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ON p-ADIC CHARACTER DEDEKIND SUMS

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Abstract Using p-adic measure theory we give explicit representations of p-adic analogues of the character Dedekind sums and their reciprocity laws.

1. Introduction

K. Rosen and W. Synder ([12]) showed that by p-adically interpolating certain partial zeta functions, it is possible to interpolate the higher order Dedekind sums

$$s_m(h,k)=\sum_{a=0}^{k-1}\frac{a}{k}\overline{B}_m\left(\frac{ha}{k}\right)$$

introduced by Apostol ([1]), thus obtain p-adic Dedekind sums. The authors then showed that there is a reciprocity law for p-adic Dedekind sums, however they are not able to obtain an explicit form for the reciprocity law for the arbitrary p-adic integers. C. Synder ([16]) obtained such an explicit form for the reciprocity law for the arbitrary p-adic integers by the use of p-adic measure theory. In a series papers A. Kudo ([8, 9, 10]) extended the results of Rosen and Synder to higher order Dedekind sums

$$s_{m+1}^{(r)}(h,k)=\sum_{a=0}^{k-1}\overline{B}_{m+1-r}\left(\frac{a}{k}\right)\overline{B}_r\left(\frac{ha}{k}\right), 0\leqslant r\leqslant m+1$$

for every h, k and $r\geqslant 1$. Kudo accomplished this by using an expression for $k^m s_{m+1}^{(r)}(h,k)$ in terms of Euler numbers and a p-adic continuous function which interpolates these numbers.

B. Berndt ([2]) gave a character transformation formula similar to those for the Dedekind eta function and defined Dedekind sums with character $s(h,k;\chi)$ by

$$s(h,k;\chi)=\sum_{a=0}^{kf-1}\chi(a)\overline{B}_1\left(\frac{a}{kf}\right)\overline{B}_{1,\chi}\left(\frac{ha}{k}\right)$$

for $(h,k)=1$, where χ is a primitive Dirichlet character of conductor f and $\overline{B}_{m,\chi}(x)$ is the mth character Bernoulli function. M. Cenkci, M. Can and V. Kurt ([5]) extended this definition as

$$s_m(h,k;\chi)=\sum_{a=0}^{kf-1}\chi(a)\overline{B}_1\left(\frac{a}{kf}\right)\overline{B}_{m,\chi}\left(\frac{ha}{k}\right) \tag{1.1}$$

and established reciprocity law.

The purpose of this paper is to define p-adic character Dedekind sums which interpolate (1.1). The basic idea is to use an expression for $s_m(h,k;\chi)$ in terms of generalized Euler numbers and a p-adic continuous function which interpolates these numbers. We also show that there is a reciprocity law for these sums for $m+1\equiv 0\,(\mathrm{mod} p-1)$, where p is an odd prime number.

2. Preliminaries

For integers m, h and k such that $m\geqslant 0$ and $k>0$ the higher order Dedekind sums are defined as

$$s_m(h,k)=\sum_{a=0}^{k-1}\frac{a}{k}\overline{B}_m\left(\frac{ha}{k}\right), \tag{2.1}$$

where $\overline{B}_m(x)$ denotes the mth periodic Bernoulli function defined by

$$\sum_{m=0}^{\infty} B_m(x) \frac{t^n}{n!} = \frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1}$$

for all real x and $\overline{B}_m(x) = \overline{B}_m(\{x\})$ with $\{x\}$ denotes the fractional part of x. For $x = 0$, $B_m(0) = B_m$ is the mth Bernoulli number. For even m, the higher order Dedekind sums are relatively uninteresting. However, for odd m, they possess a reciprocity formula.

There are other representations of Dedekind sums. Let $E_m(u)$ be the modified Euler numbers belonging to a parameter u. $E_m(u)$ is defined by ([10])

$$\frac{u}{\mathrm{e}^t - u} = \sum_{m=0}^{\infty} E_m(u) \frac{t^m}{m!}.$$

Note that $mE_{m-1}(u) = B_m$ and $\frac{1-u}{u} E_m(u) = H_m(u)$ for all $m \in \mathbb{N}$, where $H_m(u)$ is the Eulerian number with parameter u ([4]). It is known that (see [4]) for any $a \in \mathbb{Z}$, $k, m \in \mathbb{N}$ and for any kth root of unity ζ, we have

$$mE_{m-1}(\zeta) = k^{m-1} \sum_{j=0}^{k-1} \overline{B}_m\left(\frac{j}{k}\right) \zeta^{-j}$$

and

$$k^m \overline{B}_m\left(\frac{a}{k}\right) = m \sum_{\zeta^k=1} E_{m-1}(\zeta) \zeta^{-a}.$$

Now, since $\frac{a}{k} = \overline{B}_1\left(\frac{a}{k}\right)$, we have from (2.1) that

$$k^m s_m(h,k) = m \sum_{\zeta^k=1} \frac{E_{m-1}(\zeta)}{\zeta^h - 1}$$

after a little reduction ((6.6) of [3]).

There are many generalizations of Bernoulli numbers and polynomials. One of them is via a Dirichlet character. Let χ be a primitive Dirichlet character of conductor f. Then character Bernoulli polynomials $B_{m,\chi}(x)$ are defined by

$$\sum_{a=0}^{f-1} \frac{\chi(a)\, t\mathrm{e}^{(a+x)t}}{\mathrm{e}^{ft} - 1} = \sum_{m=0}^{\infty} B_{m,\chi}(x) \frac{t^n}{n!}.$$

This definition immediately leads the relation

$$B_{m,\chi}(x) = f^{m-1} \sum_{a=0}^{f-1} \chi(a) B_m\left(\frac{a+x}{f}\right).$$

Character Dedekind sums, which we are going to use for the definition of p-adic character Dedekind sums, are defined by

$$s_m(h,k;\chi) = \sum_{a=0}^{kf-1} \chi(a) \overline{B}_1\left(\frac{a}{kf}\right) \overline{B}_{m,\chi}\left(\frac{ha}{k}\right),$$

where $\overline{B}_{m,\chi}(x) = B_{m,\chi}(\{x\})$. We note that for a principal character χ this definition reduces to Apostol's.

For a primitive Dirichlet character χ of conductor f let $E_{m,\chi}(u)$ be the numbers defined by

$$\sum_{a=0}^{f-1} \frac{\chi(a)\, u^{f-a} \mathrm{e}^{at}}{\mathrm{e}^{ft} - u^f} = \sum_{m=0}^{\infty} E_{m,\chi}(u) \frac{t^m}{m!}.$$

Note that $mE_{m-1,\chi}(1) = B_{m,\chi}$ for all $m \in \mathbb{N}$. Let ζ be an arbitrary primitive fth root of unity. Then, if $(k,f) = 1$, we deduce that

$$\chi(k)\, k^m \overline{B}_{m,\chi}\left(\frac{a}{k}\right) = m \sum_{\zeta^k=1} E_{m-1,\chi}(\zeta) \zeta^a$$

(Proposition 3.1 of [7]). From this relation we may write the character Dedekind sums in terms of generalized Euler numbers as

$$\chi(k)\, k^m s_m(h,k;\chi) = \sum_{a=0}^{kf-1} \chi(a)\, \overline{B}_1\left(\frac{a}{kf}\right) m \sum_{\zeta^k=1} E_{m-1,\chi}\left(\zeta^h\right)\zeta^a.$$

Throughout this paper we use standard terminology from the p-adic theory. Let p denote a fixed prime number which, for convenience, we assume to be odd. Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the set of p-adic integers and p-adic rational numbers, respectively. Let $|\cdot|_p$ denote the p-adic absolute value on $\mathbb{Q}_p$, normalized so that $|p|_p = p^{-1}$. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ with respect to p-adic absolute value. Note that two fields $\mathbb{C}$ and $\mathbb{C}_p$ are algebraically isomorphic, and any one of the two can be embedded in the other.

The group of p-adic units is denoted by $\mathbb{Z}_p^*$. If V is the group $\left\{x \in \mathbb{Q}_p : x^{p-1} = 1\right\}$, then $\mathbb{Z}_p^* = V \times (1+p\mathbb{Z}_p)$. Thus, if $a \in \mathbb{Z}_p^*$ then $a = w(a)\langle a\rangle$, where $w(a)$ and $\langle a\rangle$ are the projections of a onto V and $1+p\mathbb{Z}_p$, respectively. Letting $w(a) = 0$ for $a \in \mathbb{Z}$ such that $(a,p) \neq 1$, we see that w is actually a Dirichlet character, called Teichmüller character, having conductor p. We note that the order of w is $p-1$.

Let f be a positive integer. We set $X_f = \varprojlim_N \left(\mathbb{Z}/fp^N\mathbb{Z}\right)$, the map from $\mathbb{Z}/fp^M\mathbb{Z}$ to $\mathbb{Z}/fp^N\mathbb{Z}$ for $M \geqslant N$, to be reduction moddp^N. In the special case $f = 1$, $X_1 = \mathbb{Z}_p$. Let $a + p^N\mathbb{Z}_p = \left\{x \in \mathbb{Q}_p : |x-a|_p \leqslant p^{-N}\right\}$ for $a \in \mathbb{Q}_p$ and $N \in \mathbb{Z}$. Then the sets of the form $a + p^N\mathbb{Z}_p$ form a basis of open sets for the metric space $\mathbb{Q}_p$. This means that any open subset of $\mathbb{Q}_p$ is a union of open sets of this type. Note that $a + fp^N\mathbb{Z}_p = \bigcup_{0\leqslant b<p}\left(a+bfp^N\right) + fp^{N+1}\mathbb{Z}_p$ and $X_f^* = X_f \backslash pX_f = \bigcup_{\substack{0<a<fp\\(a,p)=1}} a + fp\mathbb{Z}_p$ (see [6, 13]).

Let $UD(\mathbb{Z}_p,\mathbb{C}_p)$ be the Banach algebra of all uniformly (or strictly) differentiable functions $f : \mathbb{Z}_p \to \mathbb{C}_p$ under the pointwise operations and valuation (see [11, 13, 18]). If ζ satisfies the condition that $\zeta^{p^n} \neq 1$ for al $n \geqslant 0$ we can define a finitely additive measure μ_ζ on X_f by

$$\mu_\zeta\left(a + fp^N\mathbb{Z}_p\right) = \frac{\zeta^{fp^N-a}}{1-\zeta^{fp^N}}, 0 \leqslant a < fp^N, N \geqslant 0.$$

Then for a function $f \in UD(\mathbb{Z}_p,\mathbb{C}_p)$ we have

$$\int_{X_f} f(x)\,\mathrm{d}\mu_\zeta(x) = \lim_{N\to\infty}\sum_{a=0}^{fp^N-1} f(a)\,\mu_\zeta\left(a+fp^N\mathbb{Z}_p\right) = \lim_{N\to\infty}\frac{\zeta^{fp^N}}{1-\zeta^{fp^N}}\sum_{a=0}^{fp^N-1} f(a)\,\zeta^{-a}$$

using the p-adic limit of the Nth Riemann sum of f. Main results for this formula can be given as follows:

Proposition 2.1. (see [14, 15, 17]) *Let $|t|_p \leqslant p^{1/(1-p)}$, $t \in \mathbb{C}_p$ and $t \neq 0$, and let χ be a primitive Dirichlet character with conductor f. Then we have*
(1) $E_m(\zeta) = \int_{\mathbb{Z}_p} x^m \mathrm{d}\mu_\zeta(x)$.
(2) $E_m(\zeta) - p^m E_m(\zeta^p) = \int_{\mathbb{Z}_p^\times} x^m \mathrm{d}\mu_\zeta(x)$.
(3) $E_{m,\chi}(\zeta) = \int_{X_f} \chi(x)\, x^m \mathrm{d}\mu_\zeta(x)$.
(4) $E_{m,\chi}(\zeta) - \chi(p)\, p^m E_{m,\chi}(\zeta^p) = \int_{X_f^*} \chi(x)\, x^m \mathrm{d}\mu_\zeta(x)$.

3. *p*-adic Interpolation of Character Dedekind Sums and Their Reciprocity Formula

In this section we define a p-adic continuous function that will be used to interpolate $s_m(h,k;\chi)$. Let ζ be a root of unity and $\zeta^{p^n} \neq 1$ for all $n \geqslant 0$. Let

$$F_p(s;\zeta,\chi) = \int_{X_f^*} \chi(x)\, w^{-1}(x)\,\langle x\rangle^s\, \frac{1}{x}\mathrm{d}\mu_\zeta(x)$$

for $s \in \mathbb{Z}_p$. Let exp and log denote the p-adic exponential and logarithm functions respectively. Then, since $\langle x \rangle \equiv 1 \pmod{p}$ for $x \in \mathbb{Z}_p^{\times}$, $\log \langle x \rangle \equiv 0 \pmod{p}$ and $\langle x \rangle^s = \exp(\log \langle x \rangle)$. Furthermore, fixing an embedding of algebraic closure of $\mathbb{Q}$, $\overline{\mathbb{Q}}$, into $\mathbb{C}_p$, we may then consider the values of a Dirichlet character χ as lying in $\mathbb{C}_p$. Therefore $F_p(s;\zeta,\chi)$ is an analytic function of s in $\mathbb{Z}_p$ with the expansion

$$\begin{aligned} F_p(s;\zeta,\chi) &= \sum_{m=0}^{\infty} c_m(\zeta,\chi)\, s^m, \\ c_m(\zeta,\chi) &= \int\limits_{X_f^*} \chi(x)\, w^{-1}(x) \frac{(\log \langle x \rangle)^m}{m!} \frac{1}{x} \mathrm{d}\mu_\zeta(x), \\ |c_m(\zeta,\chi)|_p &\leqslant \left| \frac{p^m}{m!} \right|_p \leqslant p^{-m} p^{\frac{m}{p-1}}. \end{aligned}$$

Now, since the order of w is $p-1$, we have by (4) of Proposition 2.1 that

$$F_p(m,\zeta,\chi) = \int\limits_{X_f^*} \chi(x)\, x^{m-1} \mathrm{d}\mu_\zeta(x) = E_{m-1,\chi}(\zeta) - \chi(p)\, p^{m-1} E_{m-1,\chi}(\zeta^p)$$

for all integers $m \geqslant 1$ and $m+1 \equiv 0 \pmod{p-1}$.

Definition 3.1. Let χ be a Dirichlet character of conductor f and let $h \in \mathbb{Z}$, $k \in \mathbb{N}$ with $(k,f) = 1$. Then

$$S_p(s;h,k;\chi) = s \sum_{a=0}^{kf-1} \chi(a) \frac{a}{kf} \sum_{\zeta^k=1} F_p\left(s;\zeta^h,\chi\right) \zeta^a$$

is the p-adic character Dedekind sum for all $s \in \mathbb{Z}_p$.

We now show that the function $S_p(s;h,k;\chi)$ interpolates the character Dedekind sums.

Proposition 3.2. *Let χ be a Dirichlet character of conductor f. For any integers m, h, k such that $m \geqslant 0$, $m+1 \equiv 0\,(\mathrm{mod} p-1)$, $k > 0$ and $(k,f) = 1$ we have*

$$S_p(m;h,k;\chi) = \chi(k)\, k^m \left\{ s_m(h,k;\chi) - \chi(p)\, p^{m-1} s_m(ph,k;\chi) \right\}.$$

Proof. Proof follows from definitions of $S_p(s;h,k;\chi)$ and $F_p(s;\zeta,\chi)$. In fact we have

$$\begin{aligned} S_p(m;h,k;\chi) &= \sum_{a=0}^{kf-1} \chi(a) \frac{a}{kf} m \sum_{\zeta^k=1} F_p\left(m;\zeta^h,\chi\right) \zeta^a \\ &= \sum_{a=0}^{kf-1} \chi(a) \frac{a}{kf} m \sum_{\zeta^k=1} \left[E_{m-1,\chi}\left(\zeta^h\right) - \chi(p)\, p^{m-1} E_{m-1,\chi}\left(\zeta^{hp}\right) \right] \zeta^a \\ &= \sum_{a=0}^{kf-1} \chi(a) \frac{a}{kf} m \sum_{\zeta^k=1} E_{m-1,\chi}\left(\zeta^h\right) \zeta^a \\ &\quad -\chi(p)\, p^{m-1} \sum_{a=0}^{kf-1} \chi(a) \frac{a}{kf} m \sum_{\zeta^k=1} E_{m-1,\chi}\left(\zeta^{hp}\right) \zeta^a \\ &= \chi(k)\, k^m s_m(h,k;\chi) - \chi(k)\, k^m \chi(p)\, p^{m-1} s_m(ph,k;\chi), \end{aligned}$$

which is the result. □

Now we are going to interpolate the reciprocity law for $s_m(h,k;\chi)$. For odd integer m, coprime positive integers h and k, a non-principle primitive Dirichlet character χ of modulus f and $(f,hk) = 1$ we have the reciprocity formula (see [5])

$$\begin{aligned} & hk^m s_m(h,k;\chi) + kh^m s_m(k,h;\overline{\chi}) \\ = & \frac{1}{m+1} \sum_{j=0}^{m+1} \binom{m+1}{j} h^j k^{m+1-j} B_{j,\overline{\chi}} B_{m+1-j,\chi} + \frac{m}{f} \chi(k)\, \overline{\chi}(-h) \left(f^{m+1} - 1\right) B_{m+1}. \end{aligned}$$

Proposition 3.3. *Let m be an odd integer, h and k coprime positive integers, χ be a non-principle primitive Dirichlet character of modulus f, $k \equiv 1 \pmod{f}$, $h \equiv 1 \pmod{f}$ and p be an odd prime number with $(p, kf) = (p, hf) = 1$, $m + 1 \equiv 0 \pmod{p-1}$. Then we have*

$$hS_p(m; h, k; \chi) + kS_p(m; k, h; \overline{\chi}) = \left(1 - p^{m-1}\right)\{hk^m s_m(h, k; \chi) + kh^m s_m(k, h; \overline{\chi})\}.$$

Proof. From Proposition 3.2 we have

$$\begin{aligned} S_p(m; h, k; \chi) &= \chi(k) k^m \left\{s_m(h, k; \chi) - \chi(p) p^{m-1} s_m(ph, k; \chi)\right\}, \\ S_p(m; k, h; \overline{\chi}) &= \overline{\chi}(h) h^m \left\{s_m(k, h; \overline{\chi}) - \overline{\chi}(p) p^{m-1} s_m(pk, h; \overline{\chi})\right\}. \end{aligned}$$

Since $k \equiv 1 \pmod{f}$ and $h \equiv 1 \pmod{f}$ we have $\chi(k) = \overline{\chi}(h) = 1$. Thus

$$\begin{aligned} & hS_p(m; h, k; \chi) + kS_p(m; k, h; \overline{\chi}) \\ &\quad = hk^m s_m(h, k; \chi) + kh^m s_m(k, h; \overline{\chi}) \\ &\quad\quad -p^{m-1}\{\chi(p) hk^m s_m(ph, k; \chi) + \overline{\chi}(p) kh^m s_m(pk, h; \overline{\chi})\}. \end{aligned}$$

Now

$$\begin{aligned} \chi(p) s_m(ph, k; \chi) &= \chi(p) \sum_{a=1}^{kf} \chi(a) \overline{B}_1\left(\frac{a}{kf}\right) \overline{B}_{m,\chi}\left(\frac{pha}{k}\right) \\ &= \sum_{a=1}^{kf} \chi(pa) \overline{B}_1\left(\frac{a}{kf}\right) \overline{B}_{m,\chi}\left(\frac{hpa}{k}\right). \end{aligned}$$

Since $(p, kf) = 1$, we may write $\overline{B}_1\left(\frac{a}{kf}\right) = \overline{B}_1\left(\frac{pa}{kf}\right)$, and the values pa run through the same values of a. Therefore we have

$$\chi(p) s_m(ph, k; \chi) = s_m(h, k; \chi).$$

Similarly

$$\overline{\chi}(p) s_m(pk, h; \overline{\chi}) = s_m(k, h; \overline{\chi}),$$

which completes the proof. □

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