintegral extension, S is Artinian local and not a field with $\{N\} := \mathrm{Max}(S)$, because $R \cong S/N$ by subintegrality shows that $N \neq 0$. From [2, Theorem 3.8], we get that $S = R[\alpha]$, for some $\alpha \in S$ such that $\alpha^3 = 0$. In view of the proof of [2, Lemma 3.6(b)], $[R, S]$ is linearly ordered.

(2) We can assume that $R \neq S$ and $C = 0$ by considering $R/C \to S/C$ and using [8, Proposition 3.7(c)]. By [8, Proposition 5.16], we get that $R \subset S$ has FIP.

(3) Assume that $R \subset S$ is finite and infra-integral and set $T := {}^+_S R$. Then, T is local Artinian with maximal ideal N and $T/N \cong R/M$ is infinite. Moreover, $T \subseteq S$ is finite, seminormal, infra-integral and has FIP by (2).

If $R \subset S$ has FIP, then $R \subseteq T$ has FIP. Conversely, assume that $R \subseteq T$ has FIP. In view of [8, Theorem 5.8], $R \subset S$ has FIP. □

We will use the following result. If $R_1, \ldots, R_n$ are finitely many rings, the ring $R_1 \times \cdots \times R_n$ localized at the prime ideal $P_1 \times R_2 \times \cdots \times R_n$ is isomorphic to $(R_1)_{P_1}$ for $P_1 \in \mathrm{Spec}(R_1)$. This rule works for any prime ideal of the product.

2 FCP or FIP extensions for products of rings

We extract from the more precise result [9, Proposition 4.15] the following statement, about the canonical diagonal extension $K \subseteq K^n$, for a field K and a positive integer $n > 1$. Recall that the nth *Bell number* B_n is the number of partitions of $\{1, \ldots, n\}$ [3, p. 214]. Actually, the finiteness of $|[K, K^n]|$ comes from [5, Proposition 3, p. 29].

Proposition 2.1. *Let K be a field and n a positive integer, $n > 1$. Then $|[K, K^n]| = B_n$, where B_n is the nth Bell number and $K \subseteq K^n$ is a seminormal and infra-integral FIP extension.*

We now intend to extend the above result to diagonal ring extensions $\delta_n : R \hookrightarrow R^n$, for arbitrary rings R. We need information about some closures and give necessary conditions for the FCP or FIP properties hold. If $R \subseteq R_i,\ i = 1, \dots, n,\ n \geq 2$ are finitely many ring extensions and $\delta : R \hookrightarrow \prod_{i=1}^n R_i$ is the canonical diagonal extension, it can be factored $R \hookrightarrow R^n \hookrightarrow \prod_{i=1}^n R_i$. We can also consider that $R \hookrightarrow R^2$ is a subextension by considering the product $R \times R \to R_1 \times \prod_{i=2}^n R_i$ of the extensions $R \hookrightarrow R_1$ and $R \hookrightarrow \prod_{i=2}^n R_i$. Of course, this embedding of R^2 is not unique. A more complete study appears in Section 4 (see Proposition 4.6).

Proposition 2.2. *Let $R \subseteq R_i,\ i = 1, \dots, n,\ n \geq 2$ be finitely many ring extensions, $\mathcal{R} := \prod_{i=1}^n R_i$ and $R \subseteq \prod_{i=1}^n R_i = \mathcal{R}$ the canonical diagonal extension. Then:*

(1) $\mathrm{Supp}(\mathcal{R}/R) = \mathrm{Spec}(R)$.

(2) Assume that $R \subseteq \mathcal{R}$ has FCP (resp. FIP). Then, R is an Artinian ring and each extension $R \subseteq R_i$ has FCP (resp. FIP).

(3) Assume that $R \subseteq \mathcal{R}$ has FIP. Then, R has finitely many ideals.

Proof. We have $R^2 \subseteq \prod_{i=1}^n R_i$ and $R^n \subseteq \prod_{i=1}^n R_i$.

(1) Let $P \in \mathrm{Spec}(R)$. Then, $R_P \neq 0$ implies $(1, 0) \notin R_P$ and $P \in \mathrm{Supp}(R^2/R) \subseteq \mathrm{Supp}(\mathcal{R}/R)$, which gives (1). Indeed, $(R^2/R)_P \cong (R_P)^2/R_P$.

(2) Assume that $R \subseteq \mathcal{R}$ has FCP, so that $R \subseteq R^n$ has FCP. Then, R is an Artinian ring in view of Proposition 1.4. Statements about FCP or FIP are clear.

(3) Assume that $R \subseteq \mathcal{R}$ has FIP, so that $R \subseteq R^2$ has FIP. Let I, J be two distinct ideals of R. Then, $R + (0 \times I)$ and $R + (0 \times J)$ are two distinct R-subalgebras of R^2. Since $R \subseteq R^2$ has FIP, it follows that R has finitely many ideals. □

Rings which have finitely many ideals are characterized by D. D. Anderson and S. Chun [1], a result that will be often used.

Proposition 2.3. *[1, Corollary 2.4] A commutative ring R has only finitely many ideals if and only if R is a finite direct product of finite local rings, SPIRs, and fields, that are the local rings of R.*

From now on, a ring R with finitely many ideals is termed an FMIR and a ΣFMIR if at least a local ring of R is an infinite SPIR. We also call ΣPIR an infinite SPIR. For an arbitrary ring R, we denote by $\Sigma\mathrm{Max}(R)$ the set of all $M \in \mathrm{Max}(R)$ such that R_M is an infinite FMIR.

Proposition 2.4. *Let $R \subseteq R_i,\ i = 1, \dots, n$ be finitely many ring extensions and $\mathcal{R} := \prod_{i=1}^n R_i$. Let $\overline{R}_i$ (resp. $\overline{\mathcal{R}}$) be the integral closure of R in R_i (resp. $\mathcal{R}$). Then:*

(1) $\overline{\mathcal{R}} = \prod_{i=1}^n \overline{R}_i$.

(2) Assume that $R \subseteq R_i$ has FCP for each i. Then, $\overline{\mathcal{R}} \subseteq \mathcal{R}$ has FCP (and FIP).

Proof. (1) is [4, Proposition 9, ch. V, p. 16].

(2) Assume that $R \subseteq R_i$ has FCP for each i. In view of [8, Theorem 3.13], we get that $\overline{R}_i \subseteq R_i$ has FCP for each i. This extension has also FIP since FCP and FIP are equivalent for an integrally closed extension [8, Theorem 6.3]. Now, use [7, Proposition III.4], to get that $\prod_{i=1}^n \overline{R}_i \subseteq \prod_{i=1}^n R_i$ has FCP (and then FIP because integrally closed). □

Corollary 2.5. *Let $R \subseteq R_1$ and $R \subseteq R_2$ be two integrally closed extensions. Then, $R \subseteq R_1 \times R_2$ has FCP (resp. FIP) if and only if each $R \subseteq R_i$ has FCP and R is Artinian (resp. an FMIR).*

In particular, $R \subseteq R^2$ has FIP if and only if R is an FMIR.

Proof. One implication is obvious, since any R-subalgebra S_1 of R_1 yields an R-subalgebra $S_1 \times R_2$ of $R_1 \times R_2$. Then, use Proposition 2.2.

Conversely, assume that $R \subseteq R_1$ and $R \subseteq R_2$ have both FCP (and then FIP) and that R is Artinian. Then, $R^2 \subseteq R_1 \times R_2$ has FCP (resp. FIP) by Proposition 2.4. Moreover, $R^2 \subseteq R_1 \times R_2$

is integrally closed and $R \subseteq R^2$ is an integral extension. In view of Proposition 1.4, it follows that $R \subseteq R^2$ and so $R \subseteq R_1 \times R_2$ have FCP by [8, Theorem 3.13].

Now, assume that $R \subseteq R_1$ and $R \subseteq R_2$ have both FIP and that R is an FMIR. By Proposition 1.3, $R \subseteq R^2$ as well as $R \subseteq R_1 \times R_2$ have FIP by [8, Theorem 3.13]. $\square$

Proposition 2.6. *Let $R \subseteq R_i,\ i = 1, \dots, n$, be finitely many integral extensions, $S_i := {}_{R_i}^{+}R$, $T_i := {}_{R_i}^{t}R$ for each i, $\mathcal{R} := \prod_{i=1}^{n} R_i,\ \mathcal{S} := \prod_{i=1}^{n} S_i$ and $\mathcal{T} := \prod_{i=1}^{n} T_i$. Then:*

(1) ${}_{\mathcal{R}}^{+}R = {}_{\mathcal{S}}^{+}R$ and ${}_{\mathcal{R}}^{t}R = \mathcal{T}$.

(2) If each $T_i \subseteq R_i$ has FCP (resp. FIP), then ${}_{\mathcal{R}}^{t}R \subseteq \mathcal{R}$ has FCP (resp. FIP). This holds if each $R \subseteq R_i$ has FCP (resp. FIP).

Proof. (1) Obviously, ${}_{\mathcal{S}}^{+}R \subseteq {}_{\mathcal{R}}^{+}R$ and is subintegral. Moreover, $\mathcal{S} \subseteq \mathcal{R}$ is seminormal, since so are each $S_i \subseteq R_i$. Then, $\mathcal{S} \in [{}_{\mathcal{R}}^{+}R, \mathcal{R}]$, with ${}_{\mathcal{R}}^{+}R \subseteq \mathcal{S}$ seminormal, so that ${}_{\mathcal{S}}^{+}R \subseteq {}_{\mathcal{R}}^{+}R$ is also seminormal, then an equality.

We know that $\prod_{i=1}^{n} T_i \subseteq \prod_{i=1}^{n} R_i$ is t-closed [15, Lemma 5.6]. To conclude, it is enough to show that $R \subseteq \prod_{i=1}^{n} T_i$ is infra-integral.

The prime ideals of $\prod_{i=1}^{n} T_i$ are the $P_i \times \prod_{j=1, j\neq i}^{n} T_j$, where P_i is a prime ideal of T_i. For $P_i \in \mathrm{Spec}(T_i)$, set $Q_i := P_i \cap R$. Then, $(\prod_{i=1}^{n} T_i)/(P_i \times \prod_{j=1, j\neq i}^{n} T_j) \cong T_i/P_i \cong R/Q_i$, since $R \subseteq T_i$ is infra-integral. It follows that $R \subseteq \prod_{i=1}^{n} T_i$ is infra-integral

(2) In view of [8, Proposition 3.7(d)], we get that $\prod_{i=1}^{n} T_i = {}_{\mathcal{R}}^{t}R \subseteq \mathcal{R}$ has FCP (resp. FIP). There was a misprint in the statement of [8, Proposition 3.7(d)], where we should read: If $R = R_1 \times \cdots \times R_n$ is a finite product of rings and $R \subseteq S$ satisfies FCP, then S can be identified with a product of rings $S_1 \times \cdots \times S_n$ where $R_i \subseteq S_i$ for each i. Then $\ell[R, S] = \sum_{i=1}^{n} \ell[R_i, S_i]$. $\square$

The next proposition and Proposition 2.2 enable us to reduce our study to quasi-local rings.

Proposition 2.7. *[8, Proposition 3.7 and Corollary 3.2] Let $R \subseteq S$ be a ring extension.*

(1) If $R \subseteq S$ has FCP (FIP), then $|\mathrm{Supp}(S/R)| < \infty$.

(2) If $|\mathrm{MSupp}(S/R)| < \infty$, then $R \subseteq S$ has FCP (FIP) if and only if $R_M \subseteq S_M$ has FCP (FIP) for each $M \in \mathrm{MSupp}(S/R)$.

Proposition 2.8. *Let $R \subseteq R_i,\ i = 1, \dots, n$, be finitely many subintegral extensions and $\mathcal{R} := \prod_{i=1}^{n} R_i$, where (R, M) is a quasi-local ring. Then:*

(1) Each R_i is a quasi-local ring with $\{N_i\} := \mathrm{Max}(R_i)$ and $R \subseteq \mathcal{R}$ is infra-integral.

(2) Set $N := \prod_{i=1}^{n} N_i$ and $S := R + N$. Then (S, N) is a quasi-local ring and $\mathrm{Spec}(S) = \{P_i' \times \prod_{j=1, j\neq i}^{n} N_j \mid P_i' \in \mathrm{Spec}(R_i), i = 1, \dots, n\}$. In particular, $R \subseteq S$ is infra-integral and ${}_{\mathcal{R}}^{+}R \subseteq S$.

(3) Assume $\dim(R) = 0$. Then, ${}_{\mathcal{R}}^{+}R = S$.

(4) If each R_i is a Noetherian ring and a f.g. R-module, then S is a f.g. R-module.

Proof. (1) R_i is quasi-local since $R \subseteq R_i$ is subintegral (Definition 1.2). Now, an arbitrary prime ideal of $\mathcal{R}$ is of the form $P' := P_i' \times \prod_{j=1, j\neq i}^{n} R_j$, for some i and $P_i' \in \mathrm{Spec}(R_i)$. Setting $P := P' \cap R$, we see that $P = P_i' \cap R$. From $\mathcal{R}/P' \cong R_i/P_i' \cong R/P$, since $R \subseteq R_i$ is subintegral, we deduce that $R \subseteq \mathcal{R}$ is infra-integral.

(2) The ideals $N_i' := N_i \times \prod_{j=1, j\neq i}^{n} R_j$ are the maximal ideals of $\mathcal{R}$, for $i \in \{1, \dots, n\}$, and they all lie over M. Observe that S is an R-subalgebra of $\mathcal{R}$. From $N \cap R = M$, we infer that $S/N \cong R/M$ and that $N \in \mathrm{Max}(S)$. Since $R \subseteq \mathcal{R}$ is an integral extension, so is $S \subseteq \mathcal{R}$. Moreover, each N_i' lies over N. Hence (S, N) is a quasi-local ring.

Let $Q \in \mathrm{Spec}(S)$, there is some $P \in \mathrm{Spec}(\mathcal{R})$ lying over Q, of the form $P := P_i' \times \prod_{j=1, j\neq i}^{n} R_j$, for some $P_i' \in \mathrm{Spec}(R_i)$. Since $Q \subseteq N$, we get $Q \subseteq (P_i' \times \prod_{j=1, j\neq i}^{n} R_j) \cap (\prod_{k=1}^{n} N_k) = P_i' \times \prod_{j=1, j\neq i}^{n} N_j \subseteq S \cap P = Q$, so that $Q = P_i' \times \prod_{j=1, j\neq i}^{n} N_j$. Conversely, any ideal of the form $P_i' \times \prod_{j=1, j\neq i}^{n} N_j$, for some i and $P_i' \in \mathrm{Spec}(R_i)$ is in $\mathrm{Spec}(S)$, since $P_i' \times \prod_{j=1, j\neq i}^{n} R_j$ lies over it.

Since $R \subseteq S$ is a subextension of $R \subseteq \mathcal{R}$, (1) entails that $R \subseteq S$ is infra-integral. But $\prod_{i=1}^n N_i$ is also an ideal of $\mathcal{R}$, so that $N = (S : \mathcal{R})$. To end, $\mathcal{R}/N \cong (R/M)^n$ and $S/N \cong R/M$ give that $S/N \subseteq \mathcal{R}/N$ is seminormal by Proposition 2.1, and so is $S \subseteq \mathcal{R}$. Then, ${}_{\mathcal{R}}^{+}R \subseteq S$.

(3) Assume $\dim(R) = 0$, in which case $\mathrm{Spec}(S) = \{\prod_{i=1}^n N_i\} = \{N\}$. Then $S/N \cong R/M$ shows that $R \subseteq S$ is a subintegral extension and $S = {}_{\mathcal{R}}^{+}R$.

(4) If each R_i is Noetherian and f.g. over R, then, each N_i is a f.g. R_i-module, and also a f.g. R-module. Hence, $R + N$ is a f.g. R-module. □

Remark 2.9. Contrary to the t-closure, the seminormalization of a diagonal morphism is not the product of the seminormalizations. We can compare these results with [15, Lemma 5.6], which says that seminormalization and t-closure commute with finite products of morphisms.

Proposition 2.10. *Let $R \subseteq R_i,\ i = 1, \dots, n$ be finitely many integral extensions and $\mathcal{R} := \prod_{i=1}^n R_i$, where (R, M) is a quasi-local ring. Then:*

(1) ${}_{\mathcal{R}}^{t}R \subseteq \mathcal{R}$ has FCP (resp.FIP) if each $R \subseteq R_i$ has FCP (resp. FIP).

(2) If $\dim(R) = 0$ and each $R \subseteq R_i$ has FCP, then, ${}_{\mathcal{R}}^{+}R \subseteq {}_{\mathcal{R}}^{t}R$ has FIP.

(3) If $\dim(R) = 0$ and each $R \subseteq R_i$ has FCP (resp. FIP), then $R \subseteq \mathcal{R}$ has FCP (resp. FIP) if and only if $R \subseteq {}_{\mathcal{R}}^{+}R$ has FCP (resp. FIP).

Proof. (1) Proposition 2.6 gives that ${}_{\mathcal{R}}^{t}R \subseteq \mathcal{R}$ has FCP (resp. FIP).

(2) Set $T_i := {}_{R_i}^{t}R,\ S_i := {}_{R_i}^{+}R,\ T := \prod_{i=1}^n T_i = {}_{\mathcal{R}}^{t}R$. Now, each $R \subseteq S_i$ is subintegral. It follows from Proposition 2.8 and [15, Lemma 5.6] that $S := R + \prod_{i=1}^n N_i = {}_{\mathcal{R}}^{+}R$, where N_i is the maximal ideal of S_i for each i. Moreover, $N_i \subseteq (S_i : T_i)$ holds for each i by [8, Proposition 4.9] and S_i and T_i share the ideal N_i, since $S_i \subseteq T_i$ is seminormal and infra-integral. Actually, $N_i = (S_i : T_i)$ when $S_i \neq T_i$ and $(S_i : T_i) = S_i$ when $S_i = T_i$. Therefore we get $N := \prod_{i=1}^n N_i \subseteq (S : T)$ and N is a common ideal of S and T, maximal in S by Proposition 2.8. Set $k := R/M \cong S/N \cong S_i/N_i \cong T_i/N_{i,j}$, for each maximal ideal $N_{i,j}$ of T_i. For each i, we have $N_i = \cap_{j=1}^{n_i} N_{i,j}$, for some n_i, [8, Proposition 4.9], so that $T_i/N_i \cong \prod_{j=1}^{n_i} T_i/N_{i,j}$. Then the extension $S/N \subseteq (\prod_{i=1}^n T_i)/N \cong \prod_{i=1}^n (T_i/N_i)$ can be identified to $k \subseteq k^{\sum n_i}$, which has FIP (and then FCP) by Proposition 2.1. It follows that ${}_{\mathcal{R}}^{+}R \subseteq {}_{\mathcal{R}}^{t}R$ has FIP (and then FCP) by [8, Proposition 3.7].

(3) By [8, Theorem 4.6 and Theorem 5.8], $R \subseteq \mathcal{R}$ has FCP (resp. FIP) if and only if $R \subseteq {}_{\mathcal{R}}^{+}R$, ${}_{\mathcal{R}}^{+}R \subseteq {}_{\mathcal{R}}^{t}R$ and ${}_{\mathcal{R}}^{t}R \subseteq \mathcal{R}$ have FCP (resp. FIP) if and only if $R \subseteq {}_{\mathcal{R}}^{+}R$ has FCP (resp. FIP) by (1) and (2). □

The FCP case is now completely solved with the following theorem.

Theorem 2.11. *Let $R \subseteq R_i,\ i = 1, \dots, n,\ n \geq 2$ be finitely many extensions and $\mathcal{R} := \prod_{i=1}^n R_i$. Then $R \subseteq \mathcal{R}$ has FCP if and only if R is an Artinian ring and each extension $R \subseteq R_i$ has FCP.*

Proof. The"only if" implication is Proposition 2.2(2).

Conversely, assume that R is an Artinian ring and each $R \subseteq R_i$ has FCP. From Proposition 2.4, we infer that $\overline{\mathcal{R}} \subseteq \mathcal{R}$ has FCP. Moreover $R^n \subseteq \overline{\mathcal{R}} = \prod_{i=1}^n \overline{R}_i$ has FCP by [8, Proposition 3.7] and $R \subseteq R^n$ has FCP by Proposition 1.4, giving that $R \subseteq \overline{\mathcal{R}}$ has FCP by [8, Corollary 4.3]. To end, use [8, Theorem 3.13] to get that $R \subseteq \prod_{i=1}^n R_i$ has FCP. □

We now consider the FIP property for the product of two FIP extensions. The case of $n > 2$ FIP extensions is studied in Section 4.

Proposition 2.12. *Let $R \subset R_1, R_2$ be two subintegral FIP extensions and set $\mathcal{R} := R_1 \times R_2$. Assume that (R, M) is quasi-local such that $|R/M| = \infty$. Then $R \subseteq \mathcal{R}$ has not FIP.*

Proof. Let N_i be the maximal ideal of R_i. The infra-integrality of $R \subset R_i$ implies that $M \neq N_i$. It follows that $S_1 := R + (N_1 \times M)$ and $S_2 := R + (M \times N_2)$ are incomparable R-subalgebras of $S := R + (N_1 \times N_2)$, because $(x, 0) \in S_1 \setminus S_2$ for $x \in N_1 \setminus M$ and $(0, y) \in S_2 \setminus S_1$ for $y \in N_2 \setminus M$.

Assume now that $R \subset \mathcal{R}$ has FIP. In this case, $R \subset S$ has FIP and R is Artinian by Proposition 2.2. It follows that $S = {}_{\mathcal{R}}^{+}R$ by Proposition 2.8, so that $R \subset S$ is a subintegral extension. From Proposition 1.5, we deduce that S_1 and S_2 are comparable, a contradiction and $R \subset \mathcal{R}$ has not FIP. □

In order to settle the main Theorem 2.17 of the section, we begin to clear the way by studying when $R \subseteq \mathcal{R}$ has not FIP. We can suppose that $R_1 = R$, because $R \times R_2 \subseteq R_1 \times R_2$. By Proposition 2.2 and Proposition 2.3, we need only to consider a ΣPIR (R, M) in view of [8, Proposition 3.7]. Indeed, the case of a field R has already been studied in [2]. Note that if (R, M) is a local Artinian ring, then R is finite if and only if R/M is finite, since $M^n = 0$ for some integer n. In such a case, any finite extension of R has FIP. We first look at minimal ramified extensions. Before, we give a useful lemma.

Lemma 2.13. *Let $R \subset S$ be a ring extension, where (R, M) is a quasi-local ring with $|R/M| = \infty$. Let $\mathcal{F}$ be a set of representative elements of R/M. If there exists a family $\{R_\alpha\}$ of elements of $[R, S]$ such that $R_\alpha \neq R_\beta$ for each $\alpha \neq \beta \in \mathcal{F}$, then $R \subset S$ has not FIP.*

Proof. Obvious. ◻

Lemma 2.14. *Let $R \subset S$ be a minimal ramified extension, where (R, M) is a SPIR.*

(1) There exists $t \in M$ such that $M = Rt$ and $t^p = 0$, with $t^{p-1} \neq 0$, for some integer $p > 1$.

(2) Let N be the maximal ideal of S. There exists $x \in S \backslash R$ such that $S = R + Rx,\ N = Rt + Rx$. Moreover, there are some unique positive integers $p \geq k, q \geq 1$ and some $a, b \in R \setminus M$ such that $x^2 = at^k,\ tx = bt^q$. Then, $(R :_R x) = M = (R : S)$.

(3) $q \geq 2$ holds.

Proof. (1) is the definition of a SPIR (see Section 1). Each element of R is of the form ut^h for some unique integer $h \leq p$ and some unit u.

(2) The integers k and q exist by Theorem 1.1 or [8, Theorem 2.3 (c)] because $x^2, tx \in M$ and are unique by (1) since the ideals of R are linearly ordered.

(3) Assume $q = 1$. Then, $tx = bt$ implies $t(x - b) = 0$. But $x - b \notin N$ since $b \in R \setminus M$, so that $x - b$ is a unit in S, and then $t = 0$, a contradiction, which yields $q \geq 2$. In particular, $tx \in Rt^2$. ◻

Proposition 2.15. *Let $R \subset S$ be a minimal ramified extension, where (R, M) is a ΣPIR. We set $\mathcal{R} := R \times S$ and $\{N\} := \mathrm{Max}(S)$.*

(1) $T := {}_{\mathcal{R}}^{+}R = R + (M \times N)$.

(2) $R \subset \mathcal{R}$ has FIP if and only if $N^2 = M$ and $MN = M^2 = 0$.

Proof. (1) The value of T is given in Proposition 2.8.

(2) We keep the notation of Lemma 2.14. There exists $t \in M$ such that $M = Rt$ and $t^p = 0$, with $t^{p-1} \neq 0$, for some integer $p > 1$. There exists $x \in S \setminus R$ such that $S = R + Rx,\ N = Rt + Rx$. Moreover, there are some positive integers $p \geq k, q \geq 1$ and some $a, b \in R \setminus M$ such that $x^2 = at^k,\ tx = bt^q$, with $q \geq 2$. Then, $M^2 = Rt^2,\ MN = Rt^2 + Rtx = Rt^2$ since $tx \in Rt^2$, so that $M^2 = MN$, and $N^2 = Rt^2 + Rtx + Rx^2 = Rt^2 + Rt^k$.

Let $\mathcal{F}$ be a set of representative elements of R/M. Then $\mathcal{F}$ is infinite.

Assume first that $k > 1$, so that $x^2 \in Rt^2$. For $\alpha \in \mathcal{F}$, set $R_\alpha := R + R(0, t + \alpha x) + R(0, t^2)$. Then, $R_\alpha \in [R, T]$. Let $\beta \in \mathcal{F}$ be such that $\alpha \neq \beta$, so that $\alpha - \beta \notin M$. Assume that $R_\alpha = R_\beta$. We get that $(0, t + \alpha x) = (c, c) + (0, dt + d\beta x) + (0, et^2)$, for some $c, d, e \in R$, giving $0 = c$ and $t + \alpha x = c + dt + d\beta x + et^2 = dt + d\beta x + et^2$. Since $(\alpha - d\beta)x = (d-1)t + et^2 \in M$, we get $\alpha - d\beta \in M$ $(*)$ in view of Lemma 2.14(2). It follows that there exists $d' \in R$ such that $\alpha - d\beta = d't$, yielding $d'tx = d'bt^q = (d-1)t + et^2$, so that $(d-1)t = d'bt^q - et^2 \in Rt^2$, leading to $d - 1 \in M$ $(**)$. But $(*)$ and $(**)$ give $\alpha - \beta \in M$, a contradiction. Then, $R_\alpha \neq R_\beta$, and $R \subset \mathcal{R}$ has not FIP in view of Lemma 2.13.

It follows that when $R \subset \mathcal{R}$ has FIP, we must have $k = 1$.

Now, assume that $k = 1$. Then, $x^2t = at^2 = (tx)x = xbt^q = (xt)bt^{q-1} = b^2t^{2q-1}$, so that $at^2 - b^2t^{2q-1} = t^2(a - b^2t^{2q-3}) = 0$. But $q \geq 2$ implies $2q - 3 \geq 1$, giving $a - b^2t^{2q-3}$ is a unit in R. Then, $t^2 = 0$ and $p = q = 2$, with $tx = 0$.

So, when $R \subset \mathcal{R}$ has FIP, then $k = 1$ and $p = q = 2$, which give $M^2 = MN = 0$ and $N^2 = Rt = M$.

Assume now that $N^2 = M$ and $MN = M^2 = 0$. Then, $Rt = Rt^2 + Rt^k$, giving $k = 1$, and $Rt^2 = 0$, giving $p = q = 2$. Observe that $R \subset \mathcal{R}$ is an integral FCP extension by Theorem 2.11.

Using notation and statement of [8, Theorem 5.18], set $R_1 := R + TM = R$. Then, $T = R[(0, x)]$, $(0, x)^3 = 0 \in M$, and, with $T' := R[(0, x)^2] = R[(0, t)]$ and $T'' := R + T'M = R$, we have $T' = T''[(0, t)]$, with $(0, t) \in T$, and $(0, t)^3 = 0 \in T'M$. We can conclude that $R \subset \mathcal{R}$ has FIP. □

Corollary 2.16. *Let $R \subset S$ be a non minimal subintegral FIP extension, where (R, M) is a ΣPIR. Then, $R \subset R \times S$ has not FIP.*

Proof. Since $R \subset S$ has FIP, there is $S_1 \in [R, S]$, such that $R \subset S_1$ is a minimal extension, necessarily ramified. Assume that $R \subseteq R \times S$ has FIP, then so has $R \subset R \times S_1$. Using the notation of Lemma 2.14 and Proposition 2.15 for $R \subseteq S_1$, we have $M = Rx^2$, $S_1 = R + Rx$, $N = Rx^2 + Rx$, where N is the maximal ideal of S_1 and $x^3 = 0$, $x^2 \neq 0$. There exists $S_2 \in [S_1, S]$ such that $S_1 \subset S_2$ is a minimal extension, necessarily ramified. Let P be the maximal ideal of S_2. In view of [8, Theorem 2.3(c)], there is $y \in S_2$ such that $S_2 = S_1 + S_1 y = R + Rx + Ry + Rxy$ and $P = N + S_1 y = Rx^2 + Rx + Ry + Rxy$. Moreover, $(S_1 : y) = N$. But, $NP \subseteq N$ gives $xy \in N$ and $P^2 \subseteq N$ gives $y^2 \in N$, so that $P = Rx^2 + Rx + Ry$ and there exist $b, c, d, e \in R$ such that $y^2 = bx^2 + cx$ $(*)$ and $yx = dx^2 + ex$ $(**)$. It follows that $yx^2 = x(dx^2 + ex) = ex^2$, so that $(y - e)x^2 = 0$. If $e \notin M$, then $e \notin P$ and $e - y$ is a unit in S_2, giving $x^2 = 0$, a contradiction. But $e \in M$ implies that $ex^2 \in Rx^4 = 0$, so that $yx^2 = 0$. Now, $(*)$ gives $xy^2 = bx^2 x + cx^2 = dx^2 y + exy = cx^2$. But $e \in M = Rx^2$ entails $ex \in Rx^3 = 0$, so that $xy^2 = dx^2 y = 0$, whence $cx^2 = 0$, from which we infer that $c \in M = Rx^2$. Therefore, we get $y^2 = bx^2$ since $x^3 = 0$. Let $\mathcal{F}$ be a set of representative elements of R/M. For $\alpha \in \mathcal{F}$, set $R_\alpha := R + R(0, x + \alpha y) + R(0, x^2)$. Then, $R_\alpha \in [R, R + (R \times S_2)]$ since $(x + \alpha y)^2 = (1 + 2\alpha d + \alpha^2 b)x^2$. Let $\beta \in \mathcal{F}$ be such that $\alpha \neq \beta$, so that $\alpha - \beta \notin M$. Assume that $R_\alpha = R_\beta$. We get that $(0, x + \alpha y) = (c, c) + (0, dx + d\beta y) + (0, ex^2)$, for some $c, d, e \in R$, giving $0 = c$ and $x + \alpha y = c + dx + d\beta y + ex^2 = dx + d\beta y + ex^2$. Since $(\alpha - d\beta)y = (d - 1)x + ex^2 \in N$, we get $\alpha - d\beta \in N \cap R = M$ $(\dagger)$. It follows that there exists $d' \in R$ such that $\alpha - d\beta = d'x^2$, yielding $0 = d'x^2 y = (d - 1)x + ex^2$, so that $(d - 1)x \in M$, leading to $d - 1 \in M$ $(\dagger\dagger)$. But $(\dagger)$ and $(\dagger\dagger)$ give $\alpha - \beta \in M$, a contradiction. Then, $R_\alpha \neq R_\beta$, and $R \subset R \times S$ has not FIP in view of Lemma 2.13. □

To shorten, a minimal ramified (subintegral) extension $(R, M) \hookrightarrow (S, N)$ between quasi-local rings is called *special* if $M^2 = MN = 0$ and $N^2 = M$, as in Proposition 2.15. Such extensions exist. Any minimal ramified extension $R \subset S$ such that R is a field is special. Here is another example. Let K be a field and $R := K[T]/(T^2)$. If t is the class of T in R, let $S := R[X]/(X^2 - t, Xt)$. The natural map $R \to S$ is injective. This follows from the fact that $R[X]$ is a free $K[X]$-module with basis $\{1, t\}$ and some easy calculations. Let x be the class of X in S. Then, $M := Rt$ is the only maximal ideal of R, so that (R, M) is a quasi-local ring. Moreover, $S = R[x]$, with $x \in S \setminus R$ satisfying $x^2 \in M$ and $Mx \subseteq M$, so that $R \subset S$ is a minimal ramified extension [8, Theorem 2.3]. It follows that the only maximal ideal of S is $N := Rx + Rt$, and we have the following relations: $t^2 = xt = 0$ and $x^2 = t$, giving $N^2 = Rx^2 = Rt = M$ and $MN = Rt^2 + Rtx = Rt^2 = M^2 = 0$. Then, $R \subset S$ is a special minimal ramified extension.

Theorem 2.17. *Let $R \subseteq S_1, S_2$ be FIP extensions, $\Sigma_i := {}_{S_i}^{+}R$ for $i = 1, 2$ and $\mathcal{R} := S_1 \times S_2$. Then $R \subseteq \mathcal{R}$ has FIP if and only if R is an FMIR such that* $\mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$, *and, for each $M \in$* $\mathrm{Supp}(\Sigma_i/R) \cap \Sigma\mathrm{Max}(R)$, *$i \in \{1, 2\}$, either $R_M \subset (\Sigma_i)_M$ is a special minimal ramified extension or R_M is a field.*

Proof. For a maximal ideal M of R, we denote by $S(M)$ the seminormalization of R_M in $(S_1 \times S_2)_M$.

Assume that $R \subseteq S_1 \times S_2$ has FIP. In view of Proposition 2.2, R is an FMIR, and so is a finite direct product $\prod_{i=1}^n R_i$ of fields, finite local rings and SPIRs that are localization of R at some maximal ideal M of R by Proposition 2.3. Hence $R_M \subseteq (S_1 \times S_2)_M = (S_1)_M \times (S_2)_M$ has FIP by Proposition 2.7. Assume that R_M is not a finite ring. Then, R_M is either an infinite field or a ΣPIR.

Let $M \in \Sigma\mathrm{Max}(R)$, so that $|R_M/M'| = \infty$ for $M' := MR_M$ (see the remark before Lemma 2.13). For $j \in \{1, 2\}$, we have that $R_M \subseteq (\Sigma_j)_M$ is a subintegral FIP extension with (R_M, M') a quasi-local ring. Assume first that R_M is a ΣPIR. Using Propositions 2.12, 2.15

and Corollary 2.16, we get that $R_M = (\Sigma_j)_M$ for some $j \in \{1,2\}$, so that $M \notin \mathrm{Supp}(\Sigma_j/R)$ and, for $l \in \{1,2\} \setminus \{j\}$, either $R_M = (\Sigma_l)_M$ or $R_M \subset (\Sigma_l)_M$ is a special minimal ramified extension. Assume now that R_M is an infinite field. Using Proposition 2.12, we get that $R_M = (\Sigma_j)_M$ for some $j \in \{1,2\}$ and, for $l \in \{1,2\} \setminus \{j\}$, there exists $\alpha \in (\Sigma_l)_M$ which satisfies $(\Sigma_l)_M = R_M[\alpha]$ and $\alpha^3 = 0$ by [2, Theorem 3.8] since $R_M \subseteq (\Sigma_l)_M$ has FIP. Then, $M \notin \mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R)$ and $\mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$.

Conversely, assume that R is an FMIR, and so a finite direct product $\prod_{i=1}^n R_i$ of fields, finite local rings and SPIRs such that $\mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$, with, for each $M \in \mathrm{Supp}(\Sigma_i/R) \cap \Sigma\mathrm{Max}(R)$, $i \in \{1,2\}$, either $R_M \subset (\Sigma_i)_M$ is a special minimal ramified extension or R_M is an infinite field. Observe first that for each i, there is $M \in \mathrm{Max}(R)$ such that $R_i = R_M$.

Since R is a quasi-semilocal ring, $\mathrm{MSupp}((S_1 \times S_2)/R)$ is finite. Then, $R \subseteq S_1 \times S_2$ has FIP if and only if $R_M \subseteq (S_1 \times S_2)_M$ has FIP for each $M \in \mathrm{MSupp}((S_1 \times S_2)/R)$ by Proposition 2.7. Moreover, $R_M \subseteq (S_j)_M$ is an FIP extension for $j = 1,2$. Fix $M \in \mathrm{MSupp}((S_1 \times S_2)/R)$. Proposition 2.4 tells us that $\overline{\mathcal{R}}_M = (\overline{S_1})_M \times (\overline{S_2})_M = (\overline{S_1} \times \overline{S_2})_M \subseteq \mathcal{R}_M$ has FIP, where $\overline{\mathcal{R}}_M$ (resp. $(\overline{S_i})_M$) is the integral closure of R_M in $(S_1)_M \times (S_2)_M = (S_1 \times S_2)_M$ (resp. $(S_i)_M$). Then, in view of [8, Theorem 3.13], $R_M \subseteq (S_1 \times S_2)_M$ has FIP if and only if $R_M \subseteq (\overline{S_1} \times \overline{S_2})_M$ has FIP. From Proposition 2.10, we deduce that $R_M \subseteq (S_1 \times S_2)_M$ has FIP if and only if $R_M \subseteq S(M)$ has FIP. But, $S(M) = {}_{(\Sigma_1)_M \times (\Sigma_2)_M}^{+} R_M$ by Proposition 2.6. Therefore, $S(M)$ is module finite over the Artinian ring R_M by Proposition 2.8.

(1) If R_M is an infinite field, then $M \in \Sigma\mathrm{Max}(R)$. We have $R_M = (\Sigma_l)_M$ for some $l \in \{1,2\}$ since $\mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$. Let $j \neq l$. Since $R_M \subseteq (\Sigma_j)_M$ has FIP, there is $\alpha_j \in (\Sigma_j)_M$ such that $(\Sigma_j)_M = R_M[\alpha_j]$, with $\alpha_j^3 = 0$ by [2, Theorem 3.8]. Moreover, $R_M[\alpha_j]$ is a quasi-local ring with maximal ideal $\alpha_j R_M[\alpha_j]$. Set $\alpha_l := 0$ and $\alpha := (\alpha_1, \alpha_2)$. In view of Proposition 2.8, we get $S(M) = R_M[\alpha]$, with $\alpha^3 = 0$, so that $R_M \subseteq S(M)$ has FIP by [2, Theorem 3.8]. Indeed, $S(M) = R_M + (\alpha_j R_M[\alpha_j] \times 0) = R_M + \alpha R_M$.

(2) If R_M is a ΣPIR, then $M \in \Sigma\mathrm{Max}(R)$, there is some $j \in \{1,2\}$ such that $(\Sigma_j)_M = R_M$, with, for $l \in \{1,2\} \setminus \{j\}$, either $R_M = (\Sigma_l)_M$ or $R_M \subset (\Sigma_l)_M$ is a special minimal ramified extension. Then, $R_M \subseteq S(M)$ has FIP by either Proposition 2.15 or Corollary 2.5.

(3) If R_M is a finite ring, then $S(M)$ is a finite ring since a finitely generated R_M-module, and $R_M \subseteq S(M)$ has FIP.

In every case, $R_M \subseteq S(M)$ has FIP, and so has $R \subseteq S_1 \times S_2$. □

Corollary 2.18. *Let $R \subseteq S_1, S_2$ be seminormal FIP extensions and $\mathcal{R} := S_1 \times S_2$. Then $R \subseteq \mathcal{R}$ has FIP if and only if R is an FMIR.*

Proof. Since $R = {}_{S_i}^{+}R$ for $i = 1,2$, we get $\mathrm{Supp}(\Sigma_1/R) \cap \mathrm{Supp}(\Sigma_2/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$. Then, use Theorem 2.17. □

3 FCP or FIP extensions and the CRT

The aim of this section is to get an extension of the Chinese Remainder Theorem (CRT) in the following sense. Let R be a ring, $n > 1$ an integer and $I_1, \ldots, I_n$ ideals of R distinct from R, but not necessarily distinct, such that $\cap_{j=1}^n I_j = 0$. Such a family $\{I_1, \ldots, I_n\}$ of ideals of R is called a *separating family*, a reference to Algebraic Geometry where a finite family of morphisms $\{f_j : M \to M_j \mid j = 1, \ldots, n\}$ of R-modules is called separating if $\cap_{j=1}^n \ker f_j = 0$. We intend to study the ring extension $R \subseteq \prod_{j=1}^n (R/I_j) =: \mathcal{R}$ associated to a separating family, denoting by $C := (R : \mathcal{R})$ its conductor, also called the *conductor of the separating family*. We set $J_j := \cap_{k=1, k\neq j}^n I_k$, or more generally $J_E := \cap_{k=1, k\notin E}^n I_k$ for any subset E of $\{1, \ldots, n\}$. We also denote by e_i the element of $\mathcal{R}$ whose ith coordinate is 1 and the others are 0 and call $\{e_1, \ldots, e_n\}$ the "canonical basis". The above extension is an isomorphism if $C = R$ (Chinese Remainder Theorem). If not, either $|[R, \mathcal{R}]|$ or $\ell[R, \mathcal{R}]$ measures in some sense how R is far from $\mathcal{R}$.

Proposition 3.1. *Let R be a ring and $\{I_1, \ldots, I_n\}$ a separating family of ideals of R. Then:*

(1) $R \subseteq \mathcal{R}$ *is an infra-integral extension.*

(2) $C = \cap_{j=1}^n (I_j + J_j) = \sum_{j=1}^n J_j$.

(3) $R \subseteq \mathcal{R}$ has FCP if and only if R/C is Artinian.

Proof. (1) Clearly, $R \to \prod_{j=1}^{n}(R/I_j)$ is an integral ring extension (actually, module finite), that is infra-integral because of the form of elements of $\mathrm{Spec}(\mathcal{R})$.

(2) is [21, Lemma 2.25].

(3) In view of [8, Theorem 4.2], we have that $R \subseteq \mathcal{R}$ has FCP if and only if R/C is an Artinian. □

An immediate consequence is the following. Let R be a ring, $n > 1$ an integer and $I_1, \dots, I_n$ ideals of R distinct from R, but not necessarily distinct. Set $C := \sum_{j=1}^{n} J_j$. Then, $R/(\cap_{j=1}^{n} I_j) \subseteq \prod_{j=1}^{n}(R/I_j)$ has FCP if and only if R/C is an Artinian ring.

In the rest of the section, we examine the FIP property. The case of a separating family with two elements is easy to solve.

Proposition 3.2. *Let R be a ring, with two ideals I and J such that $I, J \neq R$ and $I \cap J = 0$. Then $R \subseteq R/I \times R/J$ is a Δ_0-extension, which has FIP if and only if $R/(I+J)$ is an FMIR.*

Proof. For $x \in R$, we denote by $\overline{x}$ its class in R/I and by $\tilde{x}$ its class in R/J. Set $e_1 := (\overline{1}, \tilde{0})$, $e_2 := (\overline{0}, \tilde{1})$, so that $\{e_1, e_2\}$ is a generating set of the R-module $R/I \times R/J$. From $e_i^2 = e_i$ and $e_1 e_2 = 0$ follow that $R/I \times R/J = R + Re_1$. Hence there is a bijection between the set of ideals of R containing $I + J$ and $[R, R/I \times R/J]$ by Proposition 1.3 and $R \subseteq R/I \times R/J$ has FIP if and only if $R/(I+J)$ is an FMIR. □

Next lemma shows that we can reduce our study to a zero conductor extension.

Lemma 3.3. *Let R be a ring and $\{I_1, \dots, I_n\}$ a separating family of ideals of R. Then $R \subseteq \mathcal{R}$ has FIP if and only if the zero conductor extension $R/(\sum_{j=1}^{n} J_j) \subseteq \prod_{j=1}^{n}(R/(I_j + J_j))$ has FIP.*

Proof. By [8, Proposition 3.7], $R \subseteq \mathcal{R}$, with conductor C, has FIP if and only if $R/C \subseteq \mathcal{R}/C$ has FIP. Since C is an ideal of $\mathcal{R}$, for each $j \in \{1, \dots, n\}$, there exists an ideal C_j of R containing I_j such that $C = \prod_{j=1}^{n} C_j/I_j$. Now, there is a natural isomorphism $\mathcal{R}/C \cong \prod_{j=1}^{n}(R/C_j)$. For each j, we get that $C_j/I_j = (I_j + J_j)/I_j$ because $I_j + \sum_{i=1}^{n} J_i = J_j + (\sum_{i=1, i \neq j}^{n} J_i) + I_j = I_j + J_j$. Then, $R/C_j \cong (R/I_j)/(C_j/I_j) \cong (R/I_j)/((I_j + J_j)/I_j) \cong R/(I_j + J_j)$ giving the wanted result. □

Proposition 3.4. *Let R be a ring and $\{I_1, \dots, I_n\}$ a separating family of ideals of R with zero conductor. Then:*

(1) $J_j = 0$ for each j.

(2) If $R \subseteq \mathcal{R}$ has FIP, then $R/(J_{\mathcal{P}_1} + J_{\mathcal{P}_2})$ is an FMIR for any partition $\{\mathcal{P}_1, \mathcal{P}_2\}$ of $\{1, \dots, n\}$ as well as R/I_j for each j. In that case, R is an Artinian ring.

Proof. (1) By Proposition 3.1, $C = \sum_{j=1}^{n} J_j$, so that $J_j = 0$.

(2) Set $K_i := J_{\mathcal{P}_i}$ for $i = 1, 2$. Then, $K_1 \cap K_2 = 0$, so that we have the extensions $R \subseteq R/K_1 \times R/K_2$ and $R/K_i \subseteq \prod_{j \in \mathcal{P}_l}(R/I_j)$ for $l \neq i$, $l \in \{1, 2\}$ leading to the composite $R \subseteq R/K_1 \times R/K_2 \subseteq \mathcal{R}$. If $R \subseteq \mathcal{R}$ has FIP, then so has $R \subseteq R/K_1 \times R/K_2$. By Proposition 3.2, $R/(K_1 + K_2)$ is an FMIR. The second statement follows from (2) and $J_j = 0$. To complete the proof, use Proposition 3.1 since $C = 0$. □

The following result shows that the case of a nonlocal Artinian ring R is very different from the local case.

Proposition 3.5. *Let R be a ring containing a set of $p > 2$ orthogonal idempotents $\{e_1, \dots, e_p\}$, generating the ideal R. Then R is an FMIR if $R \subseteq \mathcal{R}$ has FIP for each separating family $\{I_1, \dots, I_n\}$ of ideals of R. In particular, an Artinian nonlocal ring R is an FMIR if $R \subseteq \mathcal{R}$ has FIP for each separating family of ideals of R. The converse holds if no local ring of R is a SPIR.*

Proof. Consider the faithfully flat extension $R \subseteq \prod_{i=1}^{p} R/Re_i =: S$ with zero conductor (Proposition 3.1). If $R \subseteq S$ has FIP, then each R/Re_i is an FMIR by Proposition 3.4 and so is S. Then observe that if $R \to S$ is a faithfully flat ring morphism, R is an FMIR if so is S, because

$IS \cap R = I$ for each ideal I of R. Now if R is Artinian nonlocal, then R has $p > 1$ idempotents generating the ideal R by the Structure Theorem of Artinian rings. If $p > 2$, use the first part of the proof. If $p = 2$, then $\{(0), (0)\}$ is a separating family of ideals of R, so that $R \subset R^2$ has FIP and R is a FMIR by Corollary 2.5. ◻

Now let (R, M) be a local Artinian ring with $|R/M| < \infty$. Then $|R| < \infty$ (see the remark before Lemma 2.13), so that $R \subseteq \mathcal{R}$ has FIP for each separating family, since $|\mathcal{R}| < \infty$.

We know that $|\mathrm{MSupp}(S/R)| < \infty$ if $R \subseteq S$ has FIP (Proposition 2.7(1)). By Proposition 2.7 and former results of the section, the FIP property study can be reduced to the next proposition hypotheses.

If (R, M) is an Artinian local ring, we denote by $n(R)$ the nilpotency index of M.

Proposition 3.6. *Let (R, M) be an Artinian local ring with $|R/M| = \infty$ and a separating family $\{I_1, \dots, I_n\}$ of ideals, with $C = 0$.*
We set $T := R + M\mathcal{R}$, $\mathcal{C} := (R : T)$, $n(R/\mathcal{C}) = p$, and for each $i > 0$, $M_i := M + TM^i = M + \mathcal{R}M^{i+1}$, $R_i := R + TM^i = R + \mathcal{R}M^{i+1}$. Then,

(1) $T = {}_{\mathcal{R}}^{+}R$ and $R \subseteq \mathcal{R}$ has FIP *if and only if $R \subseteq T$ has FIP.*

(2) $\mathcal{C} = (0 : M)$.

(3) $R \subseteq T$ has FIP if and only if either $R = T$, or $R_1 = T$, or $R_1 \subset T$ is minimal (ramified), with, in the two last situations, either $M = (R : T)$, or $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $1 \leq i \leq p - 1$.

The case $R = T$ corresponds to an extension of the form $K \subseteq K^n$, where K is a field, and the case $M = \mathcal{C}$ to $M^2 = 0$.

Proof. Let $\{e_1, \dots, e_n\}$ be the canonical basis of the R-module $\mathcal{R}$. Since $(R : \mathcal{R}) = 0$, $J_j = 0$ for each $j \in \{1, \dots, n\}$ by Proposition 3.4.

(1) $T = {}_{\mathcal{R}}^{+}R$ follows from [8, Theorem 5.18] since $\mathrm{Rad}(\mathcal{R}) = M\mathcal{R}$ and $R \subseteq \mathcal{R}$ has FCP by Proposition 3.1. Since $R \subseteq \mathcal{R}$ is infra-integral, $R \subseteq \mathcal{R}$ has FIP if and only if $R \subseteq T$ has FIP by Proposition 1.5.

(2) is an easy calculation, because each $J_j = 0$, $\cap_{j=1}^n I_j = 0$ and the unit element of $\mathcal{R}$ is $e_1 + \dots + e_n$.

(3) Since $R \subseteq R_i \subseteq T$ is finite and subintegral, (R_i, M_i) is local Artinian for each $i > 0$. We have $TM = M + \mathcal{R}M^2 = M_1 \subseteq \mathcal{R}M \in \mathrm{Max}(T)$, $R_1 = R + \mathcal{R}M^2$, $R_2 = R + \mathcal{R}M^3$ and $M_2 = M + \mathcal{R}M^3$. Because R/M is infinite, [8, Theorem 5.18], applied with $S := \mathcal{R}$, gives that $R \subseteq T$ has FIP if and only if the next two properties hold:

(i) Either $R = T$, or $M = (R : T)$, or $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $1 \leq i \leq p - 1$;

(ii) If $R \neq T$, there exists $\alpha \in T$ such that $T = R_1[\alpha]$ and $\alpha^3 \in TM$, and, with $T' := R_1[\alpha^2]$ and $T'' := R + T'M$, there exists $\beta \in T$ such that $T' = T''[\beta]$ and $\beta^3 \in T'M$.

Assume that $T \neq R, R_1$, so that $\alpha \notin R_1$. We first show that (ii) implies that $R_1 \subset T$ is minimal. Let $\alpha \in T$ be such that $\alpha^3 \in TM = M_1 \subseteq \mathcal{R}M$, giving $\alpha \in \mathcal{R}M$, so that $\alpha^2 \in \mathcal{R}M^2 \subseteq M_1$ and $\alpha M_1 \subseteq \mathcal{R}MM_1 = \mathcal{R}M(M + \mathcal{R}M^2) \subseteq \mathcal{R}M^2 \subseteq M_1$. Then, $R_1 \subset T$ is minimal (ramified) in view of [8, Theorem 2.3(c)].

Conversely, we show that $R_1 \subset T$ is minimal (ramified), with either $M = (R : T)$, or $\mathrm{L}_R(M_i/M_{i+1}) = 1$ for all $1 \leq i \leq p - 1$ implies (ii). Actually, (i) already holds. Since $R_1 \subset T$ is minimal, there is $\alpha \in T$ such that $T = R_1[\alpha]$ and $\alpha^2 \in M_1 \subset R_1$, with $\alpha M_1 \subseteq M_1$. Then, $\alpha^3 \in M_1 = TM$. Now, we can rewrite (ii) as $T' = R_1[\alpha^2] = R_1$ and $T'' = R + T'M = R + R_1M = R + \mathcal{R}M^3 = R_2$. Assume that $M \neq (R : T) = (0 : M)$, so that $M^2 \neq 0$. Then, $M_1^2 = (M + \mathcal{R}M^2)^2 \subseteq M + \mathcal{R}M^3 = M_2 \subset M_1$ (because $\mathrm{L}_R(M_1/M_2) = 1$) implies that $R_2 \subset R_1$ is minimal ramified by Theorem 1.1. Arguing as for α, we obtain some $\beta \in T$ such that $T' = T''[\beta]$ and $\beta^3 \in T'M$ and (ii) holds.

If $T = R_1$, it is enough to take $\alpha = \beta = 0$ to get (ii).

If $R = T$, then $I_j = M$ for each j entails $M = \cap_{j=1}^n I_j = 0$ and R is a field. Then $R \subseteq \mathcal{R}$ is of the form $K \subseteq K^n$, where K is a field, and has FIP (see Proposition 2.1). Assume that $M = \mathcal{C}$, then $M^2 = 0$. ◻

By Proposition 3.4, we know that when $R \subseteq \mathcal{R}$ has FIP, then R/I_j is an FMIR for each j. It is natural to ask if the converse holds, and if not, what conditions are needed to get the FIP property. We consider here a simple case which already gives a rather complicated result.

Proposition 3.7. *Let (R, M) be an Artinian local ring such that $M^2 = 0$ and $|R/M| = \infty$. Let $\{I_1, \ldots, I_n\}$ be a separating family of ideals, with conductor 0 and $n \geq 3$. Then, $R \subseteq \mathcal{R}$ has FIP if and only if R/I_j is an FMIR and $M = I_j + \cap_{k \neq j,l} I_k$, for each $j, l \in \{1, \ldots, n\},\ j \neq l$.*

Proof. Set $T := R + M\mathcal{R}$, $\mathcal{C} := (R : T)$, and for each $i > 0$, $M_i := M + TM^i = M + \mathcal{R}M^{i+1}$, $R_i := R + TM^i = R + \mathcal{R}M^{i+1}$. Since $M^2 = 0$, we get that $R_1 = R$ and $M_1 = M = M_2$. Then, applying Proposition 3.6, we have that $R \subseteq \mathcal{R}$ has FIP if and only if $R \subseteq T$ has FIP, if and only if either $R = R_1 = T$, or $R \subset T$ is minimal (ramified), with $M = (R : T)$. This last condition is always satisfied since $\mathcal{C} = (0 : M)$. Then, $R \subseteq \mathcal{R}$ has FIP if and only if either $R = R_1 = T$, or $R \subset T$ is minimal.

We begin to remark that $M = I_k$ for at least $n-1$ ideals I_k implies that $M = 0$, so that R is a field and we are in the situation of Proposition 2.1. Indeed, if $n-1$ ideals I_k are equal to M, for instance $I_1, \ldots, I_{n-1}$, we get that $\cap_{k \neq n} I_k = M = 0$ since $(R : \mathcal{R}) = 0$. In particular, we get that $I_n = 0$. Hence, the assertion of Proposition 3.7 holds.

So, in the following, we may assume that there exist some $I_j, I_l \neq M,\ j \neq l$. Consider the following R-subextension of $(R/I_j) \times R$ defined by $R'_j := R + ((M/I_j) \times 0) = \{(\overline{x} + \overline{m}, x) \mid x \in R,\ m \in M\}$. Since $\cap_{k \neq j} I_k = 0$, we have the ring extension $R \subseteq R + \prod_{k \neq j} M/I_k$. An easy calculation shows that we have a ring extension $R'_j \subseteq T$. Moreover, $R \neq R'_j$ since $(\overline{m}, 0) \in R'_j \setminus R$ for any $m \in M \setminus I_j$. In particular, $R \neq T$. The canonical map $\varphi : R'_j \to T$ is defined by $\varphi(\overline{x} + \overline{m}, x) = (\overline{x}, \ldots, \overline{x}) + (\overline{m}, \overline{0}, \ldots, \overline{0})$ (after reindexing the components).

Assume first that $R \subseteq \mathcal{R}$ has FIP, so that $R \subset T$ is a minimal extension. Then, $R \neq R'_j$ implies that $R'_j = T$ and φ is surjective. Let $y \in M$ and $j' \in \{1, \ldots, n\},\ j' \neq j$. Consider $(\overline{0}, \ldots, \overline{y}, \ldots, \overline{0}) \in T$, where all the coordinates are $\overline{0}$ except possibly the j'th which is $\overline{y}$. Then, there exist $x \in R,\ m \in M$ such that $(\overline{0}, \ldots, \overline{y}, \ldots, \overline{0}) = (\overline{x}, \ldots, \overline{x}) + (\overline{m}, \overline{0}, \ldots, \overline{0})$. This gives $y - x \in I_{j'},\ x + m \in I_j$ and $x \in I_k$ for each $k \neq j, j'$. Then, $x \in \cap_{k \neq j,j'} I_k$ and $y \in I_{j'} + \cap_{k \neq j,j'} I_k$, giving $M = I_{j'} + \cap_{k \neq j,j'} I_k$ for any $j' \neq j$. Since there is some $l \neq j$ such that $M \neq I_l$, the same reasoning gives that $M = I_j + \cap_{k \neq j,l} I_k$. At last, if there exist $j', l' \in \{1, \ldots, n\},\ j' \neq l'$ such that $M \neq I_{j'}, I_{l'}$, the same reasoning gives again $M = I_{j'} + \cap_{k \neq j',l'} I_k$. But, when $M = I_{j'}$, we have $M = I_{j'} + \cap_{k \neq j',l'} I_k$ whatever is $I_{l'}$.

Conversely, assume that $R/I_{j'}$ is an FMIR and $M = I_{j'} + \cap_{k \neq j',l'} I_k$, for each $j', l' \in \{1, \ldots, n\},\ j' \neq l'$, with $M \neq I_j$ for some j. We are going to show that $R \subset R'_j$ is minimal ramified and that $R'_j = T$.

Since R/I_j is an FMIR with $|R/M| = \infty$ and $M \neq I_j$, there exists some $z \in M \setminus I_j$ such that $M/I_j = (R/I_j)\overline{z}$, with $\overline{z} \neq 0$ and $\overline{z}^2 = 0$. Set $t := (\overline{z}, 0) \in R'_j \setminus R$. Using the properties of R'_j, we get that $R'_j = R[t]$, with $t^2 = 0 \in M,\ tM = 0 \subseteq M$, so that $R \subset R'_j$ is a minimal ramified extension by [8, Theorem 2.3].

Let $j' \neq j$ and $x \in M$. Since $M = I_{j'} + \cap_{k \neq j',j} I_k$, there exist $a \in I_{j'}$ and $b \in \cap_{k \neq j',j} I_k$ such that $x = a + b$. Then, $\overline{x} = \overline{b}$ in $M/I_{j'}$. It follows that we get $(\overline{0}, \ldots, \overline{x}, \ldots, \overline{0}) = (\overline{b}, \ldots, \overline{b}, \ldots, \overline{b}) + (\overline{0}, \ldots, -\overline{b}, \ldots, \overline{0})$, where $\overline{x}$ stands at the j'th component in the first element, and $-\overline{b}$ stands at the jth component in the last element. Indeed, for $k \neq j, j'$, we have $\overline{b} = \overline{0}$ since $b \in \cap_{k \neq j',j} I_k$. We have $(\overline{b}, \ldots, \overline{b}, \ldots, \overline{b}) \in R$ and $(\overline{0}, \ldots, -\overline{b}, \ldots, \overline{0}) \in (M/I_j) \times 0$, so that $(\overline{0}, \ldots, \overline{x}, \ldots, \overline{0}) \in R'_j$. This holds for any $j' \neq j$ and obviously for $(\overline{0}, \ldots, \overline{x}, \ldots, \overline{0})$ where $\overline{x}$ stands at the jth component by definition of R'_j. Then, $T = R + \prod_k (M/I_k) = R + ((M/I_j) \times 0) = R'_j$, giving that $R \subset T$ is minimal, so that $R \subset \mathcal{R}$ has FIP. □

Remark 3.8. When $n = 3$, the condition of Proposition 3.7 becomes $M = I_j + I_l$, for each $j, l \in \{1, 2, 3\},\ j \neq l$. Here is an example where $I_j \not\subseteq I_l$ for each $j, l \in \{1, 2, 3\},\ j \neq l$.

Let k be an infinite field, and set $R := k[X, Y]/(X, Y)^2 = k[x, y]$, for some indeterminates X, Y. Then, R is an Artinian local ring with maximal ideal $M := (x, y)$ such that $M^2 = 0$ and $|R/M| = \infty$. Set $I_j := k(x + \lambda_j y)$, where λ_1, λ_2 and λ_3 are three distinct elements of k. Then, $I_j \cap I_l = 0$ for each $j, l \in \{1, 2, 3\},\ j \neq l$. We have $R/I_j = k[\overline{x}]$, which is a SPIR, although R is not a SPIR, with $M/I_j = k\overline{x}$.

In the following, we are going to consider a kind of converse for Proposition 3.4, taking for R a local FMIR. By Proposition 2.3, either R is a field, or a finite ring, or a ΣPIR. The case where

R is a field is Proposition 2.1. If R is a finite ring, $\mathcal{R}$ being R-module finite, $\mathcal{R}$ is also a finite ring, so that $R \subseteq \mathcal{R}$ has FIP. The last case to consider is a ΣPIR R.

Proposition 3.9. *Let (R, M) be a ΣPIR and a separating family $\{I_1, \ldots, I_n\}$ of ideals, with conductor 0. Then, $R \subseteq \mathcal{R}$ has FIP if and only if either $n = 2$, or $I_j = M$ for $n - 2$ ideals I_j.*

Proof. For $n = 2$, we get $I_1 = I_2 = 0$ and Corollary 2.5 gives that $R \subseteq R/I_1 \times R/I_2$ has FIP.

Assume that $n > 2$. The ideals of the SPIR R are linearly ordered. Thus we can assume $I_1 \subseteq \cdots \subseteq I_j \subseteq \cdots \subseteq I_n$. By Proposition 3.4, we get that $J_j = 0$ for each $j \in \{1, \ldots, n\}$. Hence, for $j = 1$, we get $I_2 = 0$ and $I_1 = 0$ for $j \neq 1$. Moreover, there is some $t \in M$ such that $M = Rt$, with $t^p = 0$, $t^{p-1} \neq 0$ for some positive integer $p > 1$ since R is not a field, and, for each $j \in \{1, \ldots, n\}$, there is an integer $p_j > 0$ such that $I_j = Rt^{p_j}$, with $I_j \neq Rt^{p_j - 1}$. In particular, we have $p = p_1 = p_2 \geq \cdots \geq p_j \geq \cdots \geq p_n$.

Assume that $I_3 \neq M$, whence $p_3 > 1$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of $\mathcal{R}$ over R and $\mathcal{F}$ a set of representative elements of R/M. For each $\alpha \in \mathcal{F}$, set $R_\alpha := R + R(t^{p-1}e_2 + \alpha t^{p_3-1}e_3)$, which is an R-subalgebra of $\mathcal{R}$. Let $\alpha, \beta \in \mathcal{F}$, $\alpha \neq \beta$, so that $\alpha - \beta \notin M$. Assume that $R_\alpha = R_\beta$. Then, $t^{p-1}e_2 + \alpha t^{p_3-1}e_3 \in R_\beta$, so that there exist $a, b \in R$ such that $t^{p-1}e_2 + \alpha t^{p_3-1}e_3 = a\sum_{j=1}^n e_j + b(t^{p-1}e_2 + \beta t^{p_3-1}e_3)$. This gives $a = 0$, $t^{p-1}(1 - b) = 0$ $(*)$ and $t^{p_3-1}(\alpha - b\beta) \in I_3$ $(**)$. But we get $1 - b \in M$ by $(*)$ and $\alpha - b\beta \in M$ by $(**)$, so that $\alpha - \beta \in M$, a contradiction; whence $R_\alpha \neq R_\beta$, and $R \subseteq \mathcal{R}$ has not FIP by Lemma 2.13.

Now, assume that $n > 2$ and $I_j = M$ for all $j \geq 3$. Using the notation of Proposition 3.6, we get that $T = R + (M \times M) \subseteq R^2$. But $R \subseteq R^2$ has FIP by Corollary 2.5, so that $R \subseteq T$ has FIP, inducing that $R \subseteq \mathcal{R}$ has FIP by Proposition 3.6. □

Corollary 3.10. *Let (R, M) be a quasi-local ring such that $|R/M| = \infty$, and a separating family $\{I_1, \ldots, I_n\}$ of ideals of R. Assume that $R/(\sum_{i=1}^n J_j)$ is a SPIR. Then, $R \subseteq \mathcal{R}$ has FIP if and only if either $n = 2$, or $I_j + J_j = M$ for $n - 2$ ideals $I_j + J_j$.*

Proof. Set $R' := R/(\sum_{i=1}^n J_j) = R/C$, where $C := (R : \mathcal{R})$, so that $R \subseteq \mathcal{R}$ has FIP if and only if $R' \subseteq \prod_{j=1}^n (R/(I_j + J_j))$ has FIP (Lemma 3.3). Then, apply Proposition 3.9 to this extension. □

Remark 3.11. Let (R, M) be a local Artinian ring such that $|R/M| = \infty$, and a separating family $I_1, \ldots, I_n$ of ideals of R different from M, with $n > 2$, associated extension $R \subseteq \mathcal{R}$ and conductor C. We give below such an extension having FIP while R/C is not an FMIR.

Let K be an infinite field, $R := K[X, Y]/(X, Y)^2$ with maximal ideal M. Then (R, M) is a local Artinian ring with $M^2 = 0$ and $R/M \cong K$ infinite. Let x, y be the classes of X, Y in R, $I_1 := Rx$, $I_2 := Ry$, $I_3 := R(x + y)$ and $\mathcal{R} := \prod_{j=1}^3 (R/I_j)$. From $I_j \cap I_k = 0$ for each $j \neq k \in \{1, 2, 3\}$, we deduce that $C = 0$ by Proposition 3.1 and also that $\{I_1, I_2, I_3\}$ is a separating family. Let $\overline{a}$ be the class of $a \in R$ in any R/I_j. Observe that $M/I_1 = (R/I_1)\overline{y}$, $M/I_2 = (R/I_2)\overline{x}$, $M/I_3 = (R/I_3)\overline{x}$, because $y = (x + y) - x$. Hence each M/I_j is a principal ideal with $(M/I_j)^2 = 0$, so that each R/I_j is a SPIR. Set $e_1 := (\overline{y}, \overline{0}, \overline{0})$, $\alpha := e_2 := (\overline{0}, \overline{x}, \overline{0})$, $e_3 := (\overline{0}, \overline{0}, \overline{x})$. Using the notation of Proposition 3.6, we have $(R : T) = M$, $T = R + \mathcal{R}M = R + \sum_{i=1}^3 Re_i$ and $R_1 = R + \mathcal{R}M^2 = R$. Since $(\overline{0}, \overline{x}, \overline{x}) = x \in R$, we get $e_2 + e_3 = x$, whence $e_3 = x - \alpha$. At last, $e_1 = (\overline{y}, \overline{0}, \overline{0}) = (\overline{x + y}, \overline{0}, \overline{0}) = (\overline{x + y}, \overline{x + y}, \overline{x + y}) - (\overline{0}, \overline{x}, \overline{0}) = (x + y) - \alpha$. It follows that $T = R[\alpha]$, with $\alpha^2 = 0$ and $M\alpha = 0$, so that $R = R_1 \subset T$ is a minimal ramified extension [8, Theorem 2.3]. Then, $R \subset T$ and $R = R_1 \subset \mathcal{R}$ have FIP by Proposition 3.6, although (R, M) is a local ring which is not a SPIR: the set of ideals $\{R(x + ay) \mid a \in \mathcal{F}\}$ is infinite, if $\mathcal{F}$ is a set of representative elements of $R/M \cong K$.

Corollary 3.12. *Let (R, M) be a quasi-local ring with $|R/M| = \infty$. Let I, J be ideals of R with $I \cap J = 0$ and such that $S := R/(I + J)$ is a SPIR with nilpotency index $n(S) = p > 0$ if $I + J \neq R$.*

(1) Assume that $I + J = R$. Then, $|[R, R/I \times R/J]| = 1$.

(2) Assume that $I + J \neq R$. Then, $|[R, R/I \times R/J]| = p + 1$.

In particular, if (R, M) is a SPIR with $n(R) = q \geq 1$, then $|[R, R^2]| = q + 1$ and $[R, R^2] = \{R + M^i R^2\}_{i=0,\ldots,q}$.

Proof. (1) If $I+J=R$, then $|[R,R/I\times R/J]|=1$ by the CRT.

(2) Assume now that $I+J\neq R$. Since (S,N) is a SPIR with $N:=M/(I+J)$, $R\subseteq R/I\times R/J$ has FIP by Proposition 3.2 and its conductor is $C:=I+J$ by Proposition 3.1. Moreover, the proof of Proposition 3.2 shows that there is a bijection between $[R,R/I\times R/J]$ and the set of ideals of $R/C=S$. Since (S,N) is a SPIR, there is some $t\in S$ such that $N=St$ and the ideals of S are linearly ordered. Then, this set of ideals is $\{St^k\mid k\in\{0,\dots,p\}\}$ and $|[R,R/I\times R/J]|=p+1$.

Now if (R,M) is a SPIR, with $n(R)=q\geq 1$, we deduce from (2) that $|[R,R^2]|=q+1$. Since (R,M) is a SPIR, there exists $x\in R$ such that $M=Rx$ and the ideals of R are the Rx^i, for $i=0,\dots,q$. Moreover, the bijection φ between the set of ideals of R and $[R,R^2]$ is given by $\varphi(Rx^i)=R+x^iR^2$. ◻

We next generalize some Ferrand-Olivier's result [10, Lemme 1.5].

Theorem 3.13. *Let R be a ring, $\{I_1,\dots,I_n\}$, $n>2$, a separating family of ideals of R. Then, $R\subseteq\mathcal{R}$ is a minimal extension if and only if the following condition $(\dagger)$ holds:*

$(\dagger)$: There exist $j_0,k_0\in\{1,\dots,n\}$, $j_0\neq k_0$ such that $I_{j_0}+I_{k_0}\in\mathrm{Max}(R)$ and $I_j+I_k=R$ for any $(j,k)\neq(j_0,k_0)$, $j\neq k$.

If $(\dagger)$ holds, then $\{I_1,\dots,I_n\}$ satisfies a weak Chinese Remainder Theorem: $I_j+\cap_{k\neq j}I_k=\cap_{k\neq j}(I_j+I_k)$ for each $j\in\{1,\dots,n\}$.

Proof. Assume first that $(\dagger)$ holds. There is no harm to suppose that $j_0=1$, $k_0=2$ and set $J:=\cap_{j=2}^n I_j$. Then $I_j+I_k=R$ for any $j,k\geq 2$, $j\neq k$ gives that $\prod_{j=2}^n(R/I_j)\cong R/J$. So, we are reduced to the extension $R\subseteq R/I_1\times R/J$. But, $I_1+I_j=R$ for each $j>2$ and $I_1+I_2=M$ give $I_1+J=M$ because $I_1+J\subseteq M$. For the reverse inclusion, consider in R/I_1 the relations $\overline{1}=\overline{x}_j$ $(*_j)$ for some $x_j\in I_j$, for any $j>2$. Let $m\in M$. There is $x_2\in I_2$ with $\overline{m}=\overline{x}_2$ in R/I_1. Using $(*_j)$, we get that $\overline{m}=\overline{x}_2\cdots\overline{x}_n$, so that $m\in I_1+J$. Then, by [10, Lemme 1.5], $R\subseteq\mathcal{R}$ is a minimal extension since $I_1\cap J=0$.

Conversely, if $R\subseteq\mathcal{R}$ is minimal (integral), then $M:=(R:\mathcal{R})\in\mathrm{Max}(R)$ is an ideal of $\mathcal{R}$. Moreover, there is some $N_1\in\mathrm{Max}(\mathcal{R})$ above M and possibly only another one N_2. There is no harm to suppose that $N_1=M/I_1\times\prod_{k=2}^n R/I_k$ with $I_1\subseteq M$ and $N_2=R/I_1\times M/I_2\times\prod_{k=3}^n R/I_k$ with $I_2\subseteq M$. Any other $M'\neq M$ in $\mathrm{Max}(R)$, is lain over by a unique element of $\mathrm{Max}(\mathcal{R})$, of the form $M'\mathcal{R}=\prod_{j=1}^n((M'+I_j)/I_j)$ by [8, Lemma 2.4]. Then, $M'+I_j=R$ for all j but one, so that there is a unique I_j contained in M'. Then, for any $j,k>2$, $j\neq k$ and $i=1,2$, we have $I_j+I_k=I_i+I_j=R$, which gives $\prod_{j=2}^n(R/I_j)\cong R/J$ where $J:=\cap_{j=2}^n I_j$. So, the minimal extension $R\subseteq R/I_1\times R/J$ is involved. By [10, Lemme 1.5], we get that $I_1+J=M''$, for some $M''\in\mathrm{Max}(R)$, whence $I_1,J\subseteq M''$. Actually, we have $M=M''$. Deny, then $I_j\not\subseteq M''$ for all $j\geq 2$ gives $J\not\subseteq M''$, a contradiction. A similar proof gives $I_2\subseteq M$ since $J\subseteq M$. From $M=I_1+J\subseteq I_1+I_2\subseteq M$, we get that $I_1+I_2=M$ and the proof is complete.

Assume that $(\dagger)$ holds, then easy calculations show that $I_j+\cap_{k=1,k\neq j}^n I_k=\cap_{k=1,k\neq j}^n(I_j+I_k)$ for each $j\in\{1,\dots,n\}$, so that $\{I_1,\dots,I_n\}$ satisfies a weak Chinese Remainder Theorem. ◻

4 The case of ring powers

In this section, we consider separating families whose ideals are zero.

Proposition 4.1. *Let (R,M) be a ΣPIR and an integer $n>1$. Then $R\subseteq R^n$ has FIP if and only if $n=2$.*

Proof. Use Proposition 3.9 with $I_j=0$ for each j. Since $(R:R^n)=0$ and $M\neq 0$, we get the result. ◻

We are now in position to get a result in the general case.

Theorem 4.2. *Let R be a ring and $n>1$ an integer. Then $R\subseteq R^n$ has FIP if and only if R is an FMIR with $n=2$ when R is a ΣFMIR.*

Proof. Assume that $R \subseteq R^n$ has FIP. Using Proposition 3.4 with $I_j = 0$ for each j and since $(R : R^n) = 0$, we get that R is an FMIR. Moreover, $R_M \subseteq (R_M)^n$ has FIP for each $M \in \mathrm{Max}(R)$ in view of Proposition 2.7 since $\mathrm{MSupp}(R^n/R) = \mathrm{Max}(R)$ by Proposition 2.2. Assume that there is some $M \in \mathrm{Max}(R)$ such that R_M is a ΣPIR. Since $MR_M \neq 0$, we get that $n \leq 2$ by Proposition 4.1, so that $n = 2$.

Conversely, if R is an FMIR, then $|\mathrm{Max}(R)| < \infty$ and $R \subseteq R^n$ has FIP if and only if $R_M \subseteq (R_M)^n$ has FIP for each $M \in \mathrm{Max}(R)$. Let $M \in \mathrm{Max}(R)$. If R_M is a field, then $R_M \subseteq (R_M)^n$ has FIP by Proposition 2.1. If R_M is a finite ring, then so is $(R_M)^n$ and $R_M \subseteq (R_M)^n$ has FIP. Assume that R_M is a ΣPIR, so that R is a ΣFMIR and $n = 2$. Then, Proposition 4.1 gives that $R_M \subseteq (R_M)^n$ has FIP. Therefore, $R \subseteq R^n$ has FIP. □

We get now a generalization of Theorem 2.17.

Theorem 4.3. *Let $R \subseteq S_j$, $j = 1, \ldots, n$ be finitely many FIP extensions, $\Sigma_j := {}_{S_j}^{+}R$ and $S := \prod_{j=1}^{n} S_j$. Then $R \subseteq S$ has FIP if and only if R is an FMIR satisfying the following conditions (B_1) and (B_2):*
(B_1) $\mathrm{Supp}(\Sigma_j/R) \cap \mathrm{Supp}(\Sigma_l/R) \cap \Sigma\mathrm{Max}(R) = \emptyset$ *for any $j, l \in \{1, \ldots, n\}$ such that $j \neq l$.*
(B_2) *If there exists $M \in \Sigma\mathrm{Max}(R)$ such that R_M is a ΣPIR, then $n = 2$ and, for each such M and each $j \in \{1, 2\}$, either $R_M \subset (\Sigma_j)_M$ is a special minimal ramified extension or $R_M = (\Sigma_j)_M$.*

Proof. The result can be written under the form (A) ⇔ R is an FMIR satisfying conditions (B_1) and (B_2) where (A) is the statement: $R \subseteq S$ has FIP.

Assume that (A) holds. Then, $R \subseteq R^n$ has FIP. In view of Theorem 4.2, R is an FMIR and $n = 2$ as soon as R is a ΣFMIR, in which case we can use Theorem 2.17.

If there exist $j, l \in \{1, \ldots, n\}$, $j \neq l$ and $M \in \mathrm{Supp}(\Sigma_j/R) \cap \mathrm{Supp}(\Sigma_l/R) \cap \Sigma\mathrm{Max}(R)$, then $R_M \neq (\Sigma_j)_M, (\Sigma_l)_M$, with R_M infinite. Moreover, $R_M \subset (\Sigma_j)_M$ and $R_M \subset (\Sigma_l)_M$ are subintegral extensions. In view of Proposition 2.12, we get that $R_M \subset (\Sigma_j)_M \times (\Sigma_l)_M$ has not FIP, and so $R_M \subset S_M$ has not FIP, a contradiction. Then, (B_1) holds.

If there exists $M \in \Sigma\mathrm{Max}(R)$ such that R_M is a ΣPIR, then R is a ΣFMIR and $n = 2$ by Theorem 4.2. Moreover, since R_M is not a field, Theorem 2.17 gives that for each $j \in \{1, 2\}$, either $R_M \subset (\Sigma_j)_M$ is a special minimal ramified extension or $R_M = (\Sigma_j)_M$. Then (B_2) holds.

Conversely, assume that R is an FMIR and that (B_1) and (B_2) hold. Clearly, $\mathrm{MSupp}(S/R)$ is finite. Then, $R \subseteq S$ has FIP if and only if $R_M \subseteq S_M$ has FIP for each $M \in \mathrm{MSupp}(S/R)$ by Proposition 2.7.

The integral closure of R in S is $\overline{S} = \prod_{j=1}^{n} \overline{S}_j$ by Proposition 2.4 and $\overline{S} \subseteq S$ has FIP. Hence, $\overline{S}_M \subseteq S_M$ has FIP for each $M \in \mathrm{MSupp}(S/R)$. Then, $R \subseteq S$ has FIP if and only if the module finite extension $R_M \subseteq \overline{S}_M$ has FIP for each $M \in \mathrm{MSupp}(S/R)$ [8, Theorem 3.13].

If R_M is finite, so is $\overline{S}_M$ and $R_M \subseteq \overline{S}_M$ has FIP. Now if R_M is an infinite field, $R_M \subseteq {}_{\overline{S}_M}^{+}R_M$ as well as $R_M \subseteq \overline{S}_M$ have FIP. To see this, mimic the proof of Theorem 2.17, using the fact that there is at most one $j \in \{1, \ldots, n\}$ such that $R_M \neq (\Sigma_j)_M$, so that $R_M = (\Sigma_l)_M$ for each $l \in \{1, \ldots, n\}$, $l \neq j$. As in the proof of Theorem 2.17, we get that $R_M \subset {}_{S_M}^{+}R_M$ has FIP, because ${}_{S_M}^{+}R_M = R_M[\alpha]$, where α is the n-uple whose all components are 0, except the jth which is α_j defining $(\Sigma_j)_M = R_M[\alpha_j]$. Lastly, if R_M is a ΣPIR, then $n = 2$ and Theorem 2.17 gives that $R_M \subseteq \overline{S}_M$ has FIP.

To conclude, $R \subseteq \overline{S}$ has FIP. □

We can rephrase Theorem 4.2 in the following way.

Corollary 4.4. *Let R be a ring and $n > 1$ an integer. Then, $R \subseteq R^n$ has FIP if and only if R is Artinian and setting $\{M_1, \ldots, M_m\} := \mathrm{Max}(R)$ and $\alpha_i := n(R_{M_i})$, then for each i, one of the following conditions holds:*

(1) $\alpha_i = 1$.

(2) $|R/M_i| < \infty$.

(3) R_{M_i} *is a SPIR and* $n = 2$ *as soon as there exists some i such that $\alpha_i > 1$ and R_{M_i} is a ΣPIR.*

Proof. By Theorem 4.2, $R \subseteq R^n$ has FIP if and only if R is a finite direct product $\prod_{i=1}^{m} R_{M_i}$ of finite local rings, SPIRs, and fields, with $n = 2$ as soon as there is some R_{M_i} which is a ΣPIR. Note that $0 = \prod_{i=1}^{m} M_i^{\alpha_i}$ and set $R_i := R_{M_i}$ so that $0 = M_i^{\alpha_i} R_i$.

Assume that $R \subseteq R^n$ has FIP and fix some i. Then R_i is a field if and only if $\alpha_i = 1$, giving (1). We know that R_{M_i} is a finite ring if and only if $|R/M_i| < \infty$, which gives (2). Assume that $\alpha_i > 1$ and $|R/M_i| = \infty$. Then, R_i is a ΣPIR, so that $n = 2$ and we have (3).

Conversely, assume that R is an Artinian ring and that for each i one of conditions (1), (2) or (3) holds. It follows that R is a finite direct product $\prod_{i=1}^m R_i$ of primary rings. We have just seen that R_i is a field when $\alpha_i = 1$. If $|R/M_i| = |R_i/M_iR_i| < \infty$, then R_i is a finite ring. At last, if $\alpha_i > 1$ and $|R/M_i| = \infty$, then R_{M_i} is a ΣPIR and $n = 2$. Now, use Theorem 4.2 to get that $R \subseteq R^n$ has FIP. □

Extensions of the form $R^p \subset R^n$, for some integers $1 < p < n$ generalize extensions $R \subseteq R^n$. For R^p and R^n endowed with their canonical structures of R-algebras, we show that $\mathrm{Homal}_R(R^p, R^n)$ has at least $S(n,p)$ elements (the *Stirling number of the second kind* $S(n,p) := |P(n,p)|$ where $P(n,p)$ is the set of partitions of $\{1,\dots,n\}$ into p subsets). We set $\mathrm{Exal}_R(R^p, R^n) := \{\varphi \in \mathrm{Homal}_R(R^p, R^n) \mid \varphi \text{ injective}\}$.

Proposition 4.5. *Let R be a ring and $1 < p < n$ two integers, then:*

(1) $|\mathrm{Exal}_R(R^p, R^n)| \geq S(n,p)$.

(2) If R is connected, $|\mathrm{Exal}_R(R^p, R^n)| = S(n,p)$.

(3) If $R \subseteq \mathrm{Tot}(R)$ is t-closed and $\mathrm{Tot}(R)$ is Artinian (for instance, if R is Artinian), then $|\mathrm{Exal}_R(R^p, R^n)| \leq S(n,p)^{|\mathrm{Min}(R)|}$.

Proof. Let $\mathcal{C} := \{f_1,\dots,f_p\}$ and $\mathcal{B} := \{e_1,\dots,e_n\}$ be the canonical bases of the R-algebras R^p and R^n, that are complete families of orthogonal idempotents.

For $\varphi \in \mathrm{Homal}_R(R^p, R^n)$, let $\lambda(\varphi) := (a_{i,j}) \in M_{n,p}(R)$ be its matrix in the bases $\mathcal{C}$ and $\mathcal{B}$ (with the rule $\varphi(f_j) = \sum_{i=1}^n a_{i,j} \cdot e_i$ for each j). Then λ defines an injective map whose image Λ we compute. Applying the ring morphism φ to the relations $f_j^2 = f_j$, $f_jf_k = 0$ for each $j \neq k$ and $\sum_{j=1}^p f_j = 1_{R^p}$, we get the conditions $(*_1)$: $a_{i,j}^2 = a_{i,j}$, $(*_2)$: $a_{i,j}a_{i,k} = 0$ for each $j \neq k$ and $(*_3)$: $\sum_{j=1}^p a_{i,j} = 1$, for each i. It is easily seen that $\Lambda = \{(a_{i,j}) \in M_{n,p}(R) \mid (*_1),(*_2),(*_3)\}$ and that $\lambda : \mathrm{Homal}_R(R^p, R^n) \to \Lambda$ is bijective. Indeed, any element of Λ is the matrix of a ring morphism by $(*_1),(*_2),(*_3)$.

(1) Let $H := \{\varphi \in \mathrm{Exal}_R(R^p, R^n) \mid \lambda(\varphi) \in M_{n,p}(\{0,1\})\}$. For $\varphi \in H$ and $\lambda(\varphi) = (a_{i,j})$, we have $a_{i,j} \in \{1,0\}$ for each (i,j) and then $a_{i,k} = 0$ as soon as $a_{i,j} = 1$ for some $j \neq k$ by $(*_2)$. For each $j \in \{1,\dots,p\}$, set $A_j := \{i \in \{1,\dots,n\} \mid a_{i,j} = 1\}$. Since φ is injective, $\varphi(f_j) \neq 0$ for all j implies that each $A_j \neq \emptyset$. Then $(*_2)$ implies $A_j \cap A_k = \emptyset$ for $j \neq k$ and $(*_3)$ that $\{1,\dots,n\} = \cup_{j=1}^p A_j$, since each $i \in \{1,\dots,n\}$ is in one (and only one) A_j, so that $\{A_1,\dots,A_p\} \in P(n,p)$. Hence, there is a map $\mu : H \to P(n,p)$, where $\mu(\varphi) = \{A_1,\dots,A_p\}$, such that $\varphi(f_j) = \sum_{i \in A_j} e_i$ for each j. Then μ is bijective because any element $\{A_1,\dots,A_p\}$ of $P(n,p)$ defines some $\varphi \in H$ by the relations $\varphi(f_j) = \sum_{i \in A_j} e_i$ for each j.

(2) If R is connected, $(*_1)$ implies that $H = \mathrm{Exal}_R(R^p, R^n)$.

(3) If $T := \mathrm{Tot}(R)$ is Artinian, then $T \cong \prod_{l=1}^m R_{M_l}$, where $\mathrm{Min}(R) := \{M_1,\dots,M_m\}$. Since $R \subseteq T$ is t-closed, the idempotents of R and T coincide. Then it is enough to use (2). □

We show that anything is possible when R is a ΣPIR.

Proposition 4.6. *Let (R,M) be a ΣPIR and p,n two integers such that $1 < p < n$ and $\varphi \in$* $\mathrm{Exal}_R(R^p, R^n)$*. The following statements hold:*

(1) If $n = p+1$, φ has FIP.

(2) If $p+2 \leq n \leq 2p$, φ has FIP in some cases and not FIP in some others.

(3) If $n \geq 2p+1$, then φ has not FIP.

Proof. We keep notation of Proposition 4.5(2). Since R is connected, any extension φ of R-algebra $R^p \subseteq R^n$ comes from some partition $\cup_{j=1}^p A_j$ of $\{1,\dots,n\}$ with $\varphi(f_j) = \sum_{i \in A_j} e_i$. In view of [7, Lemma III.3], we may identify $S := R^n$ with $\prod_{j=1}^p S_j$, where $S_j := \varphi(f_j)S$ is a ring extension of R for each j. Moreover, $R^p \subseteq R^n$ has FIP if and only if each $R \subseteq S_j$ has FIP [7,

Proposition III.4]. But S_j is the R-algebra generated by $\{e_i \mid i \in A_j\}$, and then isomorphic to $R^{|A_j|}$. Consider the following cases and use Theorem 4.2 for each $R \subseteq S_j$.
(1) $n = p + 1$. Then, $|A_j| = 1$ for all j, except one j_0 such that $|A_{j_0}| = 2$. It follows that S_j is isomorphic either to R, or R^2. In both cases, $R \subseteq S_j$ has FIP and $R^p \subseteq R^n$ has FIP.
(2) $p + 2 \leq n \leq 2p$. We consider two subcases:

(a) If $|A_j| = 1$ for all j, except one j_0 such that $|A_{j_0}| = n - p + 1 \geq 3$, then $R \subset S_{j_0}$ has not FIP, whence also $R^p \subseteq R^n$.

(b) Set $k := n - p \leq p$ and consider a partition $\{A_1, \dots, A_p\}$ such that $|A_j| = 2$ for $j \leq k$ and $|A_j| = 1$ for $j > k$. Then, $R \subseteq S_j$ has FIP for each j and so has $R^p \subseteq R^n$. We have proved that $R^p \subseteq R^n$ has FIP or not according to the structure of R^p-algebra considered for R^n.
(3) $n \geq 2p+1$. Consider a partition as above. If $|A_j| \leq 2$ for all j, then $n \leq 2p$ is a contradiction. Hence, there is j_0 such that $|A_{j_0}| > 2$. It follows that $R \subset S_{j_0}$ has not FIP and $R^p \subset R^n$ has not FIP. ◻

Proposition 4.7. *Let R be a (resp. connected) ring and $1 < p < n$ two integers. Then, $\varphi \in$ $\mathrm{Exal}_R(R^p, R^n)$ has FIP if (resp. and only if) R is an FMIR and $n \leq 2p$ when R is a ΣFMIR.*

Proof. We use the notation of the proof of Proposition 4.6 which holds for an arbitrary ring. Then, $R^p \subseteq R^n$ has FIP if and only if $R \subseteq S_j$ has FIP for each j. Fix a partition $\{A_1, \dots, A_p\}$ of $\{1, \dots, n\}$, so that $S_j \cong R^{|A_j|}$. Set $k_j = |A_j|$ and $k := \sup\{k_j\}_{j=1,\dots,p}$. It follows that $R \subseteq S_j$ has FIP for each j if and only if $R \subseteq R^k$ has FIP, since there are extensions $R^{k_j} \subseteq R^k$. But Theorem 4.2 shows that $R \subseteq R^k$ has FIP if and only if R is an FMIR and $k \leq 2$ when R is ΣFMIR. Assume that R is a ΣFMIR. An easy calculation using the discussion of the proof of Proposition 4.6 leads to a partition $\{A_1, \dots, A_p\}$ of $\{1, \dots n\}$ such that $|A_j| \leq 2$ for each j if and only if $n \leq 2p$, giving the wanted result.

If R is connected, Proposition 4.5 tells us that $\mathrm{Exal}_R(R^p, R^n)$ is in bijection with the set $P(n,p)$ of partitions $\{A_1, \dots, A_p\}$ of $\{1, \dots, n\}$. Assume that $\varphi : R^p \hookrightarrow R^n$ has FIP, so that $R \subseteq S_j$ has FIP for each $j \in \{1, \dots, p\}$. The first part of the proof shows that this holds if and only if R is an FMIR and $k \leq 2$ when R is an ΣFMIR, whatever is its associated partition. ◻

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