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Flat Contact Angle Surfaces in the Heisenberg Group H^3

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Abstract. In this paper we study the equation for the Gaussian Curvature and Laplacian equation of a constant mean curvature surface in the Heisenberg Group $\mathbb{H}^3$. Also, we provide a congruence theorem for flat constant mean curvature surfaces immersed in the Heisenberg space $\mathbb{H}^3$.

1 Introduction

Surfaces making constant angles with certain directions are interesting and they are intensively studied by several authors in different ambient spaces. An interesting characterization of constant angle surfaces in Minkowski space was showed by López and Munteanu, see [7]. Also, in [5] and [6], Dillen and others have studied constant angle surfaces in product spaces $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$, namely those surfaces for which the unit normal makes a constant angle with the tangent direction to $\mathbb{R}$. Recently, Munteanu, Fastenakels and van der Veken, in [9], extended the notion of constant angle surfaces in $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$ to general Bianchi, Cartan, Vranceanu spaces and they showed that these surfaces have constant Gaussian curvature, also they gave a complete local classification in the Heisenberg group.
In [8], we construct a family of minimal tori in S^5 with constant contact angle and constant holomorphic angle. These tori are parametrized by the following circle equation

$$a^2 + \left(b - \frac{\cos\beta}{1+\sin^2\beta}\right)^2 = 2\frac{\sin^4\beta}{(1+\sin^2\beta)^2}, \tag{1.1}$$

In particular, when $a = 0$, we recover the examples found by Kenmotsu, in [2], [3] and [4]. These examples are defined for $0 < \beta < \frac{\pi}{2}$. Also, when $b = 0$, we find a new family of minimal tori in S^5, and these tori are defined for $\frac{\pi}{4} < \beta < \frac{\pi}{2}$.
We show that the Gaussian curvature K of a surface in $\mathbb{H}^3$ with contact angle β and constant mean curvature $H \neq 0$ is given by:

$$K \quad = \quad -3\sin\beta - |\nabla\beta + e_1|^2 - 2H\beta_2$$

Moreover, the contact angle satisfies the following Laplacian equation

$$\triangle(\beta) \quad = \quad -2H_2 - \tan(\beta)(|\nabla\beta + 2e_1|^2 + 4H(H+\beta_2) + \frac{\cos^2\beta}{\sin\beta})$$

where e_1 is the characteristic field defined in section 2 and introduced by Bennequin, in [1], pages 190 - 206.
Using the equations of Gauss and Codazzi, we have proved the following theorem:

Theorem 1.1. *Consider S a flat orientable Riemannian surface and $\beta : S \to]0, \frac{\pi}{2}[$ a function over S that verifies the following equation:*

$$\triangle(\beta) \quad = \quad \tan(\beta)(|\nabla\beta|^2 - 4H^2 - 2 + 6\sin\beta - \frac{\cos^2\beta}{\sin\beta})$$

then there exist only one immersion of S into $\mathbb{H}^3$ such that $H \neq 0$ is the constant mean curvature of S, and β is the contact angle of this immersion.

2 The Contact Angle for constant mean curvature surface in the Heisenberg Group $\mathbb{H}^3$

Consider the following objects:

- The 3-dimensional Heisenberg group $\mathbb{H}^3$ can be viewed as R^3 endowed with the metric

$$dx_1^2 + dx_2^2 + (\frac{1}{2}(x_2dx_1 - x_1dx_2) + dx_3))^2 \tag{2.1}$$

- the *Reeb* vector field in $\mathbb{H}^3$, given by: $\xi(z) = iz$;
- the contact distribution in $\mathbb{H}^3$, which is orthogonal to ξ:

$$\delta_z = \{v \in T_z\mathbb{H}^3 | \langle \xi, v \rangle = 0\}.$$

We identify $\mathbb{H}^3$ as $\mathbb{R}^3$ given by the following frame:

$$\begin{array}{rcl} f_1 &=& \frac{\delta}{\delta x_1} + x_2\frac{\delta}{\delta x_3} \\ f_2 &=& \frac{\delta}{\delta x_2} - x_1\frac{\delta}{\delta x_3} \\ f_3 &=& \frac{\delta}{\delta x_3} \end{array} \tag{2.2}$$

Let $(\omega^1, \omega^2, \omega^3)$ be the coframe associated to (f_1, f_2, f_3).
Thus, from equations (2.2), it follows that:

$$\begin{array}{rcl} \omega^1 &=& dx_1 \\ \omega^2 &=& dx_2 \\ \omega^3 &=& dx_3 - x_2dx_1 + x_1dx_2 \end{array} \tag{2.3}$$

Therefore, we get:

$$\begin{array}{c} d\omega^1 - \omega^3 \wedge \omega^2 - \omega^2 \wedge \omega^3 = 0 \\ d\omega^2 + \omega^3 \wedge \omega^1 + \omega^1 \wedge \omega^3 = 0 \\ d\omega^3 + \omega^2 \wedge \omega^1 - \omega^1 \wedge \omega^2 = 0 \end{array} \tag{2.4}$$

It follows that:

$$\begin{array}{c} \omega_2^1 = -\omega^3; \omega_3^2 = \omega^1; \omega_1^3 = \omega^2; \\ d\omega_2^1 = -2\omega^1 \wedge \omega^2; d\omega_3^2 = 0; d\omega_1^3 = 0 \end{array} \tag{2.5}$$

We have the following curvature forms:

$$\begin{array}{c} \omega_2^1 = d\omega_2^1 + \omega_3^1 \wedge \omega_2^3 = -3\omega^1 \wedge \omega^2 \\ \omega_3^2 = d\omega_3^2 + \omega_1^2 \wedge \omega_3^1 = \omega^2 \wedge \omega^3 \\ \omega_1^3 = d\omega_1^3 + \omega_2^3 \wedge \omega_1^2 = \omega^3 \wedge \omega^1 \end{array} \tag{2.6}$$

Therefore:

$$\begin{array}{c} \nabla_2^1(e_1, e_2) = -3 \\ \nabla_3^2(e_2, e_3) = 1 \\ \nabla_1^3(e_3, e_1) = 1 \end{array} \tag{2.7}$$

Let now S be an immersed orientable surface in $\mathbb{H}^3$.

Definition 2.1. The *Contact angle* β is the complementary angle between the contact distribution δ and the tangent space TS of the surface.

Let (e_1, e_2) be a local frame of TS, where $e_1 \in TS \cap \delta$. Then $\cos\beta = \langle \xi, e_2 \rangle$.
Thus follows that:

$$\begin{array}{rcl} e_1 &=& f_1 \\ e_2 &=& \sin(\beta)\, f_2 + \cos(\beta)\, f_3 \\ e_3 &=& -\cos(\beta)\, f_2 + \sin(\beta)\, f_3 \end{array} \tag{2.8}$$

where β is the angle between f_3 and e_2, (e_1, e_2) are tangent to S and e_3 is normal to S

3 Equations for the Gaussian Curvature and for the Laplacian of Constant Mean Curvature Surface in $\mathbb{H}^3$

In this section, we will give formulas for the Laplacian and for the Gaussian curvature of a mean curvature $H \neq 0$ surface immersed in $\mathbb{H}^3$.
We will denote that Remark 1

Remark 3.1. $d\beta(e_1) = \beta_1$, $d\beta(e_2) = \beta_2$, $dH(e_1) = H_1$, $dH(e_2) = H_2$.

Let $(\theta^1, \theta^2, \theta^3)$ be the coframe associated to (e_1, e_2, e_3).
So we have

$$\begin{aligned} \theta^1 &= w^1 \\ \theta^2 &= \sin(\beta)w^2 + \cos(\beta)w^3 \\ \theta^3 &= -\cos(\beta)w^2 + \sin(\beta)w^3 \end{aligned} \tag{3.1}$$

We know that $\theta^3 = 0$ on S, then we obtain the following equation:

$$\cos(\beta)\, w^2 = \sin(\beta)w^3 \tag{3.2}$$

It follows from (3.1) that:

$$\begin{aligned} d\theta^1 &+ \sin(\beta)w_1^2 \wedge \theta^2 = 0 \\ d\theta^2 &+ \sin(\beta)(w_2^1 + \cos(\beta)\theta^2) \wedge \theta^1 = 0 \\ d\theta^3 &= d\beta \wedge \theta^2 - \cos(\beta)w_2^1 \wedge \theta^1 + (1 + \sin^2(\beta))\theta^1 \wedge \theta^2 \end{aligned}$$

Therefore the connection form of S is given by

$$\theta_1^2 = \sin(\beta)(w_2^1 - \cos(\beta)\theta^2) \tag{3.3}$$

Differentiating e_3 at the basis (e_1, e_2), we have fundamental second forms coeficients

$$De_3 = \theta_3^1 e_1 + \theta_3^2 e_2$$

where

$$\begin{aligned} \theta_3^1 &= -\cos(\beta)w_2^1 - \sin^2(\beta)\theta^2 \\ \theta_3^2 &= d\beta + \theta^1 \end{aligned}$$

It follows from $d\theta^3 = 0$, that

$$w_2^1(e_2) = -\frac{\beta_1}{\cos\beta} - \frac{(1 + \sin^2\beta)}{\cos\beta} \tag{3.4}$$

where $d\beta(e_1) = \beta_1$.
The condition of mean curvature is equivalent to the following equation

$$\theta_1^3 \wedge \theta^2 - \theta_2^3 \wedge \theta^1 = 2H\theta^1 \wedge \theta^2$$

we have

$$-\cos(\beta)w_2^1(e_1) + \beta_2 = 2H \tag{3.5}$$

It follows from (3.5), that

$$w_2^1(e_1) = \frac{\beta_2}{\cos\beta} - \frac{2H}{\cos\beta} \tag{3.6}$$

where $d\beta(e_2) = \beta_2$.
It follows from (3.3), (3.4) and (3.6),

$$\begin{aligned} \theta_1^2 &= -\tan(\beta)((\beta_2 - 2H)\theta^1 - (\beta_1 + 2)\theta^2) \\ \theta_1^3 &= (\beta_2 - 2H)\theta^1 - (\beta_1 + 1)\theta^2 \\ \theta_2^3 &= -(\beta_1 + 1)\theta^1 - \beta_2\theta^2 \end{aligned} \tag{3.7}$$

Gauss equation is

$$d\theta_1^2 = \nabla_1^2 + \theta_2^3 \wedge \theta_1^3$$

which implies

$$d\theta_1^2 = -(|\nabla\beta|^2 + 2\beta_1) - 1 - 2H\beta_2 - 3\sin\beta\,(\theta^2 \wedge \theta^1) \tag{3.8}$$

where:

$$\nabla_1^2(e_2, e_1) = -3\sin\beta \tag{3.9}$$

and therefore

$$K = -3\sin\beta - |\nabla\beta + e_1|^2 - 2H\beta_2 \tag{3.10}$$

Codazzi equations are

$$d\theta_1^3 + \theta_2^3 \wedge \theta_1^2 = \nabla_1^3$$
$$d\theta_2^3 + \theta_1^3 \wedge \theta_2^1 = \nabla_2^3$$

A straightforward computation in the first equation gives (4.1) and the second equation is always verified.

$$\triangle(\beta) = -2H_2 - \tan(\beta)(|\nabla\beta + 2e_1|^2 + 4H(H + \beta_2) + \frac{\cos^2\beta}{\sin\beta})$$

where:

$$\begin{aligned} \nabla_1^3(e_1, e_2) &= -\cos\beta \\ \nabla_2^3(e_1, e_2) &= -\sin\beta\cos\beta \end{aligned} \tag{3.11}$$

4 Main Result

4.1 Proof of the Theorem 1.1

Let S be an orientable surface in $\mathbb{H}^3$.
Supposing that $K = 0$ in 3.10, we get $\beta_2 = -\frac{3\sin\beta + |\nabla\beta + e_1|^2}{2H}$. Now using the Laplacian equation bellow with H constant.

$$\triangle(\beta) = -2H_2 - \tan(\beta)(|\nabla\beta + 2e_1|^2 + 4H(H + \beta_2) + \frac{\cos^2\beta}{\sin\beta})$$

So it follows the result. □

References

[1] B. Aebischer: *Sympletic Geometry*, Progress in Mathematics, Vol. 124 Springer-Verlag, Berlin-New York, 1992.

[2] K. Kenmotsu: *On compact minimal surfaces with non-negative Gaussian curvature in a space of constant curvature I,II*, Tohoku Math. J. 25 (1973), 469-479; Tohoku Math. J. 27 (1975),91-301.

[3] K. Kenmotsu, *On a parametrization of minimal immersions R^2 into S^5*,Tohoku Math. J. **27** (1975) 83-90.

[4] K. Kenmotsu: *On minimal immersion of R^2 into CP^n*, J. Math Soc. Japan 37 (1985), 665-682.

[5] F. Dillen, J. Fastenakels, J. Van der Veken and L. Vrancken, Constant angle surfaces in $S^2 \times R$, Monathsh. Math., 152, 2 (2007), 89-96.

[6] F. Dillen and M. I. Munteanu, Constant angle surfaces in $H^2 \times R$, Bull. Braz. Math. Soc., 40, 1 (2009), 85-97.

[7] R. López, M.I. Munteanu : *Constant angle surfaces in the Heisenberg group*, Bull. Belg. Math. Soc. Simon Stevin, 18 (2011) 2, 271- 286.

[8] R.R. Montes, J.A. Verderesi: *Contact Angle for Immersed Surfaces in S^{2n+1}*, Differential Geometry and its Applications, Vol. 25, (2007), 92-100.

[9] M.I. Munteanu, J. Van der Veken, J. Fastenakels : *Constant angle surfaces in the Minkowski space*, Acta Mathematica Sinica (English Series), 27 (2011) 4, 747 - 756.

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