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On the Stabilization for Timoshenko System with Past History and Frictional Damping Controls

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Abstract. In this paper, we consider vibrating system of Timoshenko type in one-dimensional bounded domain with complementary past history and frictional damping controls acting only in the equation for the rotation angle. We show that the dissipation given by this complementary controls is strong enough to guarantee the stability of the system in case of the same wave propagation speeds as well as in the opposite case. We establish in each case a general decay estimate of the solution. In the particular case when the wave propagation speeds are different and the frictional damping is linear, we give a relationship between the smoothness of the initial data and the decay rate of the energy. At the end of the paper, we discuss the well-posedness and smoothness of solution using the semigroup theory.

1 Introduction

In the present work, we are concerned with the well-posedness, smoothness and asymptotic behavior of the solution of the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi) + b(x)h(\psi_t) + \int_0^{+\infty} g(s)(a(x)\psi_x(t-s))_x ds = 0, \\ \varphi(0,t) = \psi(0,t) = \varphi(L,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t),\ \psi_t(x,0) = \psi_1(x), \end{cases} \tag{P}$$

for $(x,t) \in]0,L[\times\mathbb{R}_+$, where $\mathbb{R}_+ = [0,+\infty[$, $a,\ b : [0,L] \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R} \to \mathbb{R}$ are given functions, $L,\ \rho_i,\ k_i$ are positive constants, φ_0, φ_1, ψ_0 and ψ_1 are given initial data, and (φ,ψ) is the unknown of (P). The infinite integral term in (P) and $bh(\psi_t)$ represent, respectively, the past history (infinite memory) and the frictional damping.

Our aim is to establish a general decay estimate for the asymptotic behavior of the solutions of (P) in both cases: the equal-speed wave propagation case

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \tag{1.1}$$

and the opposite one.

During the last few years, many people have been interested in the question of stability of Timoshenko systems [23] with various kinds of (internal or boundary) controls. To focus on our motivation, let us mention here only some known results related to the stabilization with finite or infinite memory controls (for further results of stabilization, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

In the case $b = 0$ and g satisfies

$$\exists \delta_1,\, \delta_2 > 0 : \, -\delta_1 g(s) \leq g'(s) \leq -\delta_2 g(s), \quad \forall s \in \mathbb{R}_+, \tag{1.2}$$

the authors of [18] proved that (P) is exponentially stable if and only if (1.1) holds, and it is polynomially stable in general, where the decay rate depends on the smoothness of the initial data. In the case $b = 0$ and g satisfies

$$\exists \delta > 0,\, \exists p \in]1, \frac{3}{2}[: \, g'(s) \leq -\delta g^p(s), \quad \forall s \in \mathbb{R}_+, \tag{1.3}$$

it was proved in [17] that (P) is polynomially stable. No relationship between the decay rate and the smoothness of the initial data is given in [17].

The past history was also considered to stabilize the semigroup associated to a general abstract linear equation in [5], [19] and [21]. In [19], some decay estimates were proved depending on the considered operators provided that g satisfies (1.2). In [21], it was proved that the exponential stability still holds even if g has horizontal inflection points or even flat zones provided that g is equal to a negative exponential except on a sufficiently small set where g is flat. The author of [5] proved that the exponential stability does not hold if the following condition is not satisfied:

$$\exists \delta_1 \geq 1,\, \exists \delta_2 > 0 : \, g(t+s) \leq \delta_1 e^{-\delta_2 t} g(s), \quad \forall t \in \mathbb{R}_+, \text{ for a.e. } s \in \mathbb{R}_+. \tag{1.4}$$

In [8], an abstract dissipative system with pas history was considered and its stability was proved for kernels g having more general decay rate, which can be arbitrary close to the one of t^{-1}. The approach of [8] was used in [11] to get the stability of the solution of (P) in case $b = 0$, as well as some coupled Timoshenko-heat systems.

The stability of Timoshenko systems with *finite memory* have attracted considerable attention in recent years and many authors have shown various decay estimates depending on the growth of the kernel g at infinity. Using an approach introduced in [16] for a viscoelastic equation, a general decay estimate of (P) with *finite memory* was obtained in [10] for kernels satisfying

$$g'(t) \leq -\xi(t) g(t), \quad \forall t \geq 0, \tag{1.5}$$

where ξ is a positive and non-increasing function. The decay result in [10] improves earlier ones in the literature in which only the exponential and polynomial decay rates are considered.

Concerning the stability of the wave equation with *finite memory*, we mention the recent results in [20], where a general decay result was given by assuming

$$g'(t) \leq -K(g(t)), \quad \forall t \geq 0, \tag{1.6}$$

where K is a non-negative function satisfying some hypotheses.

Our goals in this paper are:

1. Investigating the effect of each control on the asymptotic behavior of the solutions of (P) and on the decay rate of its energy, where each control can vanish on the whole or in a part of the domain, and giving an explicit and general characterization of the decay rate depending on the growth of g and h at infinity and zero, respectively. This includes the particular cases $a = 0$ or

$b = 0$ (only one control is considered) treated in the literature, and then it generalizes the results of [17] and [18] ($b = 0$).

2. Considering a larger class of kernels g (satisfying (2.8) below) than the one considered in [17] and [18], and improving the decay rate in some particular cases. On the other hand, this approach can be also used in the case of *finite memory* by choosing a null past history data ($\psi_0(t) = 0$ for all $t \geq 0$).

3. Studying the case when (1.1) does not hold and giving a general decay estimate depending on the smoothness of the initial data and the growth of g at infinity. To the best of our knowledge, the unique known results in this direction are the ones given in [18] in the case $b = 0$ and (1.2) holds.

The paper is organized as follows. In section 2, we consider some hypotheses and present our stability results. The proofs of our stability results will be given in sections 3 ((1.1) holds), 4 ((1.1) does not hold) and 5 (h is linear). Finally, in section 6, we discuss the well-posedness and smoothness of the solution of (P).

2 Preliminaries

In this paper, first, we consider the following hypotheses:

(H1) $a,\, b : [0, L] \to \mathbb{R}_+$ are such that

$$a \in C^1([0, L]),\ b \in L^\infty([0, L]), \tag{2.1}$$

$$\inf_{x\in[0,L]} \{a(x) + b(x)\} > 0, \tag{2.2}$$

$$a = 0 \text{ or } a(0) + a(L) > 0. \tag{2.3}$$

(H2) $h : \mathbb{R} \to \mathbb{R}$ is a differentiable non-decreasing function such that there exist constants $\epsilon_1,\, c_1,\, c_1' > 0$, and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $H(0) = 0$, and H is linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$) such that

$$c_1|s| \leq |h(s)| \leq c_1'|s| \quad \text{if } |s| \geq \epsilon_1, \tag{2.4}$$

$$s^2 + h^2(s) \leq H^{-1}(sh(s)) \quad \text{if } |s| < \epsilon_1. \tag{2.5}$$

(H3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function such that $g(0) > 0$ and

$$l := k_2 - \|a\|_\infty \int_0^{+\infty} g(s)ds > 0. \tag{2.6}$$

(H4) There exist a positive constant c_1'' and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty$$

such that

$$g'(t) \leq -c_1'' g(t),\ \forall t \geq 0 \tag{2.7}$$

or

$$\int_0^{+\infty} \frac{g(t)}{G^{-1}(-g'(t))} dt + \sup_{t\in\mathbb{R}_+} \frac{g(t)}{G^{-1}(-g'(t))} < +\infty. \tag{2.8}$$

Remark 2.1. 1. The hypothesis (H4) was introduced in [8] and it is weaker than the classical one considered in [18] ($p = 1$) and [17] ($1 < p < \frac{3}{2}$), namely,

$$\exists c_1'' > 0,\ \exists p \in [1, \frac{3}{2}[:\ g'(t) \le -c_1'' g^p(t),\ \forall t \ge 0. \tag{2.9}$$

Indeed, for $p = 1$, (2.7) is (2.9). For $p \in]1, \frac{3}{2}[$, (2.8) is satisfied with $G(t) = t^{2p}$. On the other hand, for example, for $g(t) = q_0(1+t)^{-q}$ with $q_0 > 0$ and $q \in]1, 2]$, (2.8) is satisfied with $G(t) = t^r$ for all $r > \frac{q+1}{q-1}$, but (2.9) is not satisfied (because it implies that $p \ge 1 + \frac{1}{q} \ge \frac{3}{2}$) (see also Examples 2.1 below).

In general, all positive function g of class $C^1(\mathbb{R}_+)$ with $g' < 0$ satisfies (2.8) if it is integrable on $\mathbb{R}_+$, and it does not satisfy (2.9) if it does not converge to zero at infinity *faster* than t^{-2}. Additionally, (2.8) allows us to improve the results of [17] by getting in some particular cases stronger decay rates (see Examples 2.1 given at the end of this section).

2. Hypothesis $(H2)$ (with $\epsilon_1 = 1$) was introduced and used in [13] and [14] to get the asymptotic behavior of solutions of nonlinear wave equations with nonlinear boundary damping, where they obtained decay estimates depending on the solution of an explicit nonlinear ordinary differential equation. The function H always exists thanks to the hypotheses on h (see [13] and [14]).

3. Hypothesis $(H1)$ was considered in [4] for the wave equation and used in [10] for Timoshenko systems with finite memory.

Let us consider, as in [17] and [18] for example, the classical energy functional associated with (P) defined by

$$E(t) := \frac{1}{2} g \circ \psi_x \tag{2.10}$$
$$+\frac{1}{2}\int_0^L \Big[\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2 + \Big(k_2 - a(x)\int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big]dx,$$

where, for $v:\ \mathbb{R} \to L^2(]0, L[)$ and $\phi:\ \mathbb{R}_+ \to \mathbb{R}_+$, we note

$$\phi \circ v = \int_0^L a(x) \int_0^{+\infty} \phi(s)(v(t) - v(t-s))^2 ds dx.$$

Thanks to (2.6), the expression $\int_0^L \Big[k_1(\varphi_x + \psi)^2 + \Big(k_2 - a(x)\int_0^{+\infty} g(s)ds\Big)\psi_x^2\Big]dx$ defines a norm on $(H_0^1(]0, L[))^2$ for (φ, ψ) equivalent to the one induced by $(H^1(]0, L[))^2$.

Now, we give our first main stability result which concerns the case (1.1).

Theorem 2.1. *Assume that (1.1) and (H1)-(H4) are satisfied, and let* $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in \mathcal{H}$ *(see section 6) such that*

$$\begin{cases} (2.7)\ \textit{holds} \\ \textit{or} \\ \exists M_0 \ge 0:\ \|\eta_{0x}(s)\|_{L^2(]0,L[)} \le M_0,\ \forall s > 0. \end{cases} \tag{2.11}$$

Then there exist positive constants c', c'', τ_0 *and* ϵ_0 *(depending continuously on* $E(0)$*) for which* E *satisfies*

$$E(t) \le c'' G_1^{-1}(c't), \quad \forall t \ge 0, \tag{2.12}$$

where $G_1(t) = \displaystyle\int_t^1 \frac{1}{G_0(s)} ds,$

$$G_0(t) = \begin{cases} H_0(t) & \textit{if (2.7) holds}, \\ H_0(t) G'(\epsilon_0 H_0(t)) & \textit{if (2.8) holds} \end{cases} \tag{2.13}$$

and

$$H_0(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon_1], \\ tH'(\tau_0 t) & \text{otherwise.} \end{cases} \tag{2.14}$$

Remark 2.2. 1. Because $\lim_{t\to 0^+} G_1(t) = +\infty$, then (2.12) implies that

$$\lim_{t\to+\infty} E(t) = 0. \tag{2.15}$$

2. If (2.7) holds (or if $a = 0$) and H is linear near zero (or $b = 0$), then

$$E(t) \le c'' e^{-c't}, \quad \forall t \ge 0 \tag{2.16}$$

which is the best decay rate given by (2.12).

3. Theorem 2.1 generalizes some results of [11] ($b = 0$), and generalizes and improves the ones of [17] and [18] ($b = 0$ and (2.9) holds). See Examples 2.1. □

Now, we treat the case when (1.1) does not hold.

(H5) Assume that (H1) is satisfied such that there exist $b_0,\ b_1 > 0$ satisfying

$$|a'(x)| + |a(x) - b_0| \le b_1 \max\{b(x), \inf_{x\in[0,L]}\{a(x)\}\}, \quad \forall x \in [0, L].$$

(H6) Assume that (H2) is satisfied such that H is linear,

$$h \in C^1(\mathbb{R}) \quad \text{and} \quad \inf_{t\in\mathbf{R}} h'(t) > 0.$$

Theorem 2.2. *Assume that (H1)-(H6) are satisfied, and let (see section 6)* $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in D(A)$ *such that*

$$\begin{cases} (2.7) \text{ holds} \\ \text{or} \\ \exists M_0 \ge 0 : \|\eta_{0x}(s)\|_{L^2(]0,L[)}, \|\partial_s \eta_{0x}(s)\|_{L^2(]0,L[)} \le M_0, \forall s > 0. \end{cases} \tag{2.17}$$

Then there exist positive constants C and ϵ_0 (depending in a continuous way on $\|(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T\|_{D(A)}$) such that

$$E(t) \le G_0^{-1}(\frac{C}{t}), \quad \forall t > 0, \tag{2.18}$$

where

$$G_0(t) = \begin{cases} t & \text{if (2.7) holds,} \\ tG'(\epsilon_0 t) & \text{if (2.8) holds.} \end{cases} \tag{2.19}$$

Remark 2.3. 1. Hypothesis (H5) is satisfied if, for example, $\inf_{x\in[0,L]}\{a(x)\} > 0$ or $\inf_{x\in[0,L]}\{b(x)\} > 0$ or a is equal to a positive constant b_0 in an open interval containing the closure of $(supp\, b)^c$.

2. If (2.7) holds (or $a = 0$), then (2.18) becomes

$$E(t) \le \frac{C}{t}, \quad \forall t > 0$$

which is the best decay rate given by (2.18).

3. Theorem 2.2 generalizes some results of [11] ($b = 0$), and generalizes and improves the ones of [17] and [18] ($b = 0$ and (2.9) holds). See Examples 2.1. □

Now, in the particular case where h is linear and the initial data are more regular, we prove a more general stability result than (2.18).

Theorem 2.3. *Assume that h is linear, and (H1)-(H5) are satisfied such that $t \mapsto tG'(t)$ is strictly convex (when (2.7) does not hold). Let (see section 6) $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in D(A^n)$ for $n \in \mathbb{N}^*$ such that*

$$\begin{cases} (2.7) \text{ holds} \\ or \\ \exists M_0 \geq 0 : \|\partial_s^{(k)} \eta_{0x}(s)\|_{L^2(]0,L[)} \leq M_0, \, \forall s > 0, \, \forall k = 0, 1, \cdots, n. \end{cases} \tag{2.20}$$

Then there exist positive constants C and ϵ_0 (depending in a continuous way on $\|(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T\|_{D(A^n)}$) such that the solution of (P) satisfies:

$$E(t) \leq G_1^{-1}(\frac{C}{t^n}), \quad \forall t > 0, \tag{2.21}$$

where

$$G_1(t) = \begin{cases} t & \text{if (2.7) holds}, \\ G_0(t)\Big(G_0^{*-1}(G_0(t))\Big)^{n-1} & \text{if (2.8) holds}, \end{cases} \tag{2.22}$$

$G_0(t) = tG'(\epsilon_0 t)$ and G_0^* is the dual function of G_0 defined by

$$G_0^*(t) = \sup_{s \in \mathbb{R}_+} \{ts - G_0(s)\} = t(G_0')^{-1}(t) - G_0((G_0')^{-1}(t)), \quad \forall t \geq 0.$$

Remark 2.4. 1. If $n = 1$, then (2.18) is (2.21). On the other hand, if (2.7) holds (or $a = 0$), then (2.21) becomes

$$E(t) \leq \frac{C}{t^n}, \quad \forall t > 0 \tag{2.23}$$

which is the best decay rate given by (2.21).

2. Theorem 2.3 generalizes some results of [11] ($b = 0$), and generalizes and improves some ones of [18] ($b = 0$ and (2.7) holds) and [17] ($b = 0$, (2.9) holds and $n = 1$). See Examples 2.1.

Examples 2.1. Let us give some examples to illustrate our general decay estimates and show how they generalize and improve the ones known in the literature concerning the kernel g. For the frictional damping h, we consider in these examples the simple case when H is linear (that is (2.4) is satisfied on all $\mathbb{R}$) or $b = 0$. For various examples concerning the growth of h at zero, see [2] and [10]. On the other hand, (2.18) is (2.21) for $n = 1$.

1. Let $g(t) = \frac{d}{(2+t)(\ln(2+t))^q}$ for $q > 1$, and $d > 0$ small enough so that (2.6) is satisfied. The classical condition (2.9) is not satisfied, while (2.8) holds with

$$G(t) = \int_0^t s^{\frac{1}{p}} e^{-s^{-\frac{1}{p}}} ds \quad \text{for any } p \in]0, q-1[.$$

Indeed, here (2.8) depends only on the growth of G near zero. Using the fact that $G(t) \leq t^{\frac{1}{p}+1} e^{-t^{-\frac{1}{p}}}$, we can see that $G(t(\ln t)^r g(t)) \leq -g'(t)$ for t near infinity and for any $r \in]1, q-p[$

which implies (2.8). Then (2.12) becomes

$$E(t) \le \frac{C}{(\ln(t+2))^p}, \quad \forall t \ge 0,\ \forall p \in]0, q-1[. \tag{2.24}$$

Because G_0^{*-1} is concave, we have $G_0^{*-1}(t) \ge G_0^{*-1}(1)t$ for $t \in [0,1]$. Therefore,

$$G_1(s) \ge c(G_0(s))^n \ge e^{-cs^{-\frac{1}{p}}} \quad \text{for s near zero.}$$

Thus (2.21) implies (2.24), where C depends on n in this case.

2. Let $g(t) = \frac{d}{(1+t)^q}$ for $q > 1$, and $d > 0$ small enough so that (2.6) is satisfied. The classical condition (2.9) is not satisfied if $1 < q \le 2$, while (2.8) always holds with $G(t) = t^{\frac{1}{p}+1}$ for any $p \in]0, \frac{q-1}{2}[$ (as in Example 2.1-1, we can see that, if $p \in]0, \frac{q-1}{2}[$, then $G(t(\ln t)^r g(t)) \le -g'(t)$ for t near infinity and for any $r > 1$). Then (2.12) and (2.21) give, respectively (note that $G_0^*(s) = cs^{p+1}$),

$$E(t) \le \frac{C}{(t+1)^p}, \quad \forall t \ge 0,\ \forall p \in]0, \frac{q-1}{2}[,$$

$$E(t) \le \frac{C}{(t+1)^{\frac{np}{p+n}}}, \quad \forall t \ge 0,\ \forall p \in]0, \frac{q-1}{2}[.$$

3. Let $g(t) = de^{-(\ln(2+t))^q}$ for $q > 1$, and $d > 0$ small enough so that (2.6) is satisfied. Condition (2.8) holds with $G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}} e^{-(-\ln s)^{\frac{1}{p}}} ds$ for t near zero and for any $p \in]1, q[$, since (2.8) depends only on the growth of G at zero, and when t converges to infinity and $p \in]1, q[$, $G(t^r g(t)) \le -g'(t)$ for any $r > 1$. Then (2.12) becomes

$$E(t) \le ce^{-C(\ln(1+t))^p}, \quad \forall t \ge 0,\ \forall p \in]1, q[. \tag{2.25}$$

Condition (2.8) holds also with $G(s) = s^{\frac{1}{p}+1}$ for any $p > 0$. Then (2.21) gives

$$E(t) \le \frac{C}{(t+1)^{\frac{np}{p+n}}}, \quad \forall t \ge 0,\ \forall p > 0. \tag{2.26}$$

Here, the decay rates of E in (2.25) and (2.26) are arbitrary close to the one of g and t^{-n}, respectively. This improves the results of [17], where only the polynomial decay with $n = 1$ and $b = 0$ was obtained.

4. Let $g(t) = de^{-(1+t)^q}$ for $q > 0$, and $d > 0$ small enough so that (2.6) is satisfied. Condition (2.7) holds if $q \ge 1$, and (2.8) holds if $q \in]0, 1[$ with

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}} ds \quad \text{for } t \text{ near zero and for any } p \in]0, \tfrac{q}{2}[$$

(we can see that, if $p \in]0, \frac{q}{2}[$, then $G(t^r g(t)) \le -g'(t)$ for t near infinity and for any $r \in]1, \frac{q}{p} - 1[$). Then (2.12) becomes

$$E(t) \le \begin{cases} ce^{-Ct^p}, & \forall t \ge 0,\ \forall p \in]0, \frac{q}{2}[\quad \text{if } q \in]0,1[, \\ ce^{-Ct}, & \forall t \ge 0 \quad \text{if } q \ge 1. \end{cases} \tag{2.27}$$

Condition (2.8) holds also with $G(s) = s^{\frac{1}{p}+1}$ for any $p > 0$. Then (2.21) gives

$$E(t)) \le \begin{cases} \frac{C}{(t+1)^{\frac{np}{p+n}}}, & \forall t \ge 0,\ \forall p > 0 \quad \text{if } q \in]0,1[, \\ \frac{C}{(t+1)^n}, & \forall t \ge 0 \quad \text{if } q \ge 1. \end{cases} \tag{2.28}$$

Here, if $q \in]0,1[$, (2.27) and (2.28) are stronger than the ones given in [17], where only the polynomial decay with $n = 1$ and $b = 0$ was obtained. $\square$

Now, before starting the proofs of our stability results, we define a function α introduced in [4] to establish some needed estimates in the case $a \neq 0$. Condition (2.3) implies that $a(0) > 0$ or $a(L) > 0$, and then (a is continuous) there exists $d_0 \in]0, L[$ such that $\inf_{x\in D_0} a(x) \geq d_0$, where $D_0 = [0, d_0]$ if $a(0) > 0$, and $D_0 = [L - d_0, L]$ if $a(L) > 0$. Let $d = \min\{d_0, \inf_{x\in[0,L]}\{a(x) + b(x)\}\}$ (which is positive thanks to (2.2)) and $\alpha \in C^1([0, L])$ be such that $0 \leq \alpha \leq a$, $\alpha = 0$ if $a \leq \frac{d}{4}$, and $\alpha = a$ if $a \geq \frac{d}{2}$. By definition of α, we have

$$|\alpha'(x)| \leq \frac{4}{d}\|\alpha'\|_\infty a(x). \tag{2.29}$$

On the other hand, we have the following lemma:

Lemma 2.1. *The function α is not identically zero and satisfies*

$$\inf_{x\in[0,L]}\{\alpha(x) + b(x)\} > 0, \tag{2.30}$$

$$\exists d_1 > 0 : \Big(\int_0^L \alpha(x)\int_0^{+\infty} g(s)(\psi(t) - \psi(t - s))dsdx\Big)^2 \leq d_1 g \circ \psi_x, \tag{2.31}$$

$$\exists d_2 > 0 : \Big(\int_0^L \alpha(x)\int_0^{+\infty} g'(s)(\psi(t) - \psi(t - s))dsdx\Big)^2 \leq -d_2 g' \circ \psi_x, \tag{2.32}$$

$$\exists d_3 > 0 : \Big(\int_0^L \alpha'(x)\int_0^{+\infty} g(s)(\psi(t) - \psi(t - s))dsdx\Big)^2 \leq d_3 g \circ \psi_x. \tag{2.33}$$

Proof. The proofs of (2.30) and (2.31) are identical to the ones of [10, Lemma 3.2 and Lemma 3.3] concerning the case of finite memory control, where a version of Poincaré's inequality [4] was used. Similarly to (2.31) and using (2.29), (2.32) and (2.33) follow. $\square$

3 Proof of Theorem 2.1

We will use c (sometimes c_τ which depends on some parameter τ), throughout this paper, to denote a generic positive constant which depends continuously on the initial data and can be different from step to step.

Following the energy method ([3], [10], [17], [18] and [19] for example), we will construct a Lyapunov function F equivalent to E and satisfying (3.30) below. For this purpose we establish several lemmas for all $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in D(A)$ satisfying (2.11), so all the calculations are justified. By a simple density argument ($D(A)$ is dense in $\mathcal{H}$; see section 6), (2.12) remains valid for any $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in \mathcal{H}$ satisfying (2.11). On the other hand, if $E(t_0) = 0$ for some $t_0 \geq 0$, then $E(t) = 0$ for all $t \geq t_0$ (E is non-increasing thanks to (3.1) below) and thus (2.12), (2.18) and (2.21) hold. Then, without loss of generality, we assume that $E(t) > 0$ for all $t \in \mathbb{R}_+$.

To get estimate (3.23) below, we prove Lemma 3.1-Lemma 3.10 where the proof is inspired from the classical multipliers method used in [3], [10], [17], [18] and [19]. To estimate the nonlinear terms in (3.18), we use some arguments and ideas from [12], [13] and [15]. We use also the approach of [8] to estimate the term $g \circ \psi_x$ in (3.23) under the weaker assumption (2.8).

Lemma 3.1. *The energy functional satisfies*

$$E'(t) = \frac{1}{2}g' \circ \psi_x - \int_0^L b(x)\psi_t h(\psi_t)dx \leq 0. \tag{3.1}$$

Proof. By multiplying the first two equations in (P), respectively, by φ_t and ψ_t, integrating over $]0,L[$, and using the boundary conditions, we obtain (3.1) (note that g is non-increasing and $sh(s) \geq 0$ for all $s \in \mathbb{R}$ because h is non-decreasing and $h(0) = 0$ thanks to (2.5)). The estimate (3.1) proves that (P) is dissipative, where the entire dissipation is given by the complementary past history and frictional damping controls. □

Lemma 3.2 ([9]). *Let* $g_0 = \int_0^{+\infty} g(s)ds$. *The following inequalities hold:*

$$\Big(\int_0^{+\infty} g(s)(\psi_x(t) - \psi_x(t-s))ds\Big)^2 \leq g_0 \int_0^{+\infty} g(s)(\psi_x(t) - \psi_x(t-s))^2 ds, \tag{3.2}$$

$$\Big(\int_0^{+\infty} g'(s)(\psi_x(t) - \psi_x(t-s))ds\Big)^2 \leq -g(0) \int_0^{+\infty} g'(s)(\psi_x(t) - \psi_x(t-s))^2 ds. \tag{3.3}$$

As in [17], we consider the following functional.

Lemma 3.3. *The functional*

$$I_1(t) := -\rho_2 \int_0^L \alpha(x)\psi_t \int_0^{+\infty} g(s)(\psi(t) - \psi(t-s))\,ds\,dx \tag{3.4}$$

satisfies for any $\delta > 0$

$$I_1'(t) \leq -\rho_2\Big(\int_0^{+\infty} g(s)ds - \delta\Big) \int_0^L \alpha(x)\psi_t^2 dx + c_\delta \int_0^L b(x)h^2(\psi_t)\,dx \tag{3.5}$$

$$+\delta \int_0^L (\varphi_x + \psi)^2\,dx + \delta \int_0^L \psi_x^2 dx + c_\delta g \circ \psi_x - c_\delta g' \circ \psi_x.$$

Proof. The proof is identical to the one of [10, Lemma 3.4]. □

As in [17] and [18], we consider the following functional.

Lemma 3.4. *The functional*

$$I_2(t) := -\int_0^L (\rho_2\psi\psi_t + \rho_1\varphi\varphi_t)dx$$

satisfies

$$I_2'(t) \leq -\int_0^L (\rho_2\psi_t^2 + \rho_1\varphi_t^2)dx \tag{3.6}$$

$$+k_1 \int_0^L (\varphi_x + \psi)^2 dx + c\int_0^L \psi_x^2 dx + c\int_0^L b(x)h^2(\psi_t)dx + cg \circ \psi_x.$$

Proof. The proof is identical to the one of [10, Lemma 3.5]. □

As in [3], we consider the following functional.

Lemma 3.5. *The functional*

$$I_3(t) := \rho_2 \int_0^L \psi_t(\varphi_x + \psi)dx + \frac{k_2\rho_1}{k_1} \int_0^L \psi_x \varphi_t dx$$
$$-\frac{\rho_1}{k_1} \int_0^L a(x)\varphi_t \int_0^{+\infty} g(s)\psi_x(t-s)dsdx$$

satisfies for any $\epsilon > 0$

$$I_3'(t) \leq \frac{1}{2\epsilon}\Big(k_2\psi_x(L,t) - a(L)\int_0^{+\infty} g(s)\psi_x(L,t-s)ds\Big)^2 \tag{3.7}$$
$$+\frac{1}{2\epsilon}\Big(k_2\psi_x(0,t) - a(0)\int_0^{+\infty} g(s)\psi_x(0,t-s)ds\Big)^2$$
$$+\frac{\epsilon}{2}(\varphi_x^2(L,t) + \varphi_x^2(0,t)) - (k_1 - \epsilon)\int_0^L (\varphi_x + \psi)^2 dx + \rho_2 \int_0^L \psi_t^2 dx$$
$$+\epsilon \int_0^L \varphi_t^2 dx + c_\epsilon \int_0^L b(x)h^2(\psi_t)dx - c_\epsilon g' \circ \psi_x + (\frac{k_2\rho_1}{k_1} - \rho_2)\int_0^L \varphi_t \psi_{xt} dx.$$

Proof. Using equations (P) and argument as before, we have

$$I_3'(t) = \rho_2 \int_0^L (\varphi_{xt} + \psi_t)\psi_t dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_{xt}\varphi_t dx$$
$$+\int_0^L (\varphi_x + \psi)\Big(k_2\psi_{xx} - \int_0^{+\infty} g(s)(a(x)\psi_x(t-s))_x ds - k_1(\varphi_x + \psi) - b(x)h(\psi_t)\Big)dx$$
$$+k_2 \int_0^L \psi_x(\varphi_x + \psi)_x dx - \int_0^L a(x)(\varphi_x + \psi)_x \Big(\int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)dx$$
$$-\frac{\rho_1}{k_1}\int_0^L a(x)\varphi_t\Big(g(0)\psi_x + \int_0^{+\infty} g'(s)\psi_x(t-s)ds\Big)dx$$
$$= -k_1 \int_0^L (\varphi_x + \psi)^2 dx + \rho_2 \int_0^L \psi_t^2 dx + (\frac{k_2\rho_1}{k_1} - \rho_2)\int_0^L \varphi_t \psi_{xt} dx$$
$$+\Big[\Big(k_2\psi_x - a(x)\int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)(\varphi_x + \psi)\Big]_0^L$$
$$+\frac{\rho_1}{k_1}\int_0^L a(x)\varphi_t \int_0^{+\infty} g'(s)(\psi_x(t) - \psi_x(t-s))dsdx - \int_0^L b(x)(\varphi_x + \psi)h(\psi_t)dx.$$

By using (3.3) and Young's inequality (for the last three terms of this equality), (3.7) is established. □

To estimate the boundary terms in (3.7), we proceed as in [3].

Lemma 3.6. *Let* $m(x) = 2 - \frac{4}{L}x$*. Then, for any* $\epsilon > 0$*, the functionals*

$$I_4 = \rho_2 \int_0^L m(x)\psi_t\Big(k_2\psi_x - a(x)\int_0^{+\infty} g(s)\psi_x(t-s)ds\Big)dx$$

and

$$I_5 = \rho_1 \int_0^L m(x)\varphi_t \varphi_x dx$$

satisfy

$$I_4'(t) \leq -\Big(k_2\psi_x(L,t) - a(L)\int_0^{+\infty} g(s)\psi_x(L,t-s)ds\Big)^2 \tag{3.8}$$

$$-\Big(k_2\psi_x(0,t) - a(0)\int_0^{+\infty} g(s)\psi_x(0,t-s)ds\Big)^2 + \epsilon k_1 \int_0^L (\varphi_x+\psi)^2 dx$$

$$+c(1+\frac{1}{\varepsilon})\int_0^L \psi_x^2 dx + c_\epsilon\Big(\int_0^L b(x)h^2(\psi_t)dx + g\circ\psi_x\Big) + c\int_0^L \psi_t^2 dx - cg'\circ\psi_x$$

and

$$I_5'(t) \leq -k_1(\varphi_x^2(L,t) + \varphi_x^2(0,t)) + c\int_0^L (\varphi_t^2 + \varphi_x^2 + \psi_x^2)dx. \tag{3.9}$$

Proof. The proof is identical to the one of [10, Lemma 3.7]. □

Lemma 3.7. *For any* $\epsilon \in]0,1[$, *the functional*

$$I_6(t) := I_3(t) + \frac{1}{2\epsilon}I_4(t) + \frac{\epsilon}{2k_1}I_5(t)$$

satisfies

$$I_6'(t) \leq -(\frac{k_1}{2} - c\epsilon)\int_0^L (\varphi_x+\psi)^2 dx + c\epsilon\int_0^L \varphi_t^2 dx + \frac{c}{\epsilon}\int_0^L \psi_t^2 dx \tag{3.10}$$

$$+\frac{c}{\epsilon^2}\int_0^L \psi_x^2 dx + c_\epsilon\Big(\int_0^L b(x)h^2(\psi_t)dx + g\circ\psi_x - g'\circ\psi_x\Big) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx.$$

Proof. By using Poincaré's inequality, we have

$$\int_0^L \varphi_x^2 dx \leq 2\int_0^L (\varphi_x+\psi)^2 dx + 2\int_0^L \psi^2 dx \leq 2\int_0^L (\varphi_x+\psi)^2 dx + c\int_0^L \psi_x^2 dx.$$

Then (3.7)-(3.9) imply (3.10). □

Lemma 3.8. *For any* $\epsilon \in]0,1[$, *the functional* $I_7(t) := I_6(t) + \frac{1}{8}I_2(t)$ *satisfies*

$$I_7'(t) \leq -\frac{k_1}{4}\int_0^L (\varphi_x+\psi)^2 dx - \frac{\rho_1}{16}\int_0^L \varphi_t^2 dx + c\int_0^L (\psi_t^2 + \psi_x^2)dx \tag{3.11}$$

$$+c\Big(\int_0^L b(x)h^2(\psi_t)dx + g\circ\psi_x - g'\circ\psi_x\Big) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt}dx.$$

Proof. Inequalities (3.10) (with ϵ small enough) and (3.6) imply (3.11). □

Now, we use a function w introduced in [3] to get a crucial estimate.

Lemma 3.9. *The function*

$$w(x,t) = -\int_0^x \psi(y,t)dy + \frac{1}{L}\Big(\int_0^L \psi(y,t)dy\Big)x \tag{3.12}$$

satisfies the estimates

$$\int_0^L w_x^2 dx \le c\int_0^L \psi^2 dx, \quad \forall t \ge 0, \tag{3.13}$$

$$\int_0^L w_t^2 dx \le c\int_0^L \psi_t^2 dx, \quad \forall t \ge 0. \tag{3.14}$$

Proof. . We just have to calculate w_x and use Hölder's inequality to get (3.13). Applying (3.13) to w_t, we get

$$\int_0^L w_{xt}^2 dx \le c\int_0^L \psi_t^2 dx, \quad \forall t \ge 0.$$

Then, using Poincaré's inequality ($w_t(0,t) = w_t(L,t) = 0$), we arrive at (3.14). □

Lemma 3.10. *For any $\epsilon \in]0,1[$, the functional*

$$I_8(t) := \int_0^L (\rho_2\psi\psi_t + \rho_1 w\varphi_t)dx$$

satisfies (l is defined by (2.6))

$$I_8'(t) \le -\frac{l}{2}\int_0^L \psi_x^2 dx + \frac{c}{\epsilon}\int_0^L \psi_t^2 dx + \epsilon\int_0^L \varphi_t^2 dx \tag{3.15}$$

$$+c\int_0^L b(x)h^2(\psi_t)dx + cg\circ\psi_x.$$

Proof. . The proof is identical to the one of [10, Lemma 3.10]. □

Now, let $N_1, N_2, N_3 > 0$. We put

$$I_9(t) := N_1E(t) + N_2I_1(t) + N_3I_8(t) + I_7(t). \tag{3.16}$$

By combining (3.1), (3.5), (3.11) and (3.15), and taking $\delta = \frac{k_1}{8N_2}$ in (3.5), we obtain ($g_0 = \int_0^{+\infty} g(s)ds$ is positive and bounded thanks to (H3))

$$I_9'(t) \le -(\frac{lN_3}{2} - c)\int_0^L \psi_x^2 dx - (\frac{\rho_1}{16} - \epsilon N_3)\int_0^L \varphi_t^2 dx \tag{3.17}$$

$$-\int_0^L \Big(N_2\rho_2 g_0(\alpha(x) + b(x)) - \frac{cN_3}{\epsilon} - c\Big)\psi_t^2 dx - \frac{k_1}{8}\int_0^L (\varphi_x + \psi)^2 dx$$

$$-N_1\int_0^L b(x)\psi_t h(\psi_t)dx + c_{N_2,N_3}\Big(\int_0^L b(x)(\psi_t^2 + h^2(\psi_t))dx + g\circ\psi_x\Big)$$

$$+(\frac{N_1}{2} - c_{N_2} - c)g' \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx.$$

At this point, we choose N_3 large enough so that $\frac{lN_3}{2} - c > 0$, then $\epsilon \in]0, 1[$ small enough so that $\frac{\rho_1}{16} - \epsilon N_3 > 0$. Next, we pick N_2 large enough so that $N_2 \rho_2 g_0 \inf_{x \in [0,L]} \{\alpha(x) + b(x)\} - \frac{cN_3}{\epsilon} - c > 0$ (this is possible thanks to (2.30)). Consequently, we get from (3.17), using the definition (2.10) of E,

$$I_9'(t) \leq -cE(t) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx \tag{3.18}$$

$$+(\frac{N_1}{2} - c)g' \circ \psi_x + c \int_0^L b(x)(\psi_t^2 + h^2(\psi_t))dx - N_1 \int_0^L b(x)\psi_t h(\psi_t)dx.$$

To estimate the last two integrals of (3.18), we use some ideas from [2], [12], [13] and [15]. Let

$$\Omega_+ = \{x \in]0, L[: \ |\psi_t| \geq \epsilon_1\} \quad \text{and} \quad \Omega_- = \{x \in]0, L[: \ |\psi_t| < \epsilon_1\}, \tag{3.19}$$

where ϵ_1 is defined in (H2). Using (2.4), we get (note that $sh(s) \geq 0$)

$$c \int_{\Omega_+} b(x)(\psi_t^2 + h^2(\psi_t))dx - N_1 \int_0^L b(x)\psi_t h(\psi_t)dx \leq (c - N_1) \int_{\Omega_+} b(x)\psi_t h(\psi_t)dx.$$

Now, by definition of the functionals $I_1 - I_8$ and E, there exists a positive constant β satisfying $|N_2 I_1 + N_3 I_8 + I_7| \leq \beta E$ which implies that

$$(N_1 - \beta)E \leq I_9 \leq (N_1 + \beta)E,$$

then we choose N_1 large enough so that $c - N_1 \leq 0$, $\frac{N_1}{2} - c \geq 0$ and $N_1 > \beta$ (that is $I_9 \sim E$), we get from (3.18)

$$I_9'(t) \leq -cE(t) + cg \circ \psi_x \tag{3.20}$$

$$+(\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx + c \int_{\Omega_-} b(x)(\psi_t^2 + h^2(\psi_t))dx.$$

Case 1. H **is linear on** $[0, \epsilon_1]$: then (2.4) is satisfied on $\mathbb{R}$ and therefore

$$c \int_0^L b(x)(\psi_t^2 + h^2(\psi_t))dx - N_1 \int_0^L b(x)\psi_t h(\psi_t)dx \leq (c - N_1) \int_0^L b(x)\psi_t h(\psi_t)dx.$$

So, with the same choice of N_1, we get from (3.18), for $H_0 = Id$ in this case,

$$I_9'(t) \leq -cH_0(E(t)) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t \psi_{xt} dx. \tag{3.21}$$

Case 2. $H'(0) = 0$ **and** $H'' > 0$ **on** $]0, \epsilon_1]$: without loss of generality, we can assume that H' defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let H^* be denote the dual function of the convex function H given by

$$H^*(t) = \sup_{s \in \mathbb{R}_+} \{ts - H(s)\}, \quad \forall t \in \mathbb{R}_+.$$

For $t \in \mathbb{R}_+$, the function $s \mapsto ts - H(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(H')^{-1}(t)$. Therefore

$$H^*(t) = t(H')^{-1}(t) - H((H')^{-1}(t)), \quad \forall t \in \mathbb{R}_+.$$

Because H is convex and $H(0)=0$, then, for any $s_0\in\mathbb{R}_+$,

$$H\Big(\frac{b(x)}{\max\{1,\|b\|_\infty\}}s_0\Big)\le\frac{b(x)}{\max\{1,\|b\|_\infty\}}H(s_0)+\Big(1-\frac{b(x)}{\max\{1,\|b\|_\infty\}}\Big)H(0)\le b(x)H(s_0)$$

which implies that, for $s_0=H^{-1}(\psi_t h(\psi_t))$,

$$b(x)H^{-1}(\psi_t h(\psi_t))dx\le\max\{1,\|b\|_\infty\}H^{-1}(b(x)\psi_t h(\psi_t)).$$

Thus, using (2.5),

$$\int_{\Omega_-}b(x)(\psi_t^2+h^2(\psi_t))dx\le\int_{\Omega_-}b(x)H^{-1}(\psi_t h(\psi_t))dx\le c\int_{\Omega_-}H^{-1}(b(x)\psi_t h(\psi_t))dx.$$

Therefore, using Jensen's inequality and (3.1), we find

$$\int_{\Omega_-}b(x)(\psi_t^2+h^2(\psi_t))dx\le cH^{-1}\Big(\int_{\Omega_-}cb(x)\psi_t h(\psi_t)dx\Big)\le cH^{-1}(-cE'(t)).$$

Then, using (3.20), we get

$$I_9'(t)\le -cE(t)+cH^{-1}(-cE'(t))+cg\circ\psi_x+(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^L\varphi_t\psi_{xt}dx.$$

Let τ_0, $\tau'>0$. Using the fact that $E'\le 0$, $H''\ge 0$ and $I_9\ge 0$, we obtain

$$\Big(H'(\tau_0E(t))I_9(t)+\tau'E(t)\Big)'=\tau_0E'(t)H''(\tau_0E(t))I_9(t)+H'(\tau_0E(t))I_9'(t)+\tau'E'(t)$$

$$\le H'(\tau_0E(t))\Big(-cE(t)+cH^{-1}(-cE'(t))+cg\circ\psi_x+(\frac{\rho_1k_2}{k_1}-\rho_2)\int_0^L\varphi_t\psi_{xt}dx\Big)+\tau'E'(t).$$

Using Young's inequality:

$$H^{-1}(-cE'(t))H'(\tau_0E(t))\le -cE'(t)+H^*(H'(\tau_0E(t))),$$

and the fact that $H^*(t)\le t(H')^{-1}(t)$ and $H'(\tau_0E)$ is non-increasing, we get

$$\Big(H'(\tau_0E(t))I_9(t)+\tau'E(t)\Big)'\le(\frac{\rho_1k_2}{k_1}-\rho_2)H'(\tau_0E(t))\int_0^L\varphi_t\psi_{xt}dx$$

$$+cH'(\tau_0E(0))g\circ\psi_x-cH'(\tau_0E(t))E(t)+cH^*(H'(\tau_0E(t)))+(\tau'-c)E'(t)$$

$$\le(\frac{\rho_1k_2}{k_1}-\rho_2)H'(\tau_0E(t))\int_0^L\varphi_t\psi_{xt}dx+cH'(\tau_0E(0))g\circ\psi_x$$

$$-cH_0(E(t))+c\tau_0H_0(E(t))+(\tau'-c)E'(t),$$

where $H_0(t)=tH'(\tau_0t)$ in this case. By choosing τ_0 small enough and τ' large enough, we arrive at

$$\Big(\frac{H_0(E(t))}{E(t)}I_9(t)+\tau'E(t)\Big)'\le -cH_0(E(t))+cg\circ\psi_x \tag{3.22}$$

$$+(\frac{\rho_1k_2}{k_1}-\rho_2)\frac{H_0(E(t))}{E(t)}\int_0^L\varphi_t\psi_{xt}dx.\ \square$$

Let $I_{10} = \frac{H_0(E)}{E} I_9 + \tau' E$, where H_0 is defined by (2.14) ($I_{10} = I_9$ if H is linear on $[0, \epsilon_1]$). The functional I_{10} satisfies $I_{10} \sim E$ (because $I_9 \sim E$ and $\frac{H_0(E)}{E}$ is non-increasing) and, using (3.21) and (3.22),

$$I'_{10}(t) \leq -cH_0(E(t)) + cg \circ \psi_x + (\frac{\rho_1 k_2}{k_1} - \rho_2)\frac{H_0(E(t))}{E(t)} \int_0^L \varphi_t \psi_{xt} dx. \tag{3.23}$$

Now, we estimate the term $g \circ \psi_x$ in (3.23) in function of E' by exploiting condition (2.7)-(2.8). This is the main difficulty in the treating of the past history term. When g satisfies the classical condition (2.7) (g converges exponentially to zero at infinity), the conclusion is immediate thanks to (3.1) (Case 1 below).

Case 1. (2.7) holds: then, using (3.1),

$$g \circ \psi_x \leq -cg' \circ \psi_x \leq -cE'(t)$$

which implies, using (3.23), for $G_0 = H_0$ in this case,

$$I'_{10}(t) \leq -cG_0(E(t)) - cE'(t) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\frac{G_0(E(t))}{E(t)} \int_0^L \varphi_t \psi_{xt} dx. \tag{3.24}$$

Case 2. (2.8) holds: we apply here the approach introduced in [8]. First, we note that, if $g'(s_0) = 0$ for some $s_0 \geq 0$, then $g(s_0) = 0$ because $G^{-1}(0) = 0$ and $s \mapsto \frac{g(s)}{G^{-1}(-g'(s))}$ is bounded (thanks to (H4)), and therefore, $g(s) = 0$ for all $s \geq s_0$ because g is non-negative and non-increasing. This implies that

$$g \circ \psi_x = \int_0^L a(x) \int_0^{s_0} g(s)(\psi_x(t) - \psi_x(t-s))^2 ds dx.$$

Thus, without loss of generality, we can assume that $g' < 0$ on $\mathbb{R}_+$.

Now, because E is non-increasing (see the definition of η_0 in section 6),

$$\int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx \leq c \sup_{\tau \in \mathbb{R}} \int_0^L \psi_x^2(\tau) dx$$

$$\leq c \sup_{\tau > 0} \int_0^L \psi_{0x}^2(\tau) dx + cE(0) \leq c \sup_{\tau > 0} \int_0^L \eta_{0x}^2(\tau) dx + cE(0).$$

Thus, thanks to (2.11), there exists a positive constant $m_1 = cM_0 + cE(0)$ (where M_0 is defined in (2.11)) such that

$$\int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx \leq m_1, \quad \forall t,\, s \in \mathbb{R}_+.$$

Let $\epsilon_0, \tau_1, \tau_2 > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$ for $s \in \mathbb{R}_+$ ($K(0) = 0$ because, thanks to (H4), $\lim_{s \to 0^+} \frac{s}{G^{-1}(s)} = \lim_{t \to 0^+} \frac{G(t)}{t} = G'(0) = 0$). The function K is non-decreasing. Indeed, because G^{-1} is concave and $G^{-1}(0) = 0$ (thanks to (H4)), then, for any $0 \leq s_1 < s_2$,

$$K(s_1) = \frac{s_1}{G^{-1}(\frac{s_1}{s_2} s_2 + (1 - \frac{s_1}{s_2})0)} \leq \frac{s_1}{\frac{s_1}{s_2} G^{-1}(s_2) + (1 - \frac{s_1}{s_2}) G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2).$$

Therefore,

$$K\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big) \leq K(-m_1 \tau_2 g'(s)).$$

Using this inequality, we get

$$g \circ \psi_x = \frac{1}{\tau_1 G'(\epsilon_0 H_0(E(t)))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big)$$
$$\times \frac{\tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{-\tau_2 g'(s)} K\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big) ds$$
$$\leq \frac{1}{\tau_1 G'(\epsilon_0 H_0(E(t)))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big)$$
$$\times \frac{\tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{-\tau_2 g'(s)} K(-m_1 \tau_2 g'(s)) ds$$
$$\leq \frac{1}{\tau_1 G'(\epsilon_0 H_0(E(t)))} \int_0^{+\infty} G^{-1}\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big)$$
$$\times \frac{m_1 \tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-m_1 \tau_2 g'(s))} ds.$$

Let G^* be denote the dual function of G defined by

$$G^*(t) = \sup_{s \in \mathbb{R}_+} \{ts - G(s)\}, \quad \forall t \in \mathbb{R}_+.$$

Thanks to (H4), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then, for any $t \in \mathbb{R}_+$, the function $s \mapsto ts - G(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(G')^{-1}(t)$. Therefore

$$G^*(t) = t(G')^{-1}(t) - G((G')^{-1}(t)), \quad \forall t \in \mathbb{R}_+.$$

Using Young's inequality: $t_1 t_2 \leq G(t_1) + G^*(t_2)$ for

$$t_1 = G^{-1}\Big(-\tau_2 g'(s) \int_0^L a(x)(\psi_x(t) - \psi_x(t-s))^2 dx\Big), \quad t_2 = \frac{m_1 \tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-m_1 \tau_2 g'(s))},$$

we get

$$g \circ \psi_x \leq \frac{-\tau_2}{\tau_1 G'(\epsilon_0 H_0(E(t)))} g' \circ \psi_x$$
$$+ \frac{1}{\tau_1 G'(\epsilon_0 H_0(E(t)))} \int_0^{+\infty} G^*\Big(\frac{m_1 \tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-m_1 \tau_2 g'(s))}\Big) ds.$$

Using (3.1) and the fact that $G^*(t) \leq t(G')^{-1}(t)$, we get

$$g \circ \psi_x \leq \frac{-2\tau_2}{\tau_1 G'(\epsilon_0 H_0(E(t)))} E'(t)$$

$$+ m_1 \int_0^{+\infty} \frac{g(s)}{G^{-1}(-m_1 \tau_2 g'(s))} (G')^{-1}\Big(\frac{m_1 \tau_1 G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-m_1 \tau_2 g'(s))}\Big) ds.$$

Condition (2.8) implies that $\sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} = m_2 < +\infty$. Then, using the fact that $(G')^{-1}$ is non-decreasing (thanks to (H4)), we get, for $\tau_2 = \frac{1}{m_1}$,

$$g \circ \psi_x \leq \frac{-2}{m_1 \tau_1 G'(\epsilon_0 H_0(E(t)))} E'(t)$$

$$+m_1(G')^{-1}\Big(m_1m_2\tau_1G'(\epsilon_0H_0(E(t)))\Big)\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds.$$

Because $\int_0^{+\infty}\frac{g(s)}{G^{-1}(-g'(s))}ds<+\infty$ (thanks to (2.8)), we find, for $\tau_1=\frac{1}{m_1m_2}$,

$$g\circ\psi_x\leq\frac{-c}{G'(\epsilon_0H_0(E(t)))}E'(t)+c\epsilon_0H_0(E(t)).$$

Then, for all $\epsilon_0>0$,

$$\frac{G_0(E(t))}{H_0(E(t))}g\circ\psi_x\leq-cE'(t)+c\epsilon_0G_0(E(t)),\tag{3.25}$$

where $G_0(t)=H_0(t)G'(\epsilon_0H_0(t))$ in this case which implies, using (3.23),

$$\frac{G_0(E(t))}{H_0(E(t))}I'_{10}(t)\leq-(c-c\epsilon_0)G_0(E(t))-cE'(t)$$

$$+(\frac{\rho_1k_2}{k_1}-\rho_2)\frac{G_0(E(t))}{E(t)}\int_0^L\varphi_t\psi_{xt}dx.$$

Choosing ϵ_0 small enough, we get

$$\frac{G_0(E(t))}{H_0(E(t))}I'_{10}(t)\leq-cG_0(E(t))-cE'(t)\tag{3.26}$$

$$+(\frac{\rho_1k_2}{k_1}-\rho_2)\frac{G_0(E(t))}{E(t)}\int_0^L\varphi_t\psi_{xt}dx.\ \square$$

Let

$$F=\tau\Big(\frac{G_0(E)}{H_0(E)}I_{10}+cE\Big),\tag{3.27}$$

where $\tau>0$ and G_0 is defined by (2.13). We have $F\sim E$ (because $I_{10}\sim E$ and $\frac{G_0(E)}{H_0(E)}$ is non-increasing) and, using (3.24) and (3.26),

$$F'(t)\leq-c\tau G_0(E(t))+\tau(\frac{\rho_1k_2}{k_1}-\rho_2)\frac{G_0(E(t))}{E(t)}\int_0^L\varphi_t\psi_{xt}dx.\tag{3.28}$$

Thanks to (1.1), the last term of (3.28) vanishes. Then, for $\tau>0$ such that

$$F\leq E\quad\text{and}\quad F(0)\leq1,\tag{3.29}$$

we get for $c'=c\tau>0$

$$F'\leq-c'G_0(F).\tag{3.30}$$

Then (3.30) implies that $(G_1(F))'\geq c'$, where $G_1(t)=\int_t^1\frac{1}{G_0(s)}ds$. Then, by integrating over $[0,t]$, we get $G_1(F(t))\geq c't+G_1(F(0))$. Because $F(0)\leq1$ and G_1 is decreasing, we obtain $G_1(F(t))\geq c't$ which implies that $F(t)\leq G_1^{-1}(c't)$. The fact that $F\sim E$ gives (2.12). This completes the proof of Theorem 2.1.$\square$

4 Proof of Theorem 2.2

In this section, we treat the case when (1.1) does not hold which is more realistic physically. We will estimate the last term of (3.28) using the system $(P2)$ resulting from differentiating (P) with respect to time

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_{xt} + \psi_t)_x = 0, \\ \rho_2\psi_{ttt} - k_2\psi_{xxt} + k_1(\varphi_{xt} + \psi_t) + \int_0^{+\infty} g(s)(a(x)\psi_{xt}(t-s))_x ds + b(x)h'(\psi_t)\psi_{tt} = 0, \\ \varphi_t(0,t) = \psi_t(0,t) = \varphi_t(L,t) = \psi_t(L,t) = 0. \end{cases} \tag{P2}$$

System $(P2)$ is well posed for initial data $(\varphi_0, \psi_0(0), \varphi_1, \psi_1, \eta_0)^T \in D(A)$ (see section 6). Let E_2 be the second-order energy (the energy of $(P2)$) defined by $E_2(t) = E_1(\varphi_t, \psi_t)(t)$, where $E_1(\varphi, \psi)(t) = E(t)$ and E is defined by (2.10). A simple calculation (as for (3.1)) implies that

$$E_2'(t) = \frac{1}{2}g' \circ \psi_{xt} - \int_0^L b(x)h'(\psi_t)\psi_{tt}^2 dx. \tag{4.1}$$

Because $\inf_{t\in\mathbb{R}} h'(t) > 0$ thanks to hypothesis (H6), we have

$$E_2'(t) \leq \frac{1}{2}g' \circ \psi_{xt} - c\int_0^L b(x)\psi_{tt}^2 dx \leq 0. \tag{4.2}$$

The energy of high order is widely used in the literature (see [1], [7], [18] and [19] for example) to estimate some terms. Our main contribution in this section is estimating $g \circ \psi_{xt}$ in (4.9) under the weaker assumption (2.8) by applying here the approach introduced in [8].

Let $\tau = 1$ in (3.27). Thanks to (H6), H is linear and then (3.28) holds for $H_0 = Id$. Thus, (3.28) implies that

$$F'(t) \leq -cG_0(E(t)) + (\frac{\rho_1 k_2}{k_1} - \rho_2)\frac{G_0(E(t))}{E(t)}\int_0^L \varphi_t\psi_{xt} dx, \tag{4.3}$$

where G_0 is defined by (2.19). Now, we proceed as in [18].

Lemma 4.1. *For any $\epsilon > 0$, we have*

$$(\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \varphi_t\psi_{xt} dxdt \leq \epsilon\int_0^T G_0(E(t))dt \tag{4.4}$$

$$+c_\epsilon\int_0^T \frac{G_0(E(t))}{E(t)}(g\circ\psi_{xt} - g'\circ\psi_x)dt + c_\epsilon\frac{G_0(E(0))}{E(0)}(E(0) + E_2(0)),\ \forall T \geq 0.$$

Proof. We distinguish two cases (corresponding to hypothesis (H5)).

Case 1. $\inf_{x\in[0,L]}\{a(x)\} = a_0 > 0$: let $g_0 = \int_0^{+\infty} g(s)ds$. We have

$$(\frac{\rho_1 k_2}{k_1} - \rho_2)\int_0^L \varphi_t\psi_{xt} dx = \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0}\int_0^L a_0\varphi_t\int_0^{+\infty} g(s)\psi_{xt}(t-s)dsdx \tag{4.5}$$

$$+\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0}\int_0^L a_0\varphi_t\int_0^{+\infty} g(s)(\psi_{xt}(t) - \psi_{xt}(t-s))dsdx.$$

Using Young's inequality and (3.2) (for ψ_{xt} instead of ψ_x), we get for all $\epsilon > 0$

$$\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \varphi_t \int_0^{+\infty} g(s)(\psi_{xt}(t) - \psi_{xt}(t-s)) ds dx$$

$$\leq c \int_0^L a(x)|\varphi_t| \int_0^{+\infty} g(s)|\psi_{xt}(t) - \psi_{xt}(t-s)| ds dx \leq \frac{\epsilon}{2} E(t) + c_\epsilon g \circ \psi_{xt}.$$

On the other hand, by integrating by parts and using (3.3), we obtain

$$\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \varphi_t \int_0^{+\infty} g(s) \psi_{xt}(t-s) ds dx$$

$$= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \varphi_t \Big(g(0)\psi_x + \int_0^{+\infty} g'(s) \psi_x(t-s) ds \Big) dx$$

$$= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \varphi_t \int_0^{+\infty} (-g'(s))(\psi_x(t) - \psi_x(t-s)) ds dx$$

$$\leq \frac{\epsilon}{2} E(t) - c_\epsilon g' \circ \psi_x.$$

Inserting these last two inequalities into (4.5), multiplying by $\frac{G_0(E)}{E}$ and integrating over $[0,T]$, we obtain (4.4).

Case 2. $|a'| + |a - b_0| \leq b_1 b$: we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_t \psi_{xt} dx dt \tag{4.6}$$

$$= \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \frac{G_0(E(t))}{E(t)} \int_0^L \Big(\frac{a(x)}{b_0} + \frac{b_0 - a(x)}{b_0} \Big) \varphi_t \psi_{xt} dx dt.$$

As in Case 1 (thanks to the presence of a),

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \frac{G_0(E(t))}{E(t)} \int_0^L \frac{a(x)}{b_0} \varphi_t \psi_{xt} dx dt \tag{4.7}$$

$$\leq \frac{\epsilon}{2} \int_0^T G_0(E(t)) dt + c_\epsilon \int_0^T \frac{G_0(E(t))}{E(t)} (g \circ \psi_{xt} - g' \circ \psi_x) dt.$$

On the other hand, integration with respect to t and then with respect to x and using Poincaré's inequality, the definition of $\frac{G_0(E)}{E}$, E and E_2 and their non-increasingness, we get

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \frac{G_0(E(t))}{E(t)} \int_0^L \frac{b_0 - a(x)}{b_0} \varphi_t \psi_{xt} dx dt$$

$$= \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \Big[\frac{G_0(E(t))}{E(t)} \int_0^L \frac{b_0 - a(x)}{b_0} \varphi \psi_{xt} dx \Big]_0^T$$

$$-\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \Big(\frac{G_0(E(t))}{E(t)} \Big)' \int_0^L \frac{b_0 - a(x)}{b_0} \varphi \psi_{xt} dx dt$$

$$-\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^T \frac{G_0(E(t))}{E(t)} \int_0^L \frac{b_0 - a(x)}{b_0} \varphi \psi_{xtt} dx dt.$$

Using the fact that

$$\Big|\int_0^L \frac{b_0-a(x)}{b_0}\varphi\psi_{xt}dx\Big| \le c(E(t)+E_2(t)) \le c(E(0)+E_2(0))$$

and integrating by parts the last integral, we get

$$\begin{aligned}&(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \frac{b_0-a(x)}{b_0}\varphi_t\psi_{xt}dxdt\\ \le\; & c\frac{G_0(E(0))}{E(0)}(E(0)+E_2(0)) - c(E(0)+E_2(0))\int_0^T \Big(\frac{G_0(E(t))}{E(t)}\Big)'dt\\ &+(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \frac{b_0-a(x)}{b_0}\varphi_x\psi_{tt}dxdt\\ &-(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \frac{a'(x)}{b_0}\varphi\psi_{tt}dxdt.\end{aligned}$$

Using Poincaré's inequality again, (4.2) and the fact that $|a-b_0|\le b_1 b$ and $|a'|\le b_1 b$, we deduce that

$$\begin{aligned}&(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \frac{b_0-a(x)}{b_0}\varphi_t\psi_{xt}dxdt\\ \le\; & c\frac{G_0(E(0))}{E(0)}(E(0)+E_2(0)) + c\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L b(x)(|\varphi_x|+|\varphi|)|\psi_{tt}|dxdt.\end{aligned}$$

Therefore, using Young's inequality and (4.2) to estimate the last integral,

$$\begin{aligned}&(\frac{\rho_1 k_2}{k_1}-\rho_2)\int_0^T \frac{G_0(E(t))}{E(t)}\int_0^L \frac{b_0-a(x)}{b_0}\varphi_t\psi_{xt}dxdt\\ \le\; & c\frac{G_0(E(0))}{E(0)}(E(0)+E_2(0)) + \frac{\epsilon}{2}\int_0^T G_0(E(t))dt - c_\epsilon\frac{G_0(E(0))}{E(0)}\int_0^T E_2'(t)dt\\ \le\; & c_\epsilon\frac{G_0(E(0))}{E(0)}(E(0)+E_2(0)) + \frac{\epsilon}{2}\int_0^T G_0(E(t))dt. \qquad (4.8)\end{aligned}$$

Inserting (4.7) and (4.8) into (4.6), we obtain (4.4). □

Now, using (4.3) and (4.4) and choosing ϵ small enough, we get

$$\begin{aligned}\int_0^T F'(t)dt \le & -c\int_0^T G_0(E(t))dt + c\frac{G_0(E(0))}{E(0)}(E(0)+E_2(0))\\ &+c\int_0^T \frac{G_0(E(t))}{E(t)}(g\circ\psi_{xt} - g'\circ\psi_x)dt,\end{aligned}$$

then, using (3.1) and the fact that $F\sim E$ and $\frac{G_0(E)}{E}$ is non-increasing,

$$\begin{aligned}\int_0^T G_0(E(t))dt \le\; & c\Big(1+\frac{G_0(E(0))}{E(0)}\Big)(E(0)+E_2(0)) \qquad (4.9)\\ &+c\int_0^T \frac{G_0(E(t))}{E(t)}g\circ\psi_{xt}dt.\end{aligned}$$

To estimate the last term in (4.9), we distinguish two cases.

Case 1. (2.7) holds: we have $G_0 = Id$. Using (4.2), we get

$$\frac{G_0(E(t))}{E(t)} g \circ \psi_{xt} = g \circ \psi_{xt} \le -cg' \circ \psi_{xt} \le -cE_2'(t).$$

Case 2. (2.8) holds: using (2.17) and similarly to (3.25) for $g \circ \psi_{xt}$ instead of $g \circ \psi_x$ (here $H_0 = Id$), we get, using also (4.2),

$$\frac{G_0(E(t))}{E(t)} g \circ \psi_{xt} \le -cE_2'(t) + c\epsilon_0 G_0(E(t)), \quad \forall \epsilon_0 > 0.$$

Then we get in both cases

$$\int_0^T \frac{G_0(E(t))}{E(t)} g \circ \psi_{xt} dt \le -c \int_0^T E_2'(t)dt + c\epsilon_0 \int_0^T G_0(E(t))dt, \quad \forall \epsilon_0 > 0.$$

Inserting this inequality into (4.9), choosing ϵ_0 small enough and using the fact that $G_0(E)$ is non-increasing, we deduce that for all $T \ge 0$

$$G_0(E(T))T \le \int_0^T G_0(E(t))dt \le c\Big(1 + \frac{G_0(E(0))}{E(0)}\Big)(E(0) + E_2(0)) \tag{4.10}$$

which gives (2.18) with $C = c\Big(1 + \frac{G_0(E(0))}{E(0)}\Big)(E(0) + E_2(0))$.□

5 Proof of Theorem 2.3

Using (4.10) and (6.2) (see section 6), the proof of (2.21) becomes just a particular application of Theorem 5.1 below which gives an extension of [1, Theorem 2.1] and [7, Theorem 2.1].

As in section 6, let $A: D(A) \to \mathcal{H}$ be a linear bounded operator on a Hilbert space $\mathcal{H}$ with domain $D(A) \subset \mathcal{H}$ such that A generates a C_0-semigroup. Let $U \in C^1(\mathbb{R}_+, \mathcal{H})$ the classical solution of the first order problem

$$U'(t) + AU(t) = 0, \quad \forall t > 0$$

with initial data $U(0) = U_0 \in D(A)$ (see [12] and [22]).

Theorem 5.1. *Let $\Phi: \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ =Id or Φ is differentiable, strictly convex and increasing with $\Phi(0) = 0$. Let $\phi: \mathcal{H} \to \mathbb{R}_+$ be a continuous function and $m \in \mathbb{N}^*$. Assume that $t \mapsto \phi(U(t))$ is non-increasing and*

$$\forall U_0 \in D(A^m),\ \exists C > 0: \int_0^T \Phi(\phi(U(t)))dt \le C \sum_{k=0}^m \phi(U^{(k)}(0)),\ \forall T \ge 0. \tag{5.1}$$

Then, for any $n \in \mathbb{N}^*$ and any $U_0 \in D(A^{mn})$, there exists $C_n > 0$ such that

$$\phi(U(t)) \le \Phi_{n-1}^{-1}(\frac{C_n}{t^n}), \quad \forall t > 0, \tag{5.2}$$

where

$$\Phi_{n-1}(t) = \begin{cases} Id & \text{if } \Phi = Id, \\ \Phi(t)\Big(\Phi^{*-1}(\Phi(t))\Big)^{n-1} & \text{otherwise} \end{cases}$$

and Φ^* *is the dual function of* Φ *(defined by* $\Phi^*(t) = \sup_{s\in\mathbb{R}_+}\{ts - \Phi(s)\}$*).*

Remark 5.1. The case Φ =Id gives [1, Theorem 2.1], and the case $m = 1$ and $\Phi(t) = t^p$ with $p \geq 1$ gives [7, Theorem 2.1].

Proof of Theorem 5.1. The proof is very similar to the one of [1, Theorem 2.1]. We only have to prove that, for any $n \in \mathbb{N}^*$ and any $U_0 \in D(A^{mn})$, there exists $c_{n-1} > 0$ such that for all $0 \leq S \leq T$

$$\tilde{\Phi}_{n-1}(\phi(U(T)))\int_S^T \Phi(\phi(U(t)))\frac{(t-S)^{n-1}}{(n-1)!}dt \leq c_{n-1}\sum_{k=0}^{mn}\phi(U^{(k)}(S)), \tag{5.3}$$

where

$$\tilde{\Phi}_{n-1}(t) = \begin{cases} 1 & \text{if } \Phi = Id, \\ (\Phi^{*-1}(\Phi(t)))^{n-1} & \text{if not.} \end{cases}$$

Indeed, using the fact that $\Phi(\phi(U))$ is non-increasing, we get

$$\tilde{\Phi}_{n-1}(\phi(U(T)))\int_0^T \Phi(\phi(U(t)))\frac{t^{n-1}}{(n-1)!}dt$$

$$\geq \tilde{\Phi}_{n-1}(\phi(U(T)))\Phi(\phi(U(T)))\int_0^T \frac{t^{n-1}}{(n-1)!}dt = \frac{T^n}{n!}\Phi_{n-1}(\phi(U(T)))$$

Then, using (5.3) with $S = 0$, we get (5.2).

We prove (5.3) by induction on n. Because $\tilde{\Phi}_0 = 1$, (5.3) is a consequence of (5.1) for $n = 1$. Now, suppose that (5.3) holds. We have, using Fubini's theorem and the fact that $\tilde{\Phi}_n(\phi(U)) = \tilde{\Phi}_1(\phi(U))\tilde{\Phi}_{n-1}(\phi(U))$ is non-increasing,

$$\tilde{\Phi}_n(\phi(U(T)))\int_S^T \Phi(\phi(U(t)))\frac{(t-S)^n}{n!}dt$$

$$= \tilde{\Phi}_1(\phi(U(T)))\tilde{\Phi}_{n-1}(\phi(U(T)))\int_S^T\int_t^T \Phi(\phi(U(\tau)))\frac{(\tau-t)^{n-1}}{(n-1)!}d\tau dt$$

$$\leq \int_S^T c_{n-1}\tilde{\Phi}_1(\phi(U(t)))\sum_{k=0}^{mn}\phi(U^{(k)}(t))dt \leq c_{n-1}\sum_{k=0}^{mn}\int_S^T \tilde{\Phi}_1(\phi(U(t)))\phi(U^{(k)}(t))dt.$$

If $\Phi = Id$, then $\tilde{\Phi}_1 = 1$ and then $\tilde{\Phi}_1(\phi(U))\phi(U^{(k)}) = \phi(U^{(k)})$. Otherwise, using Young's inequality: $st \leq \Phi^*(s) + \Phi(t)$, we get

$$\tilde{\Phi}_1(\phi(U))\phi(U^{(k)}) \leq \Phi^*(\tilde{\Phi}_1(\phi(U))) + \Phi(\phi(U^{(k)})) = \Phi(\phi(U)) + \Phi(\phi(U^{(k)})).$$

Then in both cases, we get (for $\tilde{c}_{n-1} > 0$)

$$\tilde{\Phi}_n(\phi(U(T)))\int_S^T \Phi(\phi(U(t)))\frac{(t-S)^n}{n!}dt \leq \tilde{c}_{n-1}\sum_{k=0}^{mn}\int_S^T \Phi(\phi(U^{(k)}(t)))dt. \tag{5.4}$$

By applying (5.3), for $n = 1$ and $U^{(k)}$ instead of U, to estimate the last integral in (5.4), we deduce (5.3) for $n + 1$ instead of n.

Proof of Theorem 2.3. Using (6.2) and applying Theorem 5.1 in the particular case $m = 1$, $\phi = E$, Φ =Id if (2.7) holds, and $\Phi = G_0$ if (2.8) holds (where (4.10) implies (5.1) in this case), we conclude (2.21). Note that, in case of our system (6.2) such that (2.8) holds, the inequality (4.10) is satisfied under the restriction (2.17) on η_0. Because, if (5.1) is satisfied for $m = 1$ and U_0 satisfying (2.17), (5.2) is still satisfied for $m = 1$ and U_0 satisfying (2.20) (the proof is identical to the one of the general case), then (4.10) and Theorem 5.1 imply (2.21) also in the case (2.8).□

6 Well-Posedness and Smoothness

In this section, we discuss the existence, uniqueness and smoothness of solution of (P). We use semigroup theory and we adapt some arguments of [6] (see also [17] and [18]) to our system (P). Following the idea of [6], let

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s) \quad \text{for } (x,t,s) \in]0,L[\times \mathbb{R}_+ \times \mathbb{R}_+$$

(η is the relative history of ψ, and it was introduced first in [6]). Then

$$\begin{cases} \eta_t + \eta_s - \psi_t = 0, & \text{in }]0,L[\times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(0,t,s) = \eta(L,t,s) = 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta(x,t,0) = 0, & \text{in }]0,L[\times \mathbb{R}_+. \end{cases} \tag{6.1}$$

Then the second equation of (P) can be formulated as

$$\rho_2\psi_{tt} - k_2\psi_{xx} + \Big(\int_0^{+\infty} g(s)ds\Big)(a(x)\psi_x)_x - \int_0^{+\infty} g(s)(a(x)\eta_x)_x ds$$
$$+k_1(\varphi_x + \psi) + b(x)h(\psi_t) = 0.$$

Let $\eta_0(x,s) = \eta(x,0,s) = \psi_0(x,0) - \psi_0(x,s)$ for $(x,s) \in]0,L[\times \mathbb{R}_+$. This means that the history is considered as an initial data for η. Let

$$\mathcal{H} = \Big(H_0^1(]0,L[)\Big)^2 \times \Big(L^2(]0,L[)\Big)^2 \times L_g,$$

where $H_0^1(]0,L[) = \{v \in H^1(]0,L[),\, v(0) = v(L) = 0\}$ and

$$L_g = \{v : \mathbb{R}_+ \to H_0^1(]0,L[),\, \int_0^L a(x)\int_0^{+\infty} g(s)v_x^2(s)dsdx < +\infty\}.$$

L_g is a Hilbert space endowed with the inner product

$$\langle v,w\rangle_{L_g} = \int_0^L a(x)\int_0^{+\infty} g(s)v_x(s)w_x(s)dsdx.$$

Then $\mathcal{H}$ is also a Hilbert space endowed with the inner product defined for $V = (v_1,v_2,v_3,v_4,v_5)^T$, $W = (w_1,w_2,w_3,w_4,w_5)^T \in \mathcal{H}$ by

$$\langle V,W\rangle_{\mathcal{H}} = \langle v_5,w_5\rangle_{L_g} + k_1\int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)dx$$
$$+\int_0^L \Big(\Big(k_2 - a(x)\int_0^{+\infty} g(s)ds\Big)\partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4\Big)dx.$$

Now, for $U = (\varphi,\psi,\varphi_t,\psi_t,\eta)^T$ and $U_0 = (\varphi_0,\psi_0(0),\varphi_1,\psi_1,\eta_0)^T$, (P) is equivalent to the following abstract Cauchy problem:

$$\begin{cases} U'(t) + AU(t) = 0 & \text{on } \mathbb{R}_+, \\ U(0) = U_0, \end{cases} \tag{6.2}$$

where A is the operator defined by

$$AV = (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_4,\tilde{v}_5)$$

for any $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$, where

$$\begin{cases} \tilde{v}_1 = -v_3, \\ \tilde{v}_2 = -v_4, \\ \tilde{v}_3 = -\frac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2), \\ \tilde{v}_4 = -\frac{1}{\rho_2}\Big(k_2\partial_{xx}v_2 - \Big(\int_0^{+\infty} g(s)ds\Big)\partial_x(a(x)\partial_x v_2)\Big) \\ \qquad -\frac{1}{\rho_2}\int_0^{+\infty} g(s)\partial_x(a(x)\partial_x v_5(s))ds + \frac{k_1}{\rho_2}(\partial_x v_1 + v_2) + \frac{b(x)}{\rho_2}h(v_4), \\ \tilde{v}_5 = -v_4 + \partial_s v_5. \end{cases}$$

Note that, thanks to (2.4) and the fact that h is continuous, we have

$$\exists h_0 > 0 : \ |h(s)| \leq h_0(1 + |s|), \quad \forall s \in \mathbb{R}. \tag{6.3}$$

Thus $h(v_4) \in L^2(]0, L[)$ for any $v_4 \in L^2(]0, L[)$, and then A is well defined.

The domain $D(A)$ of A can be characterized by

$$D(A) = \{V = (v_1, v_2, v_3, v_4, v_5)^T \in \Big(H^2(]0, L[) \cap H_0^1(]0, L[)\Big) \times \Big(H_0^1(]0, L[)\Big)^3 \times \mathcal{L}_g,$$

$$k_2\partial_{xx}v_2 - \Big(\int_0^{+\infty} g(s)ds\Big)\partial_x(a(x)\partial_x v_2) + \int_0^{+\infty} g(s)\partial_x(a(x)\partial_x v_5(s))ds \in L^2(]0, L[)\}$$

and it is dense in $\mathcal{H}$ (see also [18] and [19] and the reference therein), where

$$\mathcal{L}_g = \{v \in L_g, \, \partial_s v \in L_g, \, v(x, 0) = 0\}.$$

We use the classical notation $D(A^0) = \mathcal{H}$, $D(A^1) = D(A)$ and

$$D(A^n) = \{V \in D(A^{n-1}), \, AV \in D(A^{n-1})\} \quad \text{for } n = 2, 3, \cdots .$$

$D(A^n)$ is endowed with the graph norm $\|V\|_{D(A^n)} = \sum_{k=0}^n \|A^k V\|_{\mathcal{H}}$.

Now, we prove that the operator $A : D(A) \to \mathcal{H}$ is a maximal monotone; that is $-A$ is dissipative and $Id + A$ is surjective. Indeed, a simple calculation implies that for any $V = (v_1, v_2, v_3, v_4, v_5)^T$, $W = (w_1, w_2, w_3, w_4, w_5)^T \in D(A)$

$$\langle AV - AW, V - W\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^L a(x)\int_0^{+\infty} g'(s)|\partial_x(v_5(s) - w_5(s))|^2 dsdx$$

$$+ \int_0^L b(x)(v_4 - w_4)(h(v_4) - h(w_4))dx \geq 0,$$

since g is non-increasing, h is non-decreasing and $h(0) = 0$ (thanks to (2.5)). This implies that $-A$ is dissipative.

On the other hand, we prove that $Id+A$ is surjective; that is for any $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathcal{H}$, there exists $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$ satisfying

$$(Id + A)V = F. \tag{6.4}$$

The first two equations of system (6.4) are equivalent to

$$v_1 = v_3 + f_1 \quad \text{and} \quad v_2 = v_4 + f_2. \tag{6.5}$$

The last equation of system (6.4) is equivalent to

$$v_5 + \partial_s v_5 = v_4 + f_5,$$

then, by integrating with respect to s and noting that $v_5(0) = 0$, we get

$$v_5(s) = \Big(\int_0^s (v_4 + f_5(\tau))e^{\tau} d\tau\Big) e^{-s}. \tag{6.6}$$

Let $\mathcal{H}_1 = \Big(H_0^1(]0, L[)\Big)^2$ and $\mathcal{H}_2 = \Big(L^2(]0, L[)\Big)^2$ endowed with the inner products

$$\langle (z_1, z_2)^T, (w_1, w_2)^T \rangle_{\mathcal{H}_1} = k_1 \int_0^L (\partial_x z_1 + z_2)(\partial_x w_1 + w_2) dx$$
$$+ \int_0^L \Big(k_2 - a(x) \int_0^{+\infty} e^{-s} g(s) ds\Big) \partial_x z_2 \partial_x w_2 dx$$

and

$$\langle (z_1, z_2)^T, (w_1, w_2)^T \rangle_{\mathcal{H}_2} = \int_0^L (\rho_1 z_1 w_1 + \rho_2 z_2 w_2) dx.$$

Thanks to (2.6) and Poincaré's inequality, $\langle , \rangle_{\mathcal{H}_1}$ defines a norm on $\mathcal{H}_1$ equivalent to the one of $\Big(H^1(]0, L[)\Big)^2$. In addition, the inclusion $\mathcal{H}_1 \subset \mathcal{H}_2$ is dense and compact.

Now, inserting (6.5) and (6.6) into the third and fourth equations of system (6.4), multiplying them, respectively, by $\rho_1 w_3$ and $\rho_2 w_4$, where $(w_3, w_4)^T \in \mathcal{H}_1$, and then integrating their sum over $]0, L[$, we get

$$\langle (v_3, v_4)^T, (w_3, w_4)^T \rangle_{\mathcal{H}_2} + \langle (v_3, v_4)^T, (w_3, w_4)^T \rangle_{\mathcal{H}_1} + \int_0^L b(x) w_4 h(v_4) dx \tag{6.7}$$
$$= \langle (f_3, f_4)^T, (w_3, w_4)^T \rangle_{\mathcal{H}_2} - \langle (f_1, f_2)^T, (w_3, w_4)^T \rangle_{\mathcal{H}_1}$$
$$+ \Big(\int_0^{+\infty} (1 - e^{-s}) g(s) ds\Big) \int_0^L a(x) \partial_x f_2 \partial_x w_4 dx$$
$$- \int_0^L a(x) \partial_x \Big(\int_0^{+\infty} e^{-s} g(s) \Big(\int_0^s f_5(\tau) e^{\tau} d\tau\Big) ds\Big) \partial_x w_4 dx, \quad \forall (w_3, w_4)^T \in \mathcal{H}_1.$$

We have just to prove that (6.7) has a solution $(v_3, v_4)^T \in \mathcal{H}_1$, and then, using (6.5) and (6.6), we find (6.4). Following the method of [12] (page 95), let $\mathcal{H}_1'$ be the dual space of $\mathcal{H}_1$ and $A_0 : \mathcal{H}_1 \to \mathcal{H}_1'$ be the duality mapping. We consider the maps A_1, $B_1 : \mathcal{H}_1 \to \mathcal{H}_1'$ defined by

$$\langle A_1 (z_1, z_2)^T, (w_1, w_2)^T \rangle_{\mathcal{H}_1', \mathcal{H}_1} = \int_0^L b(x) h(z_2) w_2 dx$$

and

$$\langle B_1 (z_1, z_2)^T, (w_1, w_2)^T \rangle_{\mathcal{H}_1', \mathcal{H}_1} = \int_0^L a(x) \partial_x z_2 \partial_x w_2 dx.$$

We identify $\mathcal{H}_2$ with its dual space $\mathcal{H}_2'$ and we note

$$\tilde{f}_5 = \Big(\int_0^{+\infty} (1 - e^{-s}) g(s) ds\Big) f_2 - \int_0^{+\infty} e^{-s} g(s) \Big(\int_0^s f_5(\tau) e^{\tau} d\tau\Big) ds.$$

We have $B_1(0,\tilde f_5)^T \in \mathcal{H}_1'$ and the formula (6.7) becomes

$$\langle (Id + A_0 + A_1)(v_3,v_4)^T, (w_3,w_4)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}$$
$$= \langle (f_3,f_4)^T - A_0(f_1,f_2)^T + B_1(0,\tilde f_5)^T, (w_3,w_4)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}, \quad \forall (w_3,w_4)^T \in \mathcal{H}_1.$$

Let

$$(\tilde f_1,\tilde f_2)^T = (f_3,f_4)^T - A_0(f_1,f_2)^T + B_1(0,\tilde f_5)^T.$$

Because $\mathcal{H}_2 = \mathcal{H}_2' \subset \mathcal{H}_1'$, then $(\tilde f_1,\tilde f_2)^T \in \mathcal{H}_1'$. Therefore, the last formula is well-defined and equivalent to

$$(Id + A_0 + A_1)(v_3,v_4)^T = (\tilde f_1,\tilde f_2)^T. \tag{6.8}$$

It is sufficient to show that (6.8) has a solution $(v_3,v_4)^T \in \mathcal{H}_1$. Let $\tilde h : \mathbb{R} \to \mathbb{R}$ and $\Gamma : \mathcal{H}_1 \to \mathbb{R}$ defined by

$$\tilde h(s) = \int_0^s h(\tau)d\tau$$

and

$$\Gamma((z_1,z_2)^T) = \frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_2}^2 + \frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_1}^2 + \int_0^L b(x)\tilde h(z_2)dx \tag{6.9}$$
$$-\langle (\tilde f_1,\tilde f_2)^T, (z_1,z_2)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}.$$

The map Γ is well-defined and differentiable such that

$$\Gamma'((z_1,z_2)^T)(w_1,w_2)^T \tag{6.10}$$
$$= \langle (Id + A_0 + A_1)(z_1,z_2)^T - (\tilde f_1,\tilde f_2)^T, (w_1,w_2)^T\rangle_{\mathcal{H}_1',\mathcal{H}_1}, \ \forall (z_1,z_2)^T,\ (w_1,w_2)^T \in \mathcal{H}_1.$$

On the other hand, since h is non-decreasing and $h(0) = 0$, $\tilde h$ is non-negative, and then, using Cauchy-Schwarz inequality to minimize the last term in (6.9),

$$\Gamma((z_1,z_2)^T) \geq \Big(\frac{1}{2}\|(z_1,z_2)^T\|_{\mathcal{H}_1} - \|(\tilde f_1,\tilde f_2)^T\|_{\mathcal{H}_1'}\Big)\|(z_1,z_2)^T\|_{\mathcal{H}_1}.$$

This implies that Γ converges to infinity when $\|(z_1,z_2)^T\|_{\mathcal{H}_1}$ converges to infinity, and therefore, Γ reaches its minimum in some point $(v_3,v_4)^T \in \mathcal{H}_1$. This point satisfies $\Gamma'((v_3,v_4)^T) = 0$ which solves (6.8) thanks to (6.10) with the choice $(z_1,z_2)^T = (v_3,v_4)^T$.

Finally, we deduce that A is an infinitesimal generator of a contraction semigroup which implies the following results of existence, uniqueness and smoothness of the solution of (P) (see [12] and [22]):

Theorem 6.1. *1. For any $(\varphi_0,\psi_0(0),\varphi_1,\psi_1,\eta_0)^T \in \mathcal{H}$, the system (P) has a unique solution*

$$(\varphi,\psi,\varphi_t,\psi_t,\eta)^T \in C(\mathbb{R}_+;\mathcal{H}).$$

2. If $(\varphi_0,\psi_0(0),\varphi_1,\psi_1,\eta_0)^T \in D(A)$, then the solution

$$(\varphi,\psi,\varphi_t,\psi_t,\eta)^T \in W^{1,\infty}(\mathbb{R}_+;\mathcal{H}) \cap L^\infty(\mathbb{R}_+;D(A)).$$

3. If h is linear (then A is linear) and $(\varphi_0,\psi_0(0),\varphi_1,\psi_1,\eta_0)^T \in D(A^n)$ (for $n \in \mathbb{N}^*$), then the solution

$$(\varphi,\psi,\varphi_t,\psi_t,\eta)^T \in \cap_{k=0}^n C^{n-k}(\mathbb{R}_+;D(A^k)).$$

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