

ON A CLASS OF TWO-INDEX REAL HERMITE POLYNOMIALS[†]

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Communicated by Som Prakash Goyal

MSC 2010 Classifications: 33E20, 33C47.

Keywords and phrases: Two-index Hermite polynomials, Runge’s addition formula, Generating function, Nielsen’s identity.

Abstract We discuss some basic properties of a class of doubly indexed real Hermite polynomials including recurrence formulae, Runge’s addition formula, generating function and Nielsen’s identity.

1 Introduction

The Burchnell’s operational formula ([2])

$$\left(-\frac{d}{dx} + 2x\right)^m (f) = m! \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{H_{m-k}(x)}{(m-k)!} \frac{d^k}{dx^k} (f), \tag{1.1}$$

where $H_m(x)$ denotes the usual Hermite polynomial ([5, 10])

$$H_m(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} (e^{-x^2}), \tag{1.2}$$

enjoy a number of remarkable properties. It is used by Burchnell [2] to give a direct proof of Nielsen’s identity ([8])

$$H_{m+n}(x) = m!n! \sum_{k=0}^{\min(m,n)} \frac{(-2)^k}{k!} \frac{H_{m-k}(x)}{(m-n)!} \frac{H_{n-k}(x)}{(n-k)!}. \tag{1.3}$$

The special case of (1.1) where $f = 1$, i.e.,

$$H_m(x) = \left(-\frac{d}{dx} + 2x\right)^m \cdot (1). \tag{1.4}$$

can be employed to recover in a easier way the generating function

$$\sum_{m=0}^{+\infty} H_m(x) \frac{t^m}{m!} = \exp(2xt - t^2) \tag{1.5}$$

as well as the Runge addition formula ([9, 7])

$$H_m(x+y) = \left(\frac{1}{2}\right)^{m/2} m! \sum_{k=0}^n \frac{H_k(\sqrt{2}x)}{k!} \frac{H_{m-k}(\sqrt{2}y)}{(m-k)!}. \tag{1.6}$$

Many generalizations of such Hermite polynomials can be found in the literature including multi-index ones [11, 6, 1, 3]. In this paper, we consider the following class of two-index Hermite polynomials of single real variable:

$$H_{m,n}(x) = \left(-\frac{d}{dx} + 2x\right)^m \cdot (x^n), \tag{1.7}$$

and we derive some of their useful properties. More essentially, we discuss the associated recurrence formulae, Runge’s addition formula, generating function and Nielsen’s identity.

[†]The research is partially supported by Hassan II Academy of Science and Technology, Morocco

2 Doubly indexed real Hermite polynomials $H_{m,n}(x)$

By taking $f(x) = x^n$ in (1.1), we obtain

$$H_{m,n}(x) := \left(-\frac{d}{dx} + 2x\right)^m (x^n) \tag{2.1}$$

$$= m!n! \sum_{k=0}^{\min(m,n)} \frac{(-1)^k}{k!} \frac{x^{n-k}}{(n-k)!} \frac{H_{m-k}(x)}{(m-k)!}. \tag{2.2}$$

It follows that $H_{m,n}(x)$ is a polynomial of degree $m + n$, since

$$Q(x) := H_{m,n}(x) - x^n H_m(x)$$

is a polynomial of degree $\deg(Q) \leq n + m - 2$. For the unity of the formulations, we shall define trivially

$$H_{m,n}(x) = 0$$

whenever $m < 0$ or $n < 0$. We call them doubly indexed real Hermite polynomials. Note that $H_{m,0}(x) = H_m(x)$, $H_{0,n}(x) = x^n$ and

$$H_{m,n}(0) = \begin{cases} 0 & m < n \\ (-1)^n \frac{m!}{(m-n)!} H_{m-n}(0) & m \geq n \end{cases}. \tag{2.3}$$

A direct computation using (2.1) gives rise to

$$H_{1,n}(x) = -nx^{n-1} + 2x^{n+1}$$

for every integer $n \geq 1$. Note also that, since $H_1(x) = 2x$, it follows

$$H_{m+1}(x) = \left(-\frac{d}{dx} + 2x\right)^m (H_1(x)) = \left(-\frac{d}{dx} + 2x\right)^m (2x) = 2H_{m,1}(x). \tag{2.4}$$

The first few values of $H_{m,n}$ are given by

$H_{m,n}$	$n = 1$	$n = 2$	$n = 3$
$m = 1$	$-1 + 2x^2 = H_2(x)$	$-2x + 2x^3$	$-3x^2 + 2x^4$
$m = 2$	$-6x + 4x^3 = H_3(x)$	$2 - 10x^2 + 4x^4$	$6x - 14x^3 + 4x^5$
$m = 3$	$6 - 24x^2 + 8x^4 = H_4(x)$	$24x - 36x^3 + 8x^5$	$-6 + 54x^2 - 48x^4 + 8x^6$

From (2.2), one can deduce easily the symmetry formula

$$H_{m,n}(-x) = (-1)^{n+m} H_{m,n}(x), \tag{2.5}$$

so that the $H_{m,n}(x)$ is odd (rep. even) if and only if $n + m$ is odd (resp. even). Furthermore, the Rodrigues formula for $H_{m,n}(x)$ is

$$H_{m,n}(x) = (-1)^m e^{x^2} \frac{d^m}{dx^m} (x^n e^{-x^2}). \tag{2.6}$$

Indeed, this can be proved easily making use of

$$\left(-\frac{d}{dx} + 2x\right)^m \cdot (f) = (-1)^m e^{x^2} \frac{d^m}{dx^m} (e^{-x^2} f). \tag{2.7}$$

Therefore, these polynomials constitute a subclass of the generalized Hermite polynomials

$$H_m^\gamma(x, \alpha, p) := (-1)^m x^{-\alpha} e^{px^\gamma} \frac{d^m}{dx^m} (x^\alpha e^{-px^\gamma}). \tag{2.8}$$

considered by Gould and Hopper in [4]. In fact, we have $H_{m,n}(x) = x^n H_m^2(x, n, 1)$.

Proposition 2.1. *The polynomials $H_{m,n}$; $m, n \geq 1$, satisfy the following recurrence formulae*

$$H'_{m,n}(x) + H_{m+1,n}(x) - 2xH_{m,n}(x) = 0, \tag{2.9}$$

$$H_{m,n}(x) + nH_{m-1,n-1}(x) - 2H_{m-1,n+1}(x) = 0, \tag{2.10}$$

$$H_{m,n}(x) + mH_{m-1,n-1}(x) - xH_{m,n-1}(x) = 0, \tag{2.11}$$

$$(m - n)H_{m-1,n-1}(x) + 2H_{m-1,n+1}(x) + xH_{m,n-1}(x) = 0. \tag{2.12}$$

Proof. The first one follows by writing the derivation operator as

$$\frac{d}{dx} = - \left(-\frac{d}{dx} + 2x \right) + 2x.$$

Indeed, we get

$$\begin{aligned} \frac{d}{dx} (H_{m,n}(x)) &= - \left(-\frac{d}{dx} + 2x \right) H_{m,n}(x) + 2xH_{m,n}(x) \\ &= -H_{m+1,n}(x) + 2xH_{m,n}(x). \end{aligned}$$

For the second one, write $H_{m,n}(x)$ as

$$\begin{aligned} H_{m,n}(x) &= \left(-\frac{d}{dx} + 2x \right)^{m-1} (H_{1,n}(x)) \\ &= \left(-\frac{d}{dx} + 2x \right)^{m-1} (-nx^{n-1} + 2x^{n+1}) \\ &= -nH_{m-1,n-1}(x) + 2H_{m-1,n+1}(x). \end{aligned}$$

To prove (2.11), we use (2.6) combined with Leibnitz formula. Indeed,

$$\begin{aligned} H_{m,n}(x) &= (-1)^m e^{x^2} \frac{d^m}{dx^m} (x \cdot x^{n-1} e^{-x^2}) \\ &= (-1)^m e^{x^2} \left[x \frac{d^m}{dx^m} (x^{n-1} e^{-x^2}) + m \frac{d^{m-1}}{dx^{m-1}} (x^{n-1} e^{-x^2}) \right] \\ &= xH_{m,n-1}(x) - mH_{m-1,n-1}(x). \end{aligned}$$

Finally, (2.12) follows from (2.10) and (2.11) by subtraction. □

Remark 2.2. According to (2.4), the (2.11) (corresponding to $n = 1$) leads to the well known recurrence formula $H_{m+1}(x) = 2xH_m(x) - 2mH_{m-1}(x)$ for $H_m(x)$. Note also that (2.9) reduces further to $H'_m(x) + H_{m+1}(x) - 2xH_m(x) = 0$ by taking $n = 0$, so that we recover the known result that $H'_m(x) = 2mH_{m-1}(x)$.

Proposition 2.3. *We have the following addition formula*

$$H_{m,n}(x + y) = m!n! \left(\frac{1}{\sqrt{2}} \right)^{m+n} \sum_{k=0}^m \sum_{j=0}^n \frac{H_{k,j}(\sqrt{2}x)}{k!j!} \frac{H_{m-k,n-j}(\sqrt{2}y)}{(m-k)!(n-j)!}. \tag{2.13}$$

Proof. We begin by writing have $H_{m,n}(x + y)$ as

$$\begin{aligned} H_{m,n}(x + y) &= \left(-\frac{d}{d(x+y)} + 2(x+y) \right)^m \cdot ((x+y)^n) \\ &= \left(-\frac{1}{2} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + 2(x+y) \right)^m \cdot ((x+y)^n) \\ &= \left(\frac{1}{\sqrt{2}} \right)^m (A_x + A_y)^m \cdot ((x+y)^n) \\ &= \left(\frac{1}{\sqrt{2}} \right)^m \sum_{j=0}^n \binom{n}{j} (A_x + A_y)^m \cdot (x^j y^{n-j}), \end{aligned}$$

where A_t stands for $A_t = -\partial/(\partial\sqrt{2}t) + 2\sqrt{2}t$. Thus, since A_x and A_y commute, we can make use of the binomial formula to get

$$H_{m,n}(x + y) = \left(\frac{1}{\sqrt{2}} \right)^m \sum_{k=0}^m \sum_{j=0}^n \binom{m}{k} \binom{n}{j} A_x^k \cdot (x^j) A_y^{m-k} \cdot (y^{n-j}),$$

whence, we obtain the asserted result according to the fact that

$$A_t^r (t^s) = 2^{-s/2} H_{r,s}(\sqrt{2}t).$$

□

Remark 2.4. We recover the Runge addition formula (1.6) for the classical real Hermite polynomials $H_m(x) = H_{m,0}(x)$ by taking $n = 0$ in (2.13).

The following identities are immediate consequence of the previous proposition.

Corollary 2.5. *The identity*

$$H_{m,n}(t) = m!n! \left(\frac{1}{\sqrt{2}}\right)^{m+n} \sum_{j=0}^n \sum_{k=j}^m \frac{(-1)^j}{j!(k-j)!} H_{k-j}(0) \frac{H_{m-k,n-j}(\sqrt{2}t)}{(m-k)!(n-j)!}$$

holds by taking $x = 0$ and setting $t = y$ in (2.13), keeping in mind (2.3). We get also

$$H_{m,n}(t) = m!n! \left(\frac{1}{\sqrt{2}}\right)^{m+n} \sum_{k=0}^m \sum_{j=0}^n \frac{H_{k,j}(t/\sqrt{2})}{k!j!} \frac{H_{m-k,n-j}(t/\sqrt{2})}{(m-k)!(n-j)!}$$

by setting $x = y = t/2$ in (2.13). While for $t = -\sqrt{2}x = \sqrt{2}y$, we obtain

$$\sum_{k=0}^m \sum_{j=0}^n (-1)^{k+j} \frac{H_{k,j}(t)}{k!j!} \frac{H_{m-k,n-j}(t)}{(m-k)!(n-j)!} = 0$$

whenever $m + n$ is odd or $m > n$.

Next, we state the following

Proposition 2.6. *The generating function of $H_{m,n}$ is given by*

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x) \frac{u^m}{m!} \frac{v^n}{n!} = \exp(-u^2 + (2u + v)x - uv). \tag{2.14}$$

Proof. According to the definition of $H_{m,n}$, we can write

$$\begin{aligned} \sum_{m,n=0}^{+\infty} H_{m,n}(x) \frac{u^m}{m!} \frac{v^n}{n!} &= \left[\sum_{m=0}^{+\infty} \frac{1}{m!} \left(-u \frac{d}{dx} + 2ux\right)^m \right] \cdot \left(\sum_{n=0}^{+\infty} \frac{v^n}{n!} x^n \right) \\ &= \exp\left(-u \frac{d}{dx} + 2ux\right) (e^{vx}). \end{aligned}$$

Making use of the Weyl identity which reads for the operators $A = 2xId$ et $B = -d/dx$ as

$$\exp(uA + uB) = \exp(uA) \exp(uB) \exp(-u^2 Id); \quad u \in \mathbb{R},$$

we get

$$\sum_{m,n=0}^{+\infty} H_{m,n}(x) \frac{u^m}{m!} \frac{v^n}{n!} = e^{2ux-u^2} \exp\left(-u \frac{d}{dx}\right) (e^{vx}).$$

Therefore, the desired result follows since

$$\exp\left(-u \frac{d}{dx}\right) (e^{vx}) = \sum_{k=0}^{\infty} \frac{(-u)^k}{k!} \left(\frac{d}{dx}\right)^k (e^{vx}) = e^{-uv} e^{vx}.$$

□

Remark 2.7. The special case of $v = 0$ (in (2.14)) infers the generating function (1.5) of the standard real Hermite polynomials H_m . Furthermore, for $y = u = -v$, we get

$$e^{xy} = \sum_{m,n=0}^{+\infty} (-1)^n H_{m,n}(x) \frac{y^{m+n}}{m!n!}. \tag{2.15}$$

Proposition 2.8. *We have the recurrence formula*

$$H'_{m,n}(x) = 2mH_{m-1,n}(x) + nH_{m,n-1}(x). \tag{2.16}$$

Proof. Differentiating the both sides of (2.14) and making appropriate changes of indices yield (2.16). \square

Corollary 2.9. *We have*

$$\frac{d^\nu}{dx^\nu} (H_{r,n}(x)) = r!n! \sum_{j=0}^\nu \alpha_{j,\nu} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!}, \tag{2.17}$$

where

$$\alpha_{j,\nu} = \begin{cases} 2^\nu & \text{for } j = 0 \\ 2\alpha_{j,\nu-1} + \alpha_{j-1,\nu-1} & \text{for } 1 \leq j < \nu \\ 1 & \text{for } j = \nu \end{cases} .$$

Proof. This can be handled by mathematical induction using (2.16). \square

Remark 2.10. The $\alpha_{j,\nu}$ are even positive numbers and their first values are

$\alpha_{j,\nu}$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$\nu = 0$	1					
$\nu = 1$	2	1				
$\nu = 2$	2 ²	4	1			
$\nu = 3$	2 ³	12	6	1		
$\nu = 4$	2 ⁴	32	24	8	1	
$\nu = 5$	2 ⁵	80	80	40	10	1

We conclude this paper by giving a formula for the two-index Hermite polynomial $H_{m,n}(x)$ expressing it as a weighted sum of a product of the same polynomials. Namely, we state the following

Proposition 2.11. *Keep notation as above. Then the Nielsen identity for $H_{m,n}$; $n \geq 1$, reads*

$$H_{m+r,n}(x) = m!r!nn! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)! \nu!} \frac{(-x)^\nu}{x^{n+k}} \frac{H_{m-k,n}(x)}{(m-k)!n!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!} .$$

Proof. Recall first that $H_m^\gamma(x, \alpha, p)$, the polynomials given through (2.8), can be rewritten in the following equivalent form ([4])

$$H_m^\gamma(x, \alpha, p) := \left(-\frac{d}{dx} + p\gamma x^{\gamma-1} - \frac{\alpha}{x} \right)^m (1). \tag{1}$$

Now, since for the special values $p = 1, \gamma = 2$ and $\alpha = n$, we have

$$\begin{aligned} H_{m+r,n}(x) &= x^n H_{m+r}^2(x, n, 1) \\ &= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x} \right)^m (H_r^2(x, n, 1)) \\ &= x^n \left(-\frac{d}{dx} + 2x - \frac{n}{x} \right)^m (x^{-n} H_{r,n}(x)), \end{aligned}$$

we can make use of the Burchnall’s formula extension proved by Gould and Hopper [4], to wit

$$\left(-\frac{d}{dx} + p\gamma x^{\gamma-1} - \frac{\alpha}{x} \right)^m (f) = m! \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{H_{m-k}^\gamma(x, \alpha, p)}{(m-k)!} \frac{d^k}{dx^k} (f).$$

Thus, for $f = x^{-n} H_{r,n}$, we obtain

$$H_{m+r,n}(x) = m! \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \frac{d^k}{dx^k} (x^{-n} H_{r,n}(x)). \tag{2.18}$$

Therefore, by applying the Leibnitz formula and appealing the result of Corollary 2.9, we get

$$\begin{aligned} H_{m+r,n}(x) &= m! \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{H_{m-k,n}(x)}{(m-k)!} \sum_{\nu=0}^k \binom{k}{\nu} \frac{d^{k-\nu}}{dx^{k-\nu}}(x^{-n}) \frac{d^\nu}{dx^\nu}(H_{r,n}(x)) \\ &= m! r! n n! \sum_{k,\nu,j=0}^{m,k,\nu} \alpha_{j,\nu} \frac{\Gamma(n+k-\nu)}{(k-\nu)! \nu!} \frac{(-x)^\nu}{x^{n+k}} \frac{H_{m-k,n}(x)}{(m-k)! n!} \frac{H_{r-\nu+j,n-j}(x)}{(r-\nu+j)!(n-j)!} \end{aligned}$$

for every integer $n \geq 1$. Note that for $n = 0$, (2.18) reads simply

$$H_{m+r}(x) = m! \sum_{k=0}^m \frac{(-1)^k}{k!} \frac{H_{m-k}(x)}{(m-k)!} \frac{d^k}{dx^k}(H_r(x)).$$

In this case, we recover the usual Nielsen formula (1.3) for the real Hermite polynomials H_m . \square

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Received: May 22, 2013

Accepted: June 12, 2013