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ON SOME NEW MODULAR RELATIONS FOR RAMANUJAN'S $\kappa(q)-$FUNCTION and $\nu(q)-$FUNCTION

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Abstract. In his second and 'lost' notebooks, S. Ramanujan introduced parameters $\kappa(q)$ and $\nu(q)$ related to the Rogers-Ramanujan continued fraction. In this paper, we establish several new $P-Q$ modular equations for the ratios of Ramanujan's theta function. We establish several general formulas for explicit evaluations of the ratios of Ramanujan's theta function $\psi(q)$. We establish several new modular relations connecting $\kappa(q)$ with $\kappa(q^n)$ and $\nu(q)$ with $\nu(q^n)$ for different positive integer $n > 1$. We also establish relations between $\kappa(q)$, $\nu(q)$ and $\mu(q)$.

1 Introduction

The Rogers-Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots, \quad |q| < 1, \tag{1.1}$$

was first studied by L. J Rogers [14]. Later, this continued fraction was rediscovered by S. Ramanujan and recorded many interesting results involving $R(q)$. For more details on $R(q)$ one can see [2], [3], [6], [15], [16] and [17].

In his 'lost' notebook Ramanujan [13], introduced the parameters $\mu(q) := R(q)R(q^4)$ and $\kappa(q) := R(q)R^2(q^2)$ which are related to Rogers-Ramanujan continued fraction. Ramanujan stated several interesting identities involving the parameters $\mu(q)$ and $\kappa(q)$. These results were studied in detail by S. -Y. Kang [9]. S. -Y. Kang also introduced a new parameter $\nu(q) := R^2(q^{1/2})R(q)/R(q^2)$ which is analogous to $\mu(q)$ and $\kappa(q)$ and established some identities. Recently, C. Gugg [8] established certain identities of Ramanujan using the parameter $\kappa(q)$. S. Cooper [7], also systematically studied several results involving the parameter $\kappa(q)$.

The Ramanujan's theta function is defined by

$$\begin{aligned} f(a,b) :&= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1, \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \end{aligned} \tag{1.2}$$

Three special cases of $f(a,b)$ are as follows:

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) := \sum_{n=-\infty}^{\infty} q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.5}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

Now we define a modular equation in brief. The ordinary hypergeometric series ${}_2F_1(a,b;c;x)$ is defined by

$${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n,$$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $|\ x\ |< 1$. Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right) \tag{1.6}$$

and

$$q := q(x) := \exp\left(-\pi\,\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right), \tag{1.7}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)}. \tag{1.8}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (1.8). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

In [4] and [18], the authors have defined two parameters $l_{k,n}$ and $l'_{k,n}$ as follows:

$$l_{k,n} := \frac{\psi(-e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(-e^{-\pi\sqrt{nk}})}, \tag{1.9}$$

and

$$l'_{k,n} := \frac{\psi(e^{-\pi\sqrt{n/k}})}{k^{1/4}e^{-\frac{(k-1)\pi}{8}\sqrt{n/k}}\psi(e^{-\pi\sqrt{nk}})}. \tag{1.10}$$

They have established several properties and some explicit evaluations of $l_{k,n}$ and $l'_{k,n}$ for different positive rational values of n and k. Recently, M. S. Mahadeva Naika, S. Chandankumar, K. Sushan Bairy [10, 11] have established several new modular equations and also established general formulas for explicit evaluations of the ratios of Ramanujan's theta function ψ.

In Section 2, we collect several results which are useful to prove our main Theorems. In Section 3, we prove several new $P-Q$ modular equations for the ratios of Ramanujan's theta function. In Section 4, we establish some general formulas for explicit evaluations $l_{5,n}$ and $l'_{5,n}$. In Section 5, we establish several new modular relations connecting $\kappa(q)$ with $\kappa(q^n)$. In Section 6, we establish several new modular relations connecting $\nu(q)$ with $\nu(q^n)$. In Section 7, we establish some new modular relations connecting $\kappa(q)$, $\nu(q)$ and $\mu(q)$.

2 Preliminary results

In this section, we collect several identities which are useful in proving our main results.

Lemma 2.1. *[13, p. 56] [9] We have*

$$\frac{f^3(-q)}{f^3(-q^5)}=\frac{\psi(q)}{\psi(q^5)}\left(\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)}\right), \tag{2.1}$$

$$\frac{f^6(-q^2)}{f^6(-q^{10})}=\frac{\psi^4(q)}{\psi^4(q^5)}\left(\frac{\psi^2(q)-5q\psi^2(q^5)}{\psi^2(q)-q\psi^2(q^5)}\right), \tag{2.2}$$

$$\frac{f^3(-q^2)}{qf^3(-q^{10})}=\frac{\varphi(q)}{\varphi(q^5)}\left(\frac{5\varphi^2(q^5)-\varphi^2(q)}{\varphi^2(q)-\varphi^2(q^5)}\right), \tag{2.3}$$

$$\frac{f^6(-q)}{qf^6(-q^5)}=\frac{\varphi^4(-q)}{\varphi^4(-q^5)}\left(\frac{5\varphi^2(-q^5)-\varphi^2(-q)}{\varphi^2(-q)-\varphi^2(-q^5)}\right). \tag{2.4}$$

Lemma 2.2. *[13, p. 56] [9] We have*

$$\frac{\psi^2(q)}{q\psi^2(q^5)}=\frac{1+\kappa(q)-\kappa^2(q)}{\kappa(q)}. \tag{2.5}$$

Lemma 2.3. *[3, Entry 1.8.1, p. 33] [9] We have*

$$\frac{\psi(q)}{q^{1/2}\psi(q^5)}=\frac{1+\nu(q)}{1-\nu(q)}. \tag{2.6}$$

Lemma 2.4. *[13, p.26] [9] We have*

$$\frac{\varphi(q)}{\varphi(q^5)}=\frac{1+\mu(q)}{1-\mu(q)}. \tag{2.7}$$

Lemma 2.5. *[13, Entry 1.6.2(i), p. 50] We have*

$$16qf^2(-q^2)f^2(-q^{10})=(\varphi^2(q)-\varphi^2(q^5))(5\varphi^2(q^5)-\varphi^2(q)). \tag{2.8}$$

Lemma 2.6. *[5, Ch. 16, Entry 24(i), p. 39] We have*

$$\frac{\psi^2(q)}{\psi^2(-q)}=\frac{\varphi(q)}{\varphi(-q)}. \tag{2.9}$$

Lemma 2.7. *[5, Ch. 16, Corollary (ii), p. 74] We have*

$$\psi(q^5)\psi(q^{11})-q^5\psi(q)\psi(q^{55})=\psi(-q^5)\psi(-q^{11})+q^5\psi(q)\psi(q^{55}). \tag{2.10}$$

Lemma 2.8. *[13, p. 55]*
If $x:=\dfrac{f(-q)}{q^{1/6}f(-q^5)}$ *and* $y:=\dfrac{f(-q^2)}{q^{1/3}f(-q^{10})}$, *then*

$$xy+\frac{5}{xy}=\left(\frac{x}{y}\right)^3+\left(\frac{y}{x}\right)^3. \tag{2.11}$$

Lemma 2.9. *[13, p. 55]*
If $x:=\dfrac{f(-q)}{q^{1/6}f(-q^5)}$ *and* $y:=\dfrac{f(-q^4)}{q^{2/3}f(-q^{20})}$, *then*

$$\begin{aligned}(xy)^3+\left(\frac{5}{xy}\right)^3&=\left(\frac{x}{y}\right)^5+\left(\frac{y}{x}\right)^5-8\left\{\left(\frac{x}{y}\right)^3+\left(\frac{y}{x}\right)^3\right\}\\&+4\left(\frac{x}{y}+\frac{y}{x}\right)+\frac{4}{\left(\dfrac{x}{y}+\dfrac{y}{x}\right)}.\end{aligned} \tag{2.12}$$

Lemma 2.10. *[13, p. 55]*
If $x := \dfrac{f(-q)}{q^{1/6}f(-q^5)}$ *and* $y := \dfrac{f(-q^5)}{q^{5/6}f(-q^{25})}$, *then*

$$(xy)^2 + \left(\frac{5}{xy}\right)^2 + 5\left(xy + \frac{5}{xy}\right) + 15 = \left(\frac{y}{x}\right)^3. \tag{2.13}$$

Lemma 2.11. *[5, Ch 20, Entry 18(vi) and (vii), p. 423]*
If β, γ *and* δ *are of degrees 5, 7 and 35 respectively over* α, *then*

$$\begin{aligned}\left(\frac{\alpha\delta}{\beta\gamma}\right)^{1/8} + \left(\frac{(1-\alpha)(1-\delta)}{(1-\beta)(1-\gamma)}\right)^{1/8} - \left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/8}\\ + 2\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/12} = \sqrt{\frac{m'}{m}},\end{aligned} \tag{2.14}$$

$$\begin{aligned}\left(\frac{\beta\gamma}{\alpha\delta}\right)^{1/8} + \left(\frac{(1-\beta)(1-\gamma)}{(1-\alpha)(1-\delta)}\right)^{1/8} - \left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/8}\\ + 2\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/12} = -\sqrt{\frac{m}{m'}}.\end{aligned} \tag{2.15}$$

Lemma 2.12. *[5, Ch. 17, Entry 10(i) and Entry 11(ii), pp. 122–123] We have*

$$\varphi(q) = \sqrt{z}, \tag{2.16}$$

$$\sqrt{2}q^{1/8}\psi(-q) = \sqrt{z}\{\alpha(1-\alpha)\}^{1/8}. \tag{2.17}$$

Lemma 2.13. *[1, Theorem 5.1]*
If $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q := \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$Q^2 + P^2Q^2 = 5 + P^2. \tag{2.18}$$

Lemma 2.14. *[3, Ch. 25, Entry 66, p. 233]*
If $P = \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q = \dfrac{\psi(q^3)}{q^{3/2}\psi(q^{15})}$, *then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \tag{2.19}$$

Lemma 2.15. *[4]*
If $P := \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q := \dfrac{\psi(q^2)}{q\psi(q^{10})}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = P^2 + \frac{5}{P^2}. \tag{2.20}$$

Lemma 2.16. *[4]*
If $P := \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$, *then*

$$\frac{P^2}{Q^2} + \frac{Q^2}{P^2} + \left(P^2 + \frac{5}{P^2}\right) = \left(Q^2 + \frac{5}{Q^2}\right) + 6. \tag{2.21}$$

3 Modular equations for ratios of Ramanujan's theta function

In this section, we establish some new modular equations for ratios of Ramanujan's theta function.

Theorem 3.1. *If* $P := \dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q := \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 4 = P^2 + \frac{5}{P^2}. \tag{3.1}$$

Proof. Cubing the equation (2.11) and using the equations (2.1) and (2.3), we find that

$$\begin{aligned}
&(P^4 - 5P^2 + 4P^2Q^2 - P^2Q^4 + Q^4)(P^4Q^2 - P^4 - 4P^2Q^2 + 5Q^2 - Q^4)\\
&(P^2Q^2 - P^2 + 5 - Q^2)(25 - 10P^2 - 10Q^2 + P^4 - 2P^4Q^2 + P^4Q^4 - 4P^2Q^2\\
&- 16P^3Q + 16Q^3P - 2P^2Q^4 + Q^4)(25 - 10P^2 - 10Q^2 + P^4 - 2P^4Q^2\\
&+ P^4Q^4 - 4P^2Q^2 + 16P^3Q - 16Q^3P - 2P^2Q^4 + Q^4) = 0.
\end{aligned} \tag{3.2}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 3.2. *If* $P := \dfrac{\psi(-q)\psi(-q^2)}{q^{3/2}\psi(-q^5)\psi(-q^{10})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{10})}{q^{-1/2}\psi(-q^5)\psi(-q^2)}$, *then*

$$\begin{aligned}
Q^4 + \frac{1}{Q^4} + \left(Q^2 + \frac{1}{Q^2}\right) &= 30 + P^2 + \frac{25}{P^2}\\
&+ \left(P + \frac{5}{P}\right)\left[5\left(Q + \frac{1}{Q}\right) - \left(Q^3 + \frac{1}{Q^3}\right)\right].
\end{aligned} \tag{3.3}$$

Proof. Cubing the equation (2.12), we deduce that

$$\begin{aligned}
&a^{36} - 22500a^6b^{24} - 1020a^{24}b^{18} - 7420b^{24}a^{12} - 22500a^{24}b^6 - a^{30}b^{24}\\
&- 7420a^{24}b^{12} - 1953125a^{12}b^6 - 127500a^{12}b^{18} - 1953125b^{12}a^6\\
&- 391b^6a^{30} - 391a^6b^{30} - a^{24}b^{30} - 937500a^{12}b^{12} - 375000a^6b^{18}\\
&- 16380a^{18}b^{18} - 60a^{24}b^{24} - 24a^{18}b^{30} - 24a^{30}b^{18} - 180a^{30}b^{12}\\
&- 180a^{12}b^{30} - 127500b^{12}a^{18} - 375000b^6a^{18} - 1020a^{18}b^{24} + b^{36} = 0.
\end{aligned} \tag{3.4}$$

where

$$a := \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \text{and} \quad b := \frac{f(-q^4)}{q^{2/3}f(-q^{20})}.$$

Using the equations (2.1) and (2.2) in the above equation (3.4), we find that

$$
\begin{aligned}
&(N^2M^4+M^4-4N^2M^2+5N^2+N^4)(M^2N^8-5N^6+5M^6-M^8-N^8+4N^6M^8\\
&-25N^2M^2+25N^2M^4-25N^4M^2+30N^4M^4+M^2N^6+5M^4N^6+N^2M^6+N^8\\
&-N^2M^8-5N^4M^6-N^6M^6)(625+500N^2+150N^4+150M^4+20N^6-4N^8M^6\\
&-20M^6+M^8-4M^2N^8+40N^2M^4-40N^4M^2+660N^4M^4+6M^4N^8+N^8M^8\\
&+224M^2N^6+8M^4N^6+224N^2M^6+4N^2M^8-500M^2-8N^4M^6+6N^4M^8)\\
&(-N^{16}-109375M^4+65625M^6-21875M^8+4375M^{10}-525M^{12}-7M^{12}N^{16}\\
&+804N^8M^{14}+304M^2N^{14}+7M^2N^{16}-21M^4N^{16}-1024M^4N^{14}+35M^6N^{16}\\
&-896M^8N^{14}+400M^{10}N^{14}-128M^{12}N^{14}+16M^{14}N^{14}-35M^8N^{16}+21M^{10}N^{16}\\
&+27600N^{10}M^2+100500M^2N^8+250000N^2M^2+78125M^2-400000N^2M^4\\
&-87680N^{10}M^4+4280N^{12}M^2-14336N^{12}M^4+250000N^2M^6-112000N^2M^8\\
&+110672N^6M^{10}-17536N^6M^{12}+1104N^6M^{14}+110672N^{10}M^6-67968N^{10}M^8\\
&+21104N^{10}M^{10}-4096N^{10}M^{12}+368N^{10}M^{14}+17976N^{12}M^6-10816N^{12}M^8\\
&+33200N^2M^{10}-5120N^2M^{12}+304N^2M^{14}+469000N^4M^6-270400N^4M^8\\
&-600000N^4M^4+230000M^2N^6-277000M^4N^8-512000M^4N^6+M^{14}N^{16}\\
&+3752N^{12}M^{10}-960N^{12}M^{12}+104N^{12}M^{14}+337020N^8M^6-217648N^8M^8\\
&+89880N^4M^{10}-14336N^4M^{12}+856N^4M^{14}+527600N^6M^6-339840N^6M^8\\
&+67404N^8M^{10}-11080N^8M^{12}+325000N^4M^2-M^{16}+1328M^6N^{14})=0.
\end{aligned}
\tag{3.5}
$$

where

$$M:=\frac{\psi(q)}{q^{1/2}\psi(q^5)}\quad\text{and}\quad N:=\frac{\psi(-q^2)}{q\psi(-q^{10})}.$$

By examining the behavior of the above factors near $q=0$, we can find a neighborhood about the origin, where the second factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem second factor vanishes identically. Changing q to $-q$ in the second factor, we arrive at the equation (3.3). □

Theorem 3.3. *If* $P:=\dfrac{\psi(q)}{q^{1/2}\psi(q^5)}$ *and* $Q:=\dfrac{\psi(q^4)}{q^2\psi(q^{20})}$, *then*

$$
\begin{aligned}
&\frac{P^4}{Q^4}+\frac{Q^4}{P^4}+24\left(\frac{P^2}{Q^2}+\frac{Q^2}{P^2}\right)+8\left(P^2Q^2+\frac{25}{P^2Q^2}\right)-20\left(Q^2+\frac{5}{Q^2}\right)+120\\
&+3\left(P^4+\frac{25}{P^4}\right)-32\left(P^2+\frac{5}{P^2}\right)=P^4\left(Q^2+\frac{3}{Q^2}\right)+\frac{5}{P^4}\left(3Q^2+\frac{25}{Q^2}\right).
\end{aligned}
\tag{3.6}
$$

Proof. Using the equations (2.1) and (2.12), we arrive at (3.6). □

Theorem 3.4. *If* $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q := \dfrac{\psi(-q^4)}{q^2\psi(-q^{20})}$, *then*

$$\begin{aligned}
&Q^8 + \frac{1}{Q^8} - 19\left(Q^6 + \frac{1}{Q^6}\right) - 419\left(Q^4 + \frac{1}{Q^4}\right) - 1327\left(Q^2 + \frac{1}{Q^2}\right) - 2332 \\
&+ \left(P + \frac{5}{P}\right)\left[\left(Q^7 + \frac{1}{Q^7}\right) - 44\left(Q^5 + \frac{1}{Q^5}\right) - 295\left(Q^3 + \frac{1}{Q^3}\right) - 672\left(Q + \frac{1}{Q}\right)\right] \\
&- \left(P^2 + \frac{5^2}{P^2}\right)\left[28\left(Q^4 + \frac{1}{Q^4}\right) + 109\left(Q^2 + \frac{1}{Q^2}\right) + 132\right] - \left(P^3 + \frac{5^3}{P^3}\right) \\
&\times\left[13\left(Q + \frac{1}{Q}\right) + 9\left(Q^3 + \frac{1}{Q^3}\right)\right] - \left(P^4 + \frac{5^4}{P^4}\right)\left[\left(Q^2 + \frac{1}{Q^2}\right)\right] = 0.
\end{aligned} \tag{3.7}$$

Proof. Using the equations (2.21) and (3.6), we arrive at the equation (3.7). □

Theorem 3.5. *If* $P := \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q := \dfrac{\psi(-q^5)}{q^{5/2}\psi(-q^{25})}$ *then*

$$\begin{aligned}
&\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} - 5\left(PQ + \frac{5}{PQ}\right) - 5\left(Q^2 + \frac{5}{P^2}\right) \\
&- P^2Q^2 + \frac{5^2}{P^2Q^2} - 15 = 0.
\end{aligned} \tag{3.8}$$

Proof. Using the equation (2.1) in the equation (2.13), we deduce that

$$\begin{aligned}
&(-90P^3Q^2 + 75PQ^2 + 2Q^3P^2 - 10QP^2 - 75P + 15P^5Q^2 + 90P^3 + 5Q \\
&- Q^3 - 15P^5 - P^4Q^3 + 5P^4Q)x^2y^2 + (25Q^3P^2 - 125QP^2 + 50P^4Q \\
&+ 25P^5Q^2 + P^6Q^3 + 125PQ^2 + 150P^3 - 125P - 150P^3Q^2 - 25P^5 - 5P^6Q \\
&- 10P^4Q^3)xy - 150P^3Q^2 + 250P^4Q + 25P^5Q^2 + 5P^6Q^3 - 125P - 25P^5 \\
&- 625QP^2 + 125Q^3P^2 + 150P^3 - 50P^4Q^3 + 125PQ^2 - 25P^6Q = 0,
\end{aligned} \tag{3.9}$$

where

$$x := \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \text{and} \quad y := \frac{f(-q^5)}{q^{5/6}f(-q^{25})}.$$

Solving the above quadratic equation for xy and then cubing both sides, we find that

$$\begin{aligned}
&(-Q^5 + 5PQ^4 - 25PQ^2 + 25P + 15Q^3P^2 - 25QP^2 - 5Q^4P^3 + 15Q^2P^3 \\
&- 5P^4Q^3 + Q^4P^5)(3125(1 - Q^2) + 105600Q^5P^3 - 94800Q^5P^5 + Q^{10}P^{10} \\
&- 13125P^2 + 45625Q^2P^2 - 16250P^6 + 21250P^4 + 1250Q^4 - 625P^{10} \\
&+ 40000(P^3Q - Q^3P^7) + 4000Q^3P^9 - 12500QP - 250Q^6 + 28900P^6Q^6 \\
&- 45000P^5Q - 95250Q^2P^4 + 144500P^4Q^4 + 37650P^2Q^6 - 55250P^2Q^4 \\
&+ 67250P^6Q^2 - 77700P^6Q^4 + 20000(Q^3P + QP^7) - 15625Q^2P^8 - Q^{10} \\
&+ 13450Q^4P^8 - 3810Q^6P^8 + 24000Q^5P^7 + 5625P^8 + 120000P^5Q^3 \\
&- 650Q^4P^{10} + 170P^{10}Q^6 + 21120Q^7P^5 - 4160Q^7P^7 - 32320Q^7P^3 \\
&- 2210Q^8P^6 + 365Q^8P^8 - 21Q^8P^{10} + 320Q^7P^9 + 4800Q^7P + 25Q^8 \\
&- 5Q^{10}P^8 - 10Q^{10}P^4 + 160Q^9P^7 - 296Q^9P^5 - 20Q^9P^9 - 7400PQ^5 \\
&+ 960P^3Q^9 + 220PQ^9 - 104000P^3Q^3 - 72900P^4Q^6 + 1125Q^2P^{10} \\
&- 2500P^9Q - 4665Q^8P^2 + 5P^2Q^{10} - 1800Q^5P^9 + 10Q^{10}P^6 + 7530Q^8P^4) \\
&(-Q^3P^2 - 75PQ^2 - 15P^3 + 15Q^2P^3 + Q^3 + 5QP^2 + 75P - 5Q)^3 = 0
\end{aligned} \tag{3.10}$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. Replacing q to $-q$ in the first factor, we arrive at the equation (3.8). □

Theorem 3.6. *If* $P := \dfrac{\psi(-q)\psi(-q^7)}{q^4\psi(-q^5)\psi(-q^{35})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{35})}{q^{-3}\psi(-q^5)\psi(-q^7)}$, *then*

$$\begin{aligned} &Q^4 - \frac{1}{Q^4} + 14\left[\left(Q^3 + \frac{1}{Q^3}\right) + \left(Q^2 - \frac{1}{Q^2}\right) + 10\left(Q + \frac{1}{Q}\right)\right] + P^3 + \frac{5^3}{P^3} \\ &+ 7\left\{\left(P^2 + \frac{5^2}{P^2}\right)\left(Q + \frac{1}{Q}\right) + \left(P + \frac{5}{P}\right)\left[2\left(Q^2 + \frac{1}{Q^2}\right) + 9\right]\right\} = 0. \end{aligned} \tag{3.11}$$

Proof. Using the equations (2.14), (2.15), (2.16) and (2.17), we deduce that

$$1 + r - 2Ar + sr - s + 2A^2r = 0, \tag{3.12}$$

where

$$r := \frac{q^3\psi(-q)\psi(-q^{35})}{\psi(-q^5)\psi(-q^7)}, \quad s := \frac{\varphi(q)\varphi(q^{35})}{\varphi(q^5)\varphi(q^7)} \text{ and } A := (s/r)^{1/3}.$$

On simplification of the equation (3.12), we find that

$$2A = 1 + M, \tag{3.13}$$

where

$$M := \pm\sqrt{\frac{2s - r - 2sr - 2}{r}}.$$

Cubing both sides of the equation (3.13) and eliminating M, we deduce that

$$\begin{aligned} &14d^3c^2 + d^5c^2 - 2d^5s - 8cd^4 + 6c^3d^2 - c^3d^4 + 6c^4d + 3c^4d^3 + 4sc^5 + 5c^2ds \\ &+ c^5d^2 + 2c^5 - 5sd^3 - 5cd^2 + 15c^2d + 4d^5 + 5c^3 + 5c^3s + 5d^3 - 6d^3c^2s \\ &- d^5c^2s + 6cd^4s + 14c^3d^2s + 3c^3d^4s + 8c^4ds + c^4d^3s + c^5d^2s + 15cd^2s = 0, \end{aligned} \tag{3.14}$$

where

$$c := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad d := \frac{\psi(-q^7)}{q^{7/2}\psi(-q^{35})}.$$

Collecting the terms containing s on one side of the above equation (3.14) and squaring both sides and then using the equation (2.18), we arrive at the equation (3.11). □

Theorem 3.7. *If* $P := \dfrac{\psi(-q)\psi(-q^{11})}{q^6\psi(-q^5)\psi(-q^{55})}$ *and* $Q := \dfrac{\psi(-q)\psi(-q^{55})}{q^{-5}\psi(-q^5)\psi(-q^{11})}$, *then*

$$\begin{aligned} &Q^6 + \frac{1}{Q^6} - 33\left(Q^5 + \frac{1}{Q^5}\right) - 99\left(Q^4 + \frac{1}{Q^4}\right) - 1529\left(Q^3 + \frac{1}{Q^3}\right) \\ &- 1683\left(Q^2 + \frac{1}{Q^2}\right) - 8800\left(Q + \frac{1}{Q}\right) - 6534 - \left(P^5 + \frac{5^5}{P^5}\right) \\ &- 11\left\{\left(P^4 + \frac{5^4}{P^4}\right)\left(Q + \frac{1}{Q}\right) + \left(P^3 + \frac{5^3}{P^3}\right)\left[11 + 4\left(Q^2 + \frac{1}{Q^2}\right)\right]\right. \\ &+ \left(P^2 + \frac{5^2}{P^2}\right)\left[18 + 56\left(Q + \frac{1}{Q}\right) + 3\left(Q^2 + \frac{1}{Q^2}\right) + 8\left(Q^3 + \frac{1}{Q^3}\right)\right] \\ &+ \left(P + \frac{5}{P}\right)\left[324 + 126\left(Q + \frac{1}{Q}\right) + 160\left(Q^2 + \frac{1}{Q^2}\right) + 18\left(Q^3 + \frac{1}{Q^3}\right)\right. \\ &\left.\left. +9\left(Q^4 + \frac{1}{Q^4}\right)\right]\right\} = 0. \end{aligned} \tag{3.15}$$

Proof. Replacing q by $-q$ in the equation (2.8), we deduce that

$$-16qf^2(-q^2)f^2(-q^{10}) = \varphi^4(-q^5)\left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right]\left[5 - \frac{\varphi^2(-q)}{\varphi^2(-q^5)}\right]. \tag{3.16}$$

Using the equations (3.16) and (2.8), we find that

$$\frac{\varphi^4(q^5)}{\varphi^4(-q^5)} = \frac{\left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right]\left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 5\right]}{\left[\frac{\varphi^2(q)}{\varphi^2(q^5)} - 1\right]\left[5 - \frac{\varphi^2(q)}{\varphi^2(q^5)}\right]}. \tag{3.17}$$

Replacing q by q^{11} in the above equation (3.17), we deduce that

$$\frac{\varphi^4(q^{55})}{\varphi^4(-q^{55})} = \frac{\left[\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 1\right]\left[\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 5\right]}{\left[\frac{\varphi^2(q^{11})}{\varphi^2(q^{55})} - 1\right]\left[5 - \frac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right]}. \tag{3.18}$$

Employing the equation (2.9) along with the equations (3.17) and (3.18), we deduce that

$$\begin{aligned}&\left(\frac{\psi(q^5)\psi(q^{55})}{\psi(-q^5)\psi(-q^{55})}\right)^8\\ &= \frac{\left(\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right)\left(\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 5\right)}{\left(\frac{\varphi^2(q)}{\varphi^2(q^5)} - 1\right)\left(5 - \frac{\varphi^2(q)}{\varphi^2(q^5)}\right)}\frac{\left(\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 1\right)\left(\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 5\right)}{\left(\frac{\varphi^2(q^{11})}{\varphi^2(q^{55})} - 1\right)\left(5 - \frac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right)}.\end{aligned} \tag{3.19}$$

The equation (2.10) can be re arranged as,

$$\left[\frac{\psi(q^5)\psi(q^{55})}{\psi(-q^5)\psi(-q^{55})}\right]^8 = \frac{\left[\frac{\psi(-q^{11})}{q^{11/2}\psi(-q^{55})} + \frac{\psi(-q)}{q^{1/2}\psi(-q^5)}\right]^8}{\left[\frac{\psi(q^{11})}{q^{11/2}\psi(q^{55})} - \frac{\psi(q)}{q^{1/2}\psi(q^5)}\right]^8}. \tag{3.20}$$

Using the equations (3.19) and (3.20), we deduce that

$$\begin{aligned}&\frac{\left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 1\right]\left[\frac{\varphi^2(-q)}{\varphi^2(-q^5)} - 5\right]}{\left[\frac{\varphi^2(q)}{\varphi^2(q^5)} - 1\right]\left[5 - \frac{\varphi^2(q)}{\varphi^2(q^5)}\right]}\frac{\left[\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 1\right]\left[\frac{\varphi^2(-q^{11})}{\varphi^2(-q^{55})} - 5\right]}{\left[\frac{\varphi^2(q^{11})}{\varphi^2(q^{55})} - 1\right]\left[5 - \frac{\varphi^2(q^{11})}{\varphi^2(q^{55})}\right]}\\ &= \frac{\left[\frac{\psi(-q^{11})}{q^{11/2}\psi(-q^{55})} + \frac{\psi(-q)}{q^{1/2}\psi(-q^5)}\right]^8}{\left[\frac{\psi(q^{11})}{q^{11/2}\psi(q^{55})} - \frac{\psi(q)}{q^{1/2}\psi(q^5)}\right]^8}.\end{aligned} \tag{3.21}$$

Using the equations (2.18), (2.21) and (3.1) in the above equation (3.21), we arrive at the equation (3.15). This completes the proof. □

4 General formulas for explicit evaluations for ratios of Ramanujan's theta-functions

In this section, we establish some general formulas for the explicit evaluations of the ratios of Ramanujan's theta function $\psi(q)$.

Theorem 4.1. *If* $X := l_{5,n}l_{5,4n}$ *and* $Y := \dfrac{l_{5,n}}{l_{5,4n}}$, *then*

$$\begin{aligned} Y^4 + \frac{1}{Y^4} + \left(Y^2 + \frac{1}{Y^2}\right) &= 30 + 5\left(X^2 + \frac{1}{X^2}\right) \\ &+ \sqrt{5}\left(X + \frac{1}{X}\right)\left[5\left(Y + \frac{1}{Y}\right) - \left(Y^3 + \frac{1}{Y^3}\right)\right]. \end{aligned} \tag{4.1}$$

Proof. Using the equation (3.3) along with the equation (1.9), we arrive at the equation (4.1). □

Corollary 4.2. *We have*

$$l_{5,2} = \sqrt{\sqrt{2}+1}, \tag{4.2}$$

$$l_{5,1/2} = \sqrt{\sqrt{2}-1}, \tag{4.3}$$

$$l_{5,4} = \left(1 + \frac{\sqrt{11+5\sqrt{5}}}{2} + \frac{\sqrt{4\sqrt{11+5\sqrt{5}}+11+5\sqrt{5}}}{2}\right)^{1/2}, \tag{4.4}$$

$$l_{5,1/4} = \left(1 + \frac{\sqrt{11+5\sqrt{5}}}{2} - \frac{\sqrt{4\sqrt{11+5\sqrt{5}}+11+5\sqrt{5}}}{2}\right)^{1/2}, \tag{4.5}$$

$$l_{5,8} = \left(\sqrt{32+14\sqrt{5}+10\sqrt{10}+22\sqrt{2}} + \sqrt{34+15\sqrt{5}+11\sqrt{10}+24\sqrt{2}}\right)^{1/2}, \tag{4.6}$$

$$l_{5,1/8} = \left(\frac{\sqrt{34+15\sqrt{5}+11\sqrt{10}+24\sqrt{2}} - \sqrt{32+14\sqrt{5}+10\sqrt{10}+22\sqrt{2}}}{(\sqrt{5}+2)(1+\sqrt{2})}\right)^{1/2}. \tag{4.7}$$

Proofs of (4.2) *and* (4.3). Putting $n = 1/2$ in the equation (4.1) and using the fact that $l_{5,2}l_{5,1/2} = 1$, we deduce that

$$(l_{5,2}^4 - 2l_{5,2}^2 - 1)(l_{5,2}^4 + 2l_{5,2}^2 - 1)(2l_{5,2}^2 - 1 + \sqrt{5})^2(2l_{5,2}^2 + 1 + \sqrt{5})^2 = 0. \tag{4.8}$$

We observe that the first factor of the equation (4.8) vanishes for the specific value of $q = e^{-\pi\sqrt{2/5}}$, but the other two factors does not vanish. Hence , we deduce that

$$l_{5,2}^4 - 2l_{5,2}^2 - 1 = 0. \tag{4.9}$$

Solving the above equation (4.9), we arrive at the equations (4.2) and (4.3). □

Proofs of (4.4) *and* (4.5). Putting $n = 1$ in the equation (4.1) and by using the fact that $l_{5,1} = 1$, we deduce that

$$l_{5,4}^8 - 4l_{5,4}^6 - l_{5,4}^4(5\sqrt{5}+5) - 4l_{5,4}^2 + 1 = 0. \tag{4.10}$$

The above equation (4.10) can be rewritten as

$$x^2 - 4x - 5\sqrt{5} - 7 = 0, \quad \text{where } x := l_{5,4}^2 + \frac{1}{l_{5,4}^2}. \tag{4.11}$$

Solving the above equation for x and $x > 1$, we deduce that

$$l_{5,4}^2 + \frac{1}{l_{5,4}^2} = 2 + (11+5\sqrt{5})^{1/2}. \tag{4.12}$$

On solving the above equation (4.12), we arrive at the equations (4.4) and (4.5). □

Proofs of (4.6) *and* (4.7). Using the equation (4.2) in the equation (4.1), we obtain the equations (4.6) and (4.7). □

Theorem 4.3. *If* $X := l_{5,n}$ *and* $Y := l_{5,25n}$, *then*

$$\begin{aligned}&\frac{Y^3}{X^3} - \frac{5Y^2}{X^2} - \frac{15Y}{X} - 5\sqrt{5}\left(XY + \frac{1}{XY}\right) - 5\sqrt{5}\left(Y^2 + \frac{5}{X^2}\right)\\&- 5\left(X^2Y^2 + \frac{1}{X^2Y^2}\right) - 15 = 0.\end{aligned} \tag{4.13}$$

Proof. Using the equation (3.8) along with the equation (1.9), we arrive at the equation (4.13). □

Corollary 4.4. *We have*

$$l_{5,5} = \sqrt{5 + 2\sqrt{5}}, \tag{4.14}$$

$$l_{5,1/5} = \frac{\sqrt{5 - 2\sqrt{5}}}{\sqrt{5}}. \tag{4.15}$$

Proofs of (4.14) *and* (4.15). Putting $n = 1/5$ in the equation (4.13) and using the fact that $l_{5,5}l_{5,1/5} = 1$, we deduce that

$$(l_{5,5}^2 - 5 - 2\sqrt{5})(l_{5,5}^2 + \sqrt{5})^2 = 0. \tag{4.16}$$

Since $l_{5,5} > 0$, hence by solving the equation $l_{5,5}^2 - 5 - 2\sqrt{5} = 0$, we arrive at the equations (4.14) and (4.15). □

Theorem 4.5. *If* $X := l_{5,n}l_{5,121n}$ *and* $Y := \dfrac{l_{5,n}}{l_{5,121n}}$, *then*

$$\begin{aligned}&Y^6 + \frac{1}{Y^6} - 33\left(Y^5 + \frac{1}{Y^5}\right) - 99\left(Y^4 + \frac{1}{Y^4}\right) - 1529\left(Y^3 + \frac{1}{Y^3}\right)\\&- 1683\left(Y^2 + \frac{1}{Y^2}\right) - 8800\left(Y + \frac{1}{Y}\right) - 6534 - 25\sqrt{5}\left(X^5 + \frac{1}{X^5}\right)\\&- 11\sqrt{5}\left\{5\sqrt{5}\left(X^4 + \frac{1}{X^4}\right)\left(Y + \frac{1}{Y}\right) + 5\left(X^3 + \frac{1}{X^3}\right)\left[11 + 4\left(Y^2 + \frac{1}{Y^2}\right)\right]\right.\\&+\sqrt{5}\left(X^2 + \frac{1}{X^2}\right)\left[18 + 56\left(Y + \frac{1}{Y}\right) + 3\left(Y^2 + \frac{1}{Y^2}\right) + 8\left(Y^3 + \frac{1}{Y^3}\right)\right]\\&+\left(X + \frac{1}{X}\right)\left[324 + 126\left(Y + \frac{1}{Y}\right) + 160\left(Y^2 + \frac{1}{Y^2}\right) + 18\left(Y^3 + \frac{1}{Y^3}\right)\right.\\&\left.\left.+9\left(Y^4 + \frac{1}{Y^4}\right)\right]\right\} = 0.\end{aligned} \tag{4.17}$$

Proof. Using the equation (3.15) along with the equation (1.9), we arrive at the equation (4.17). □

Corollary 4.6. *We have*

$$l_{5,11} = \sqrt{12 + 5\sqrt{5} + 2\sqrt{67 + 30\sqrt{5}}}, \tag{4.18}$$

$$l_{5,1/11} = \sqrt{12 + 5\sqrt{5} - 2\sqrt{67 + 30\sqrt{5}}}. \tag{4.19}$$

Proofs of (4.18) *and* (4.19). Putting $n = 1/11$ in the equation (4.17) and using the fact that $l_{5,11}l_{5,1/11} = 1$, we deduce that

$$\begin{aligned}&(2l_{5,11}^4 + (3+\sqrt{5})l_{5,11}^2 + 2)(-l_{5,11}^4 + (24+10\sqrt{5})l_{5,11}^2 - 1)\\&(2l_{5,11}^8 + (3+3\sqrt{5})l_{5,11}^6 + (30-6\sqrt{5})l_{5,11}^4 + (3+3\sqrt{5})l_{5,11}^2 + 2)\\&(l_{5,11}^4 - (6-4\sqrt{5})l_{5,11}^2 + 1)^2 = 0.\end{aligned} \tag{4.20}$$

We observe that the second factor of the equation (4.20) vanishes for the specific value of $q = e^{-\pi\sqrt{11/5}}$, but the other factors does not vanish. Hence , we deduce that

$$l_{5,11}^4 - 24l_{5,11}^2 - 10l_{5,11}^2\sqrt{5} + 1 = 0. \tag{4.21}$$

Solving the above equation (4.21), we arrive at the equations (4.18) and (4.19). □

5 Modular Relations Between $\kappa(q)$ and $\kappa(q^n)$

In this section, we establish several new modular relations connecting $\kappa(q)$ with $\kappa(q^n)$ using the $P-Q$ modular equations obtained in the Section 3.

Theorem 5.1. *If* $u := \kappa(q)$ *and* $v := \kappa(q^3)$, *then*

$$\begin{aligned}&v^4u + (-u^4 + 3u + 3u^2 - 3u^3)v^3 + (-3u + 3u^3)v^2\\&+ (-3u - 3u^2 + 3u^3 + 1)v - u^3 = 0.\end{aligned} \tag{5.1}$$

Proof. Using the equation (2.5) and (2.19), we arrive at the equation (5.1). □

Theorem 5.2. *If* $u := \kappa(q)$ *and* $v := \kappa(q^4)$, *then*

$$\begin{aligned}&(v^3 - v^2 - v + 1)u^4 + (4v^3 - 4v^2 - 8v)u^3 + (2v - 2v^3 + 8v^2)u^2\\&+ (-8v^3 + 4v^2 + 4v)u + v^3 + v^4 - v^2 - v = 0.\end{aligned} \tag{5.2}$$

Proof. Using the equation (2.5) and (3.6), we arrive at the equation (5.2). □

Theorem 5.3. *If* $u := \kappa(q)$ *and* $v := \kappa(q^5)$, *then*

$$\begin{aligned}&(-2v + v^2 - 1 - v^4 + 2v^3)u^5 - (v^4 - v^2 - v^3)5u^4 - (3v^3 + 2v^2 - 3v)5u^3\\&+ (3v^4 - 3v^2 + 2v^3)5u^2 + (v^3 - v^2 - v)5u + 2v^2 + v - 2v^4 + v^5 - v^3 = 0.\end{aligned} \tag{5.3}$$

Proof. Using the equation (2.5) and (3.8), we arrive at the equation (5.3). □

Theorem 5.4. *If* $u := \kappa(q)$ *and* $v := \kappa(q^7)$, *then*

$$\begin{aligned}&7\{(v^7 + 2v^6 - 4v^5 - 4v^4 + 4v^3 + v^2 - v)u^7 + (v^7 + 6v^6 - 11v^5 - 12v^4\\&+ 14v^3 + 6v^2 - 2v)u^6 - (4v^7 + 14v^6 - 31v^5 - 23v^4 + 31v^3 + 11v^2 - 4v)u^5\\&+ (4v - 12v^6 - 23v^3 - 12v^2 - 4v^7 + 23v^5 + 24v^4)u^4 + (14v^2 + 31v^3 + 4v^7\\&- 23v^4 - 31v^5 - 4v + 11v^6)u^3 + (2v^7 + 6v^6 - 14v^5 + 11v^3 - 12v^4 + 6v^2 - v)u^2\\&+(-4v^3 + 4v^5 - 2v^2 + v - v^6 + 4v^4 - v^7)u\} + v^7u^8 + u^7 - v^8u - v = 0.\end{aligned} \tag{5.4}$$

Proof. Using the equation (2.5) and (3.11), we arrive at the equation (5.4). □

Theorem 5.5. *If* $u := \kappa(q)$ *and* $v := \kappa(q^{11})$, *then*

$$\begin{aligned}
&u^{12} - uv + 11\left\{(8v^2 + 2v^8 - v^{10} - 3v^9 - 16v^4 + 7v^6 + 24v^3 - 37v^5 - 4v + 19v^7)u^{11}\right.\\
&-(v^{11} + 7v^{10} + 5v^9 - 26v^8 - v^7 - 49v^6 + 19v^5 + 100v^4 - 38v^2 - 32v^3 + 8v)u^{10}\\
&+(88v^8 - 2v^4 - 5v^{10} - 511v^7 + 79v^9 - 354v^3 + 745v^5 - 49v^6 + 24v - 3v^{11}\\
&-32v^2)u^9 + (152v^5 + 26v^8 + 611v^4 + 2v^3 + 16v - 260v^7 + 88v^9 + 26v^{10} - 100v^2\\
&-518v^6 + 2v^{11})u^8 + (-260v^8 + 2252v^7 - 511v^9 - 2540v^5 - 152v^4 + 745v^3 + v^{10}\\
&+19v^2 - 37v + 392v^6 + 19v^{11})u^7 + (49v^3 + 49v^2 + 1022v^6 - 518v^4 - 7v - 392v^5\\
&+392v^7 - 49v^9 - 518v^8 + 7v^{11} + 49v^{10})u^6 + (260v^4 - v^2 - 37v^{11} + 19v + 2252v^5\\
&-392v^6 - 2540v^7 - 511v^3 + 745v^9 + 152v^8 - 19v^{10})u^5 + (26v^4 + 260v^5 - 88v^3\\
&-518v^6 + 26v^2 + 611v^8 - 2v - 16v^{11} - 2v^9 - 100v^{10} - 152v^7)u^4 + (32v^{10} - 354v^9\\
&+5v^2 + 79v^3 - 511v^5 + 2v^8 + 49v^6 + 745v^7 + 24v^{11} - 88v^4 - 3v)u^3 + (38v^{10}\\
&+49v^6 + 8v^{11} - 32v^9 + 5v^3 - 100v^8 + v - v^5 - 7v^2 + 26v^4 + 19v^7)u^2 + (19v^5\\
&\left.-37v^7 - 4v^{11} + 16v^8 - 8v^{10} - 2v^4 + 24v^9 + v^2 - 7v^6 - 3v^3)u\right\} + v^{12} - uv^{11} = 0.
\end{aligned} \tag{5.5}$$

Proof. Using the equation (2.5) and (3.15), we arrive at the equation (5.5). □

6 Modular Relations Between $\nu(q)$ and $\nu(q^n)$

In this section, we establish modular relations for $\nu(q)$ using the $P - Q$ modular equations obtained in the Section 3.

Theorem 6.1. *If* $u := \nu(q)$ *and* $v := \nu(-q)$, *then*

$$\begin{aligned}
&(1 + 4u^3 + 4u + u^4 - 18u^2)(1 + v^4) + (6u^4 - 8u - 12u^2 - 8u^3 + 6)v^2\\
&+ (8u - 4 + 8u^3 + 8u^2 - 4u^4)(v + v^3) = 0.
\end{aligned} \tag{6.1}$$

Proof. Using the equation (2.6) and (2.21), we arrive at the equation (6.1). □

Theorem 6.2. *If* $u := \nu(q)$ *and* $v := \nu(q^2)$, *then*

$$v^2(u^4 + 1) + (1 - 2v - 2v^2 - 2v^3 + v^4)(u^3 + u) + (-2v^4 + 10v^2 - 2)u^2 = 0. \tag{6.2}$$

Proof. Using the equations (2.6) and (2.20), we arrive at the equation (6.2). □

Theorem 6.3. *If* $u := \nu(q)$ *and* $v := \nu(q^3)$, *then*

$$(3u + 3u^2 - 3u^3 + u^4)v^3 - 3v^2(u + u^3) + (3u^3 - 3u + 3u^2 + 1)v = u^3 + v^4u. \tag{6.3}$$

Proof. Using the equation (2.6) and (2.19), we arrive at the equation (6.3). □

Theorem 6.4. *If* $u := \nu(q)$ *and* $v := \nu(q^4)$, *then*

$$\begin{aligned}
&(v^8 + 6v^2 + 6v^6 - 4v^7 - 4v - 14v^4)(u^7 + u) + (32[v^7 - v^5 + v - v^3] - 6v^8\\
&- 6 - 48v^6 - 48v^2 + 136v^4)(u^6 + u^2) + (15v^8 - 108v^7 - 108v + 202v^6 + 15\\
&+ 80v^5 - 434v^4 + 202v^2 + 80v^3)(u^3 + u^5) + (160v^7 - 20 + 160v - 20v^8\\
&- 160v^3 - 160v^5 - 320v^6 - 320v^2 + 750v^4)u^4 + v^4 + u^8v^4 + u^7 + u = 0.
\end{aligned} \tag{6.4}$$

Proof. Using the equation (2.6) and (3.6), we arrive at the equation (6.4). □

Theorem 6.5. *If* $u := \nu(q)$ *and* $v := \nu(q^5)$, *then*

$$\begin{aligned}&(u^5+10u^3+6-5u^2-5u^4-10u)v^4-(6u^5-35u+25u^3-25u^4+20u^2)v^3\\&+(11u^5+25u^2-35u^4-25u+20u^3)v^2-(6u^5-5u-10u^4-5u^3+10u^2)v\\&+v+u^5-v^5-11v^3+6v^2=0.\end{aligned}\tag{6.5}$$

Proof. Using the equation (2.6) and (3.8), we arrive at the equation (6.5). □

Theorem 6.6. *If* $u := \nu(q)$ *and* $v := \nu(q^7)$, *then*

$$\begin{aligned}&v+u^8v^7-v^8u-u^7+7\{(u-2u^2-u^7+4u^3+3u^6-4u^5)v^7+(2u^7-3u+26u^5\\&-29u^3-14u^6+14u^2)v^6+(29u^6-26u^2+5u^4+55u^3-55u^5+4u-4u^7)v^5\\&-5(u^3+u^5)v^4+(4u^7-55u^3+29u^2+55u^5-4u-26u^6+5u^4)v^3+(-29u^5\\&-14u^2+26u^3+14u^6-3u^7+2u)v^2+(3u^2+u^7+4u^5-2u^6-u-4u^3)v\}=0.\end{aligned}\tag{6.6}$$

Proof. Using the equation (2.6) and (3.11), we arrive at the equation (6.6). □

Theorem 6.7. *If* $u := \nu(q)$ *and* $v := \nu(q^{11})$, *then*

$$\begin{aligned}&v^{12}-uv-u^{11}v^{11}+11\{(14u^8-4u+8u^2+u^{10}-21u^7-32u^4+21u^5+8u^3\\&+9u^6-5u^9)v^{11}+(8u-214u^8+83u^9-15u^{10}+247u^7+6u^2-211u^5-39u^6\\&+u^{11}-128u^3+268u^4)v^{10}+(8u+538u^3-489u^9+83u^{10}-81u^6+1292u^8\\&-1393u^7+1321u^5-1166u^4-5u^{11}-128u^2)v^9+(-214u^{10}-4284u^5+268u^2\\&-3638u^8+1292u^9-1166u^3+4176u^7+3251u^4+14u^{11}-32u+378u^6)v^8\\&+(-211u^2-5452u^7+4176u^8+5452u^5-1393u^9-21u^{11}-4284u^4+247u^{10}\\&+21u+1321u^3+72u^6)v^7+(-81u^9-81u^3+9u-39u^2+378u^8+378u^4\\&+72u^7+72u^5-39u^{10}+9u^{11}-594u^6)v^6+(21u^{11}+247u^2-5452u^5+5452u^7\\&-21u+4176u^4+1321u^9+72u^6-1393u^3-211u^{10}-4284u^8)v^5+(-3638u^4\\&+14u+3251u^8+378u^6-214u^2-32u^{11}+268u^{10}+1292u^3+4176u^5-4284u^7\\&-1166u^9)v^4+(-128u^{10}-5u+1292u^4-489u^3-1393u^5+1321u^7-1166u^8\\&+8u^{11}+83u^2-81u^6+538u^9)v^3+(8u^{11}-15u^2-211u^7+6u^{10}+83u^3-214u^4\\&+268u^8+u+247u^5-39u^6-128u^9)v^2+(8u^{10}+21u^7-32u^8-5u^3-21u^5\\&+9u^6+u^2-4u^{11}+8u^9+14u^4)v\}+u^{12}=0.\end{aligned}\tag{6.7}$$

Proof. Using the equation (2.6) and (3.15), we arrive at the equation (6.7). □

7 Modular Relations Between $\kappa(q)$, $\mu(q)$ and $\nu(q)$

Theorem 7.1. *If* $k := \kappa(q)$ *and* $u := \nu(q)$, *then*

$$(k^2-1)u^2+2(1+2k-k^2)u+k^2=1.\tag{7.1}$$

Proof. Using the equations (2.5) and (2.6), we arrive at the equation (7.1). □

Theorem 7.2. *If* $u := \nu(q)$ *and* $v := \mu(q)$, *then*

$$u^4v^2+v^2+(1-2v+v^4-2v^2-2v^3)(u^3+u)=2(1-5v^2+v^4)u^2.\tag{7.2}$$

Proof. Using the equations (2.6), (2.7) and (3.1), we arrive at the equation (7.2). □

Theorem 7.3. *If $k := \kappa(q)$ and $v := \mu(q)$, then*

$$(-1+v)k^2 + (v^2+1)k + v^2 = v. \tag{7.3}$$

Proof. Using the equations (2.5), (2.7) and (3.1), we find that

$$(k^2v^2 - kv^2 + v - 1 - k - k^2v)(kv^2 + v^2 - v + k - k^2 + k^2v) = 0. \tag{7.4}$$

By examining the behaviour of the factors of the equation (7.4) near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is zero, whereas the other factors are not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 7.4. *If $k := \kappa(q)$ and $u := \nu(q^2)$, then*

$$(u-1)k^2 + (u^2+1)k + u^2 = u. \tag{7.5}$$

Proof. Using the equations (2.1), (2.2), (2.5) and (2.6), we find that

$$\begin{aligned}&(-u^2k - u + 2ku - k + k^2u)(u^2k^2 - u^2k + u - 1 - k - k^2u)\\ &(u^2k + u^2 - u + k - k^2 + k^2u) = 0.\end{aligned} \tag{7.6}$$

By examining the behaviour of the factors of the equation (7.6) near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the third factor is zero, whereas the other factors are not zero in this neighbourhood. By the Identity Theorem third factor vanishes identically. This completes the proof. □

Theorem 7.5. *If $u := \nu(q)$ and $k := \kappa(q^2)$, then*

$$\begin{aligned}&(k^2-1)^2(1+u^4) + 2(3 + 16k^3 - 22k^2 - 16k + 3k^4)u^2\\ &= 4(k^4 - 2k + 2k^3 + 1 - 6k^2)(u + u^3).\end{aligned} \tag{7.7}$$

Proof. Using the equations (2.1), (2.2), (2.5) and (2.6), we find that

$$\begin{aligned}&(-ku^2 - u + 2ku - k + k^2u)(k^4 - 4k^4u + 6k^4u^2 - 4k^4u^3 + k^4u^4 - 8k^3u^3\\ &+ 32k^3u^2 - 8k^3u - 2k^2u^4 + 24k^2u^3 - 44k^2u^2 + 24k^2u - 2k^2 + 8ku^3 + 1\\ &- 32ku^2 + 8ku - 4u + 6u^2 - 4u^3 + u^4) = 0.\end{aligned} \tag{7.8}$$

By examining the behaviour of the factors of the equation (7.8) near $q = 0$, it can be seen that there is a neighbourhood about the origin, where the second factor is zero, whereas the first factor is not zero in this neighbourhood. By the Identity Theorem second factor vanishes identically. This completes the proof. □

Theorem 7.6. *If $k := \kappa(q)$ and $u := \nu(q^3)$, then*

$$\begin{aligned}&8(8k^5 - 8k^3 - 1 - 2k^6 + 6k^4 - 2k^2 - k^8)(u + u^7) + 4(12(k + k^3 - k^7 - k^5)\\ &- 78k^4 + 7 + 32k^6 + 32k^2 + 7k^8)(u^2 + u^6) + 8(42k^4 - 14k^2 + 32k^7 - 14k^6\\ &- 7 - 7k^8 + 72k^3 - 32k - 72k^5)(u^3 + u^5) + 2(35k^8 + 4k^2 + 560(k^5 - k^3)\\ &+ 50k^4 + 35 + 208(k - k^7) + 4k^6)u^4 = (4k^6 - 1 - 6k^4 + 4k^2 - k^8)(u^8 + 1).\end{aligned} \tag{7.9}$$

Proof. Using the equations (2.5) , (2.6) and (2.19), we arrive at the equation (7.9). □

Theorem 7.7. *If* $k := \kappa(q)$ *and* $u := \nu(q^4)$, *then*

$$(k-1)u^4 - (k^3+k^2-k-1)u^3 - 3(1-k^2)ku^2 + (k^3-k^2-k+1)ku = k^3+k^4. \quad (7.10)$$

Proof. Using the equations (2.5) , (2.6) and (3.6), we arrive at the equation (7.10). □

Theorem 7.8. *If* $u := \nu(q)$ *and* $k := \kappa(q^4)$, *then*

$$\begin{aligned} &8(8k^6 - 2k^7 - 1 - k^8 + 2k - 26k^5 + 8k^2 + 26k^3 + 18k^4)(u + u^7) + 4(68k^6 + 7 \\ &+ 32k^7 + 160k^5 - 160k^3 + 7k^8 - 32k - 534k^4 + 68k^2)(u^2 + u^6) + 8(-264k^6 \\ &- 7 - 264k^2 + 1022k^4 + 78k + 138k^5 - 138k^3 - 7k^8 - 78k^7)(u^3 + u^5) \\ &+ 2(1780(k^2 + k^6) - 6190k^4 + 1920(k^3 - k^5) + 640(k^7 - k) + 35(k^8 + 1))u^4 \\ &+ (1 - 4k^6 + k^8 - 4k^2 + 6k^4)(1 + u^8) = 0. \end{aligned} \quad (7.11)$$

Proof. Using the equations (2.5) , (2.6) and (3.6), we arrive at the equation (7.11). □

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