Palestine Journal of Mathematics

Vol. 3(Spec 1) (2014) , 374–382

A Study On Semi-projective Covers, Semi-projective Modules and Formal Triangular Matrix Rings

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Dedicated to Patrick Smith and John Clark on the occasion of their 70th birthdays.

Communicated by Ayman Badawi

MSC 2010 Classifications: 16D40, 16D80.

Keywords and phrases: semi-projective module, (semi-)perfect ring.

We would like to express our special thanks to Prof.Dr. Pedro A. Guil Asensio and Prof.Dr. Ashish Srivastava for their valuable comments on this paper. We also thank them and Prof.Dr. Huanyin Chen providing us the M.Sc. Thesis [4] and Xiangrui Li for translating some Chinese statements in 4.1.1, 4.1.3 and 4.1.4 in English in [4].

Abstract. We show that a ring R is right perfect if and only if every right R-module has a semi-projective cover. We characterize (semi)hereditary and semisimple rings via semi-projective modules. Finally we investigate the relative projectivity of modules over a formal triangular matrix ring $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. We also prove that if a right T-module $(X \oplus Y)_T$ is lifting, then $(X/YM)_A$ and Y_B are lifting.

1 Introduction

In what follows R will always denote an associative ring with identity and modules will always be taken as unitary right R-modules. A module M has a projective cover P, if there is an epimorphism $f : P \longrightarrow M$ such that P is projective and Ker$f$ is small in P. A ring R is called *right perfect* if every right R-module has a projective cover. Perfect rings were characterized by H. Bass in [1]. In 1967, Wu and Jans introduced the quasi-projective cover as follows in [23]: The module P is called a *quasi-projective cover* of a module M if, there exists an epimorphism $f : P \longrightarrow M$ such that (1) P is quasi-projective (2) Ker$f$ is small in P (3) if $0 \neq B \subseteq$ Ker$f$, then P/B is not quasi-projective. Note that as projective covers, quasi-projective covers of a module need not exist. For example, the $\mathbb{Z}$-module $M = \oplus_k \mathbb{Z}/p^k\mathbb{Z}$ does not have a quasi-projective cover (see [6, Example 4]). Also, it is not known whether quasi-projective cover of a module (if it exists) is unique up to isomorphism. Wu and Jans proved in [23, Proposition 2.6] that when the projective cover $f : P \longrightarrow M$ exists, then the quasi-projective cover of M exists and is unique. This quasi-projective cover is given by the induced map $f' : P/T \longrightarrow M$, where T is the largest fully invariant submodule of P contained in Ker$f$.

In 1970; K.R. Fuller and D.A. Hill [5, Theorem 4.1], J. Golan [7, Theorem 3.1] and A. Koehler [19, Corollary 1.2] proved that (the condition (3) is not needed) a ring R is right perfect if every right R-module has a quasi-projective cover and they also investigated semiperfect rings via quasi-projective covers of finitely generated modules. After that, in 1983, T.G. Faticoni studied quasi-projective covers in [6] and in 1996, W. Xue defined the locally projective cover (without the condition (3)) and proved that a ring R is right perfect if and only if every right R-module has a locally projective cover in [25, Theorem 3.10]; he also investigated semiperfect rings via locally projective covers.

In this paper firstly we define semi-projective covers and investigate right perfect rings. Let M be a module. M is called *semi-projective* if, for all endomorphisms α and β of M with $\beta(M) \subseteq \alpha(M)$ there exists an endomorphism γ of M such that $\beta = \alpha\gamma$ (see, [2], [18], [21] and [22]). An R-module M is called *direct projective* if for every direct summand K of M every epimorphism from M to K splits. Note that we have the following hierarchy:

$$\text{projective} \Rightarrow \text{quasi-projective} \Rightarrow \text{semi-projective} \Rightarrow \text{direct projective}.$$

We say that a module P is a *semi-projective cover* of any module M if, there exists an epimorphism $f : P \longrightarrow M$ such that P is semi-projective and Ker$f$ is small in P. According to our definition, there may be a nonzero submodule B of P contained in Ker$f$ with P/B semi-projective, where $f : P \longrightarrow M$ is a semi-projective cover. Then P/B is another semi-projective cover of M. Clearly P/B and P are not isomorphic to each other. Therefore semi-projective covers may not be unique up to isomorphism in the sense of our definition. Clearly, every (quasi-)projective cover is a semi-projective cover. On the other hand, since the $\mathbb{Z}$-module $\mathbb{Q}$ is semi-projective (see, for example, [18, Corollary 2.6]), $\mathbb{Q}_{\mathbb{Z}}$ is a semi-projective cover of itself and of the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$. In this paper we obtain that a ring R is right perfect if and only if every right R-module has a semi-projective cover and R is semiperfect if and only if every finitely generated right (left) R-module has a semi-projective cover. We also observe that a ring R is semisimple if and only if every (finitely generated) right R-module is semi-projective. As a last work, we study lifting modules and the relative projectivity of modules over a formal triangular matrix ring $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$. We prove that if a right T-module $(X \oplus Y)_T$ is lifting, then $(X/YM)_A$ and Y_B are lifting. We also prove that if a right T-module $(X \oplus Y)_T$ is quasi-projective, then $(X/YM)_A$ and Y_B are quasi-projective and if a right T-module $(X \oplus Y)_T$ has a quasi-projective cover, then $(X/YM)_A$ and Y_B have semi-projective covers.

2 Semi-projective Modules and Semi-projective Covers

In this part of the paper, we give some characterizations of semiperfect, perfect, semihereditary, hereditary and semisimple rings using semi-projective modules and semi-projective covers. This characterizations have been completely inspired by the earlier related studies from [1], [5]-[9], [19] and [23]-[25].

The following theorem is an analogue of [8, Theorem 2.2] and the proof follows the same pattern. We give it here for the convenience of the readers.

Theorem 2.1. Let M be a module and let $f : P \longrightarrow M$ be an epimorphism with P projective. Then

(i) M is projective if and only if $P \oplus M$ is semi-projective.

(ii) M has a projective cover if and only if $P \oplus M$ has a semi-projective cover.

Proof. (1) Assume M is projective. Then clearly $P \oplus M$ is semi-projective. Conversely assume that $P \oplus M$ is semi-projective. Then the epimorphism f splits (see [18, Lemma 2.8]). Thus M is projective.

(2) The necessity is clear. For the sufficiency we will use the Koehler's technique in [19, Theorem 1.1]. Consider the right R-module $X = P \oplus M$. By hypothesis, there exists an epimorphism $g : Q \longrightarrow X$ such that Q is semi-projective and Ker$g$ is small in Q. Let π be the projection map from X to P. Since P is projective, there is a monomorphism $\alpha : P \longrightarrow Q$ such that $\pi g \alpha = 1_P$ and $Q = \mathrm{Im}\alpha \oplus \mathrm{Ker}(\pi g)$. Let $\overline{M} = \mathrm{Ker}(\pi g)$ and $g_1 = g \mid_{\overline{M}}$. Then we can assume $Q = P \oplus \overline{M}$. Note that $g_1(\overline{M}) = g(\overline{M}) = g(g^{-1}(M)) = M$ implies that g_1 is an epimorphism from $\overline{M}$ to M. Now we will prove that $\overline{M}$ is the projective cover of M with the epimorphism g_1. Since Ker$g$ = Kerg_1, Kerg_1 is small in $\overline{M}$. Since P is projective, there is a homomorphism $f' : P \longrightarrow \overline{M}$ such that $g_1 f' = f$, namely the following diagram is commutative:

$$\begin{tikzcd} & P \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & \\ \overline{M} \arrow[r, "g_1"'] & M \arrow[r] & 0 \end{tikzcd}$$

Since Kerg_1 is small in $\overline{M}$ and f is epic, f' is epic. Therefore by (1), $\overline{M}$ is projective. □

Corollary 2.2. If every (finitely generated) module has a semi-projective cover, then every (finitely generated) module has a projective cover.

Proof. Take into account that every (finitely generated) module is an epimorphic image of a (finitely generated) free, and so projective module. □

Corollary 2.3. (i) A ring R is semiperfect if and only if every finitely generated right (left) R-module has a semi-projective cover.

(ii) A ring R is right perfect if and only if every right R-module has a semi-projective cover.

Now applying the same proof of [8, Theorem 3.1], we get the following, where R_n is the ring of n by n matrices over R:

Theorem 2.4. The following conditions are equivalent for a ring R:

(i) R is semiperfect.

(ii) For all $n \geq 1$, every cyclic right (left) R_n-module has a semi-projective cover.

(iii) There exists an $n > 1$ such that every cyclic right (left) R_n-module has a semi-projective cover.

Golan in [8] proved that a ring R is right hereditary if and only if every submodule of a projective right R-module is quasi-projective if and only if every principal right ideal of End(F) is quasi-projective for any free right R-module F ([8, Theorem 4.4]) and R is right semihereditary if and only if every finitely generated submodule of a projective right R-module is quasi-projective if and only if every principal right ideal of R_n is quasi-projective, for all $n \geq 1$ ([8, Theorem 4.3]). Combining these facts and Theorems 4 and 5 in [24] we have the following two theorems:

Theorem 2.5. The following conditions are equivalent for a ring R:

(i) R is right hereditary.

(ii) Every submodule of a projective right R-module is semi-projective.

(iii) Every principal right ideal of End(F) is semi-projective for any free right R-module F.

Theorem 2.6. The following conditions are equivalent for a ring R:

(i) R is right semihereditary.

(ii) Every finitely generated submodule of a (finitely generated) projective right R-module is semi-projective.

(iii) Every finitely generated (principal) right ideal of R_n is semi-projective for all $n \geq 1$.

Submodules of semi-projective modules need not be semi-projective as the following example shows.

Example 2.7. Let M be the semi-projective $\mathbb{Z}$-module $\mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$, where p is any prime integer. Let N be the submodule $p\mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$. Since the epimorphism $f : \mathbb{Z}/p^3\mathbb{Z} \longrightarrow p\mathbb{Z}/p^3\mathbb{Z}$ defined by $f(x + p^3\mathbb{Z}) = px + p^3\mathbb{Z}$ does not split, N is not semi-projective.

Using Theorems 2.5 and 2.6, we can give the following theorem which is an analogue of [8, Theorem 5.1].

Theorem 2.8. Let R be a ring. If every (finitely generated) submodule of a semi-projective right R-module is semi-projective, then every factor ring of R is right (semi)hereditary.

The arguments in [9, Proposition 2.2, Theorem C, Corollary 2.4, Lemma 2.5, Corollary 2.6 and Theorem D] may be adapted to obtain the following useful results.

Proposition 2.9. Let R be a ring. If every submodule of a semi-projective right R-module is semi-projective and H is a right T-nilpotent two-sided ideal of R, then $H^2 = 0$.

Recall that the singular submodule $Z(M)$ of a module M is given by $Z(M) = \{m \in M \mid ann_R(m)$ is an essential right ideal of $R\}$. M is called *singular* if $Z(M) = M$ and *nonsingular* if $Z(M) = 0$.

Theorem 2.10. If R is right perfect and every submodule of a semi-projective right R-module is semi-projective, then every singular right R-module is injective.

Corollary 2.11. If R is right perfect and every submodule of a semi-projective right R-modules is semi-projective, then $Z(M)$ is a direct summand of M for every right R-module M.

Lemma 2.12. Let R be a left perfect ring. Assume that every finitely generated submodule of a semi-projective right R-module is semi-projective. If e and f are idempotents of R with eR and fR indecomposables, and eRf and fRe nonzero, then $eR \cong fR$ and in fact this isomorphism is given by left multiplication by any nonzero element of eRf or fRe.

Corollary 2.13. Let R be a left perfect ring. Assume that every finitely generated submodule of a semi-projective right R-module is semi-projective and e is an idempotent of R with eR indecomposable. Then eRe is a division ring.

Theorem 2.14. If R is left perfect and every finitely generated submodule of a semi-projective right R-module is semi-projective, then R has a decomposition $R = S \oplus J(R)$ over $\mathbb{Z}$, where S is a semisimple subring of R containing 1.

Finally, using the same proof of Theorem 7 in [24], we can give the following result which generalizes Theorem 7 in [24].

Theorem 2.15. If R is a ring over which submodules of Σ-semi-projective modules are direct-projective, then every factor ring of R is right hereditary.

Theorem 2.16. The following conditions are equivalent for a ring R:

(i) R is semisimple.

(ii) Every (finitely generated) right R-module is semi-projective.

(iii) Every 2-generated right R-module is semi-projective.

(iv) The direct sum of two semi-projective right R-modules is semi-projective.

(v) The direct sum of two quasi-projective right R-modules is semi-projective.

(vi) For all $n \geq 1$, every cyclic right R_n-module is semi-projective.

(vii) There exists some $n > 1$ such that every cyclic right R_n-module is semi-projective.

Proof. By [24, Theorem 9]. □

3 Some study of modules over formal triangular matrix rings

This section is devoted to the study of modules over formal triangular matrix rings and the results focus on relative projectivity and lifting properties of modules. This part has been partly inspired by the earlier related studies of modules over formal triangular matrix rings in [3] and [10]-[17].

Given a formal triangular matrix $T = \begin{bmatrix} A & 0 \\ M & B \end{bmatrix}$ it is well known that ([10]) the category Mod-T and a category Ω of triples $(X,Y)_f$ are equivalent where $X \in$ Mod-A, $Y \in$ Mod-B and $f : Y \otimes M \longrightarrow X$ is a homomorphism in Mod-A. If $(X,Y)_f$ and $(U,V)_g$ are two objects in Ω, then the morphisms from $(X,Y)_f$ to $(U,V)_g$ in Ω are pairs (φ_1, φ_2) where $\varphi_1 : X \longrightarrow U$ is an A-homomorphism, $\varphi_2 : Y \longrightarrow V$ is a B-homomorphism satisfying the condition $\varphi_1 f = g(\varphi_2 \otimes 1_M)$. The right T-module corresponding to the triple $(X,Y)_f$ is the additive group $X \oplus Y$ with the right action given by

$$(x,y)\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} = (xa + f(y \otimes m), yb).$$

Then we write $(X \oplus Y)_T$ for this right T-module. Furthermore, if $(\varphi_1, \varphi_2) : (X,Y)_f \longrightarrow (U,V)_g$ is a map in Ω, the associated T-homomorphism $\varphi : (X \oplus Y)_T \longrightarrow (U \oplus V)_T$ is given by $\varphi(x,y) = (\varphi_1(x), \varphi_2(y))$ for any $x \in X$ and $y \in Y$. It is clear that φ is injective (resp. surjective) if and only if $\varphi_1 : X \longrightarrow U$, $\varphi_2 : Y \longrightarrow V$ are injective (resp. surjective). It is convenient to view such triples as T-modules and the morphisms between them as T-homomorphisms. Here we should note that the T-module T_T corresponds to $(A \oplus M, B)_f$, where f is the A-homomorphism $B \otimes M \longrightarrow A \oplus M$ given by $f(b \otimes m) = (0, bm)$.

Let $(X,Y)_f \in Obj(\Omega)$ and $(X \oplus Y)_T$ be the associated right T-module. Under the right T-action on $X \oplus Y$ we have $(0 \oplus Y)\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f(Y \otimes M), 0)$. In general the submodule $f(Y \otimes M)$ of X_A is denoted by YM. Now consider $Y' \leq Y_B$ and let $j_2 : Y' \longrightarrow Y$ denote the inclusion map. Then $(0 \oplus Y')\begin{bmatrix} 0 & 0 \\ M & 0 \end{bmatrix} = (f(j_2 \otimes 1_M)(Y' \otimes M), 0)$. In general, the submodule $f(j_2 \otimes 1_M)(Y' \otimes M)$ of X_A is denoted by $Y'M$. Let $X' \leq X_A$ satisfy $Y'M \subseteq X'$. Writing f' for $f(j_2 \otimes 1_M)$ and denoting the inclusion $X' \longrightarrow X$ by j_1 we see that $(X', Y')_{f'} \in Obj(\Omega)$ and $(j_1, j_2) : (X', Y')_{f'} \longrightarrow (X,Y)_f$ is a map in Ω realizing $(X' \oplus Y')_T$ as a T-submodule of $(X \oplus Y)_T$. Therefore when we take a submodule $(X' \oplus Y')_T$ of $(X \oplus Y)_T$ we have $X' \leq X_A$, $Y' \leq Y_B$, $f(j_2 \otimes 1_M)(Y' \otimes M) \leq X'$. The map $f' : Y' \otimes M \longrightarrow X'$ is completely determined; it has to be $f(j_2 \otimes 1_M)$. Let X'' (resp. Y'') be a quotient of X_A (resp. Y_B) with $\eta_1 : X \longrightarrow X''$

(resp. $\eta_2 : Y \longrightarrow Y''$) the canonical maps. Let Ker$\eta_1 = X'$ and Ker$\eta_2 = Y'$. Assume that $Y'M \subseteq X'$. Let $j_1 : X' \longrightarrow X$, $j_2 : Y' \longrightarrow Y$ be the inclusion maps. Clearly we have the A-homomorphism $f'' : Y'' \otimes M \longrightarrow X''$ rendering the following diagram commutative

$$\begin{tikzcd}
Y' \otimes M \arrow[r, "j_2 \otimes 1_M"] \arrow[d, "f'"] & Y \otimes M \arrow[r, "\eta_2 \otimes 1_M"] \arrow[d, "f"] & Y'' \otimes M \arrow[r] \arrow[d, "f''"] & 0 \\
X' \arrow[r, "j_1"] & X \arrow[r, "\eta_1"] & X'' \arrow[r] & 0
\end{tikzcd}$$

In this diagram $f' = f(j_2 \otimes 1_M)$ and the rows are exact. Also it is clear that $(\eta_1, \eta_2) : (X,Y)_f \longrightarrow (X'', Y'')_{f''}$ is a map in Ω realizing $(X'' \oplus Y'')_T$ as a quotient of $(X \oplus Y)_T$. The kernel of the associated T-homomorphism $\eta : (X \oplus Y)_T \longrightarrow (X'' \oplus Y'')_T$ is precisely $(X' \oplus Y')_T$. Now when we deal with a quotient $(X'' \oplus Y'')_T$ of $(X \oplus Y)_T$ the A-homomorphism $f'' : Y'' \otimes M \longrightarrow X''$ is completely determined. The above backgrounds were taken from [11] and [13]. For more details on formal triangular matrix rings we refer to [11]-[17].

As an easy observation we can give the following:

Proposition 3.1. $(X' \oplus Y')_T$ is a direct summand of $(X \oplus Y)_T$ if and only if $X_A = X' \oplus X''$, $Y_B = Y' \oplus Y''$ with $f(j_2' \otimes 1_M) = f'$, $f(Y' \otimes M) \subseteq X'$ and $f(j_2'' \otimes 1_M) = f''$, $f(Y'' \otimes M) \subseteq X''$ where $j_2' : Y' \longrightarrow Y$, $j_2'' : Y'' \longrightarrow Y$ are the inclusion maps.

Any module N is called *lifting* if for any submodule H of N, there exists a decomposition $N = N_1 \oplus N_2$ such that $N_1 \subseteq H$ and $N_2 \cap H$ is small in N_2. Now we will give a characterization of lifting modules over the ring T:

Theorem 3.2. If the right T-module $(X \oplus Y)_T$ determined by $(X,Y)_f$ is lifting, then $(X/YM)_A$ and Y_B are lifting.

Proof. Assume that $(X \oplus Y)_T$ is lifting. Let $Y' \leq Y_B$. Consider the submodule $(X \oplus Y')_T$ of $(X \oplus Y)_T$ with the A-homomorphism $f' = f(j_2 \otimes 1_M)$ such that $j_2 : Y' \longrightarrow Y$ is the inclusion map. Since $(X \oplus Y)_T$ is lifting, there exists a decomposition $(X \oplus Y)_T = (H' \oplus K')_T \oplus (H'' \oplus K'')_T$ such that $(H' \oplus K')_T \subseteq (X \oplus Y')_T$ and $(H'' \oplus K'')_T \cap (X \oplus Y')_T = (H'' \oplus (K'' \cap Y'))_T \ll (H'' \oplus K'')_T$. Assume that $(H'' \oplus K'')_T$ and $(H' \oplus K')_T$ associate with the objects $(H'', K'')_{f''}$ and $(H', K')_{f'}$ in Ω such that $f' = f(j_2' \otimes 1_M)$, $f'' = f(j_2'' \otimes 1_M)$, where $j_2' : K' \longrightarrow Y$ and $j_2'' : K'' \longrightarrow Y$ are the inclusion maps and $f(K' \otimes M) \subseteq H'$ and $f(K'' \otimes M) \subseteq H''$. Now we have that $X = H' \oplus H''$, $Y = K' \oplus K''$, $K' \leq Y'$. By [11, Proposition 1.3], $K'' \cap Y' \ll K''$ and $H'' = f(K'' \otimes M)$. Thus Y_B is lifting.

Now let X'/YM be an A-submodule of X/YM. Then $(X' \oplus Y)_T$ is a submodule of $(X \oplus Y)_T$. Then there is a decomposition $(X \oplus Y)_T = (L_1 \oplus K_1)_T \oplus (L_2 \oplus K_2)_T$ such that $(L_1 \oplus K_1)_T \subseteq (X' \oplus Y)_T$, $(L_2 \oplus K_2)_T \cap (X' \oplus Y)_T = ((L_2 \cap X') \oplus K_2)_T \ll (L_2 \oplus K_2)_T$. Now $X = L_1 \oplus L_2$ and $Y = K_1 \oplus K_2$. By [11, Proposition 1.3], $K_2 = 0$ and $L_2 \cap X' \ll L_2$ ($f(K_2 \otimes M) = 0$). Then $X/YM = L_1/YM \oplus (L_2 \oplus YM)/YM$, $L_1/YM \subseteq X'/YM$ and $[X' \cap (L_2 \oplus YM)]/YM = [YM \oplus (X' \cap L_2)]/YM \ll (L_2 \oplus YM)/YM$. Thus $(X/YM)_A$ is lifting. □

Now we can give a part of a well-known fact in the following:

Corollary 3.3. If T is a generalized uniserial ring with $J(T)^2 = 0$, then B is a generalized uniserial ring with $J(B)^2 = 0$.

Proof. By Theorem 3.2 and [20, Corollary 2.5]. □

Example 3.4. Let R be a ring and M a right R-module. Let $T = \begin{bmatrix} R & 0 \\ M & \mathbb{Z} \end{bmatrix}$. Consider the right T-module $V_T = (M \oplus \mathbb{Z})_T$ associated to the triple $(M, \mathbb{Z})_f$ where $f : \mathbb{Z} \otimes M \longrightarrow M$ defined by $n \otimes m \mapsto nm$ for all $n \in \mathbb{Z}$ and $m \in M$. Since $\mathbb{Z}$ is not lifting, V_T is not lifting.

Let $V_T = (X \oplus Y)_T$ be a right T-module corresponds to $(X,Y)_f$ in Ω. Then we can define the following B-homomorphism:
$\tilde{f} : Y \longrightarrow \mathrm{Hom}(M, X)$ given by $\tilde{f}(y)(m) = f(y \otimes m)$ for $y \in Y$, $m \in M$.
If the right T-module $V_T = (X \oplus Y)_T$ corresponds to $(X,Y)_f$ in Ω and $(X' \oplus Y')_T$ is a submodule of $(X \oplus Y)_T$ with the homomorphism $f' = f(j_2 \otimes 1_M)$ such that $j_2 : Y' \longrightarrow Y$ is

the inclusion map and $Y'M \subseteq X'$, then we will have the B-homomorphism:
$\tilde{f}_{|Y'} : Y' \longrightarrow \mathrm{Hom}(M, X')$ given by $\tilde{f}_{|Y'}(y')(m) = f(y' \otimes m)$ for $y' \in Y'$, $m \in M$.

Haghany and Varadarajan give the complete description of the projective right T-modules in [11, Theorem 3.1]. Also in [12] Haghany and in [3] Chen and Zang investigate the relatively injectivity of right T-modules. Now we investigate the relatively projectivity of right T-modules in the following two theorems.

Theorem 3.5. Let V_1 and V_2 be two right T-modules with $(X_1, Y_1)_{f_1}$, $(X_2, Y_2)_{f_2}$ the corresponding triples. If X_2 is X_1-projective in Mod-A and $\tilde{f}_{1|Y_1'}$ is an isomorphism for every submodule $(X_1' \oplus Y_1')_T$ of V_1, then V_2 is V_1-projective in Mod-T.

Proof. Take a quotient $V_1'' = (X_1'', Y_1'')_{f_1''}$ of V_1. Then $X_1'' = X_1/X_1', Y_1'' = Y_1/Y_1'$, $\eta_1 : X_1 \longrightarrow X_1''$ and $\eta_2 : Y_1 \longrightarrow Y_1''$ are the natural epimorphisms, $(X_1', Y_1')_{f_1'}$ is a submodule of V_1 with the homomorphism $f_1' = f_1(j_2' \otimes 1_M)$ ($j_2' : Y_1' \longrightarrow Y$ is the inclusion map) and $f_1'' : Y_1'' \otimes M \longrightarrow X_1''$ is the A-homomorphism which makes the following diagram commutative:

$$\begin{tikzcd}
Y_1' \otimes M \arrow[r, "j_2' \otimes 1_M"] \arrow[d, "f_1'"] & Y_1 \otimes M \arrow[r, "\eta_2 \otimes 1_M"] \arrow[d, "f_1"] & Y_1'' \otimes M \arrow[r] \arrow[d, "f_1''"] & 0 \\
X_1' \arrow[r, "j_1'"] & X_1 \arrow[r, "\eta_1"] & X_1'' \arrow[r] & 0
\end{tikzcd}$$

where $j_1' : X_1' \longrightarrow X_1$ is the inclusion map. Now the corresponding natural T-homomorphism η from V_1 to V_1'' is the map (η_1, η_2). Let $\sigma : V_2 \longrightarrow V_1''$ be any T-homomorphism . Then σ corresponds to the pair (σ_1, σ_2) such that $\sigma_1 : X_2 \longrightarrow X_1''$ is an A-homomorphism, $\sigma_2 : Y_2 \longrightarrow Y_1''$ is a B-homomorphism and $\sigma_1 f_2 = f_1''(\sigma_2 \otimes 1_M)$ and $\sigma(x_2, y_2) = (\sigma_1(x_2), \sigma_2(y_2))$. Since X_2 is X_1-projective, there exists an A-homomorphism $\overline{\sigma_1} : X_2 \longrightarrow X_1$ such that $\eta_1\overline{\sigma_1} = \sigma_1$. Now we want to define a B-homomorphism $\overline{\sigma_2} : Y_2 \longrightarrow Y_1$ such that the pair $(\overline{\sigma_1}, \overline{\sigma_2})$ lifts σ with the corresponding T-homomorphism $\overline{\sigma}$. Take any element $y_2 \in Y_2$. Then we can define a homomorphism $\theta : M \longrightarrow X_1$ with $\theta(m) = \overline{\sigma_1} f_2(y_2 \otimes m)$. Since $\tilde{f}_1$ is an isomorphism, there exists a unique $y_1 \in Y_1$ such that $\tilde{f}_1(y_1) = \theta$. Now let $\overline{\sigma_2}(y_2) = y_1$. Clearly $\overline{\sigma_2}$ is a B-homomorphism. Let $y_2 \in Y_2$ and $m \in M$. Then $f_1(\overline{\sigma_2} \otimes 1_M)(y_2 \otimes m) = f_1(\overline{\sigma_2}(y_2) \otimes m) = f_1(y_1 \otimes m) = \tilde{f}_1(y_1)(m) = \theta(m) = \overline{\sigma_1} f_2(y_2 \otimes m)$, where $\overline{\sigma_2}(y_2) = y_1$ and $\tilde{f}_1(y_1) = \theta$. Therefore $f_1(\overline{\sigma_2} \otimes 1_M) = \overline{\sigma_1} f_2$. Thus $(\overline{\sigma_1}, \overline{\sigma_2}) : (X_2, Y_2)_{f_2} \longrightarrow (X_1, Y_1)_{f_1}$ is a morphism in Ω which corresponds to a T-homomorphism $\overline{\sigma} : V_2 \longrightarrow V_1$, namely $\overline{\sigma}(x_2, y_2) = (\overline{\sigma_1}(x_2), \overline{\sigma_2}(y_2))$. Now we should see that $\eta\overline{\sigma} = \sigma$. It is enough to show that $\eta_2\overline{\sigma_2} = \sigma_2$. Let $y_2 \in Y_2$. Since $\sigma_1 f_2 = f_1''(\sigma_2 \otimes 1_M)$, for all $m \in M$, $(\sigma_1 f_2)(y_2 \otimes m) = \sigma_1(f_2(y_2 \otimes m)) = f_1''(\sigma_2(y_2) \otimes m)$, hence $\eta_1\overline{\sigma_1}(f_2(y_2 \otimes m)) = f_1''(\sigma_2(y_2) \otimes m)$. Let $\sigma_2(y_2) = z_1 + Y_1'$ ($z_1 \in Y_1$). On the other hand, $f_1''(\eta_2 \otimes 1_M) = \eta_1 f_1$. Thus, $f_1''((\eta_2 \otimes 1_M)(z_1 \otimes m)) = \eta_1 f_1(z_1 \otimes m) = \eta_1 \tilde{f}_1(z_1)(m) = \eta_1\overline{\sigma_1} f_2(y_2 \otimes m)$, for all $m \in M$. Since $f_1(\overline{\sigma_2} \otimes 1_M) = \overline{\sigma_1} f_2$, $\eta_1\overline{\sigma_1} f_2(y_2 \otimes m) = \eta_1 f_1(\overline{\sigma_2} \otimes 1_M)(y_2 \otimes m) = \eta_1 f_1(\overline{\sigma_2}(y_2) \otimes m) = \eta_1 \tilde{f}_1(\overline{\sigma_2}(y_2))(m)$, for all $m \in M$. Now $\eta_1 \tilde{f}_1(z_1)(m) = \eta_1 \tilde{f}_1(\overline{\sigma_2}(y_2))(m)$, for all $m \in M$. This means that $\tilde{f}_1(z_1 - \overline{\sigma_2}(y_2))$ is an A-homomorphism from M to X_1'. Since $\tilde{f}_{1|Y_1'}$ is an isomorphism, there exists an element $y_1' \in Y_1'$ such that $\tilde{f}_{1|Y_1'}(y_1') = \tilde{f}_1(z_1 - \overline{\sigma_2}(y_2))$ and so $y_1' = z_1 - \overline{\sigma_2}(y_2)$. Thus $\sigma_2(y_2) = \eta_2\overline{\sigma_2}(y_2)$, namely $\sigma_2 = \eta_2\overline{\sigma_2}$. □

Note that in [4, 4.1.1], it is proven that if Y_1 is Y_2-projective and $f_1 : Y_1 \otimes M \longrightarrow X_1$ is an A-isomorphism, then V_1 is V_2-projective. Therefore we deduce that the converse of Theorem 3.5 may not be true. Namely there exist right T-modules V_1 and V_2 such that V_2 is V_1-projective but X_2 is not X_1-projective:

Example 3.6. Let R be a ring and M a right R-module such that ${}_{\mathbb{Z}}M$ is torsion-free which is not quasi-projective. Again let $T = \begin{bmatrix} R & 0 \\ M & \mathbb{Z} \end{bmatrix}$ and consider the right T-module $V_T = (M \oplus \mathbb{Z})_T$ associated to the triple $(M, \mathbb{Z})_f$ where $f : \mathbb{Z} \otimes M \longrightarrow M$ defined by $n \otimes m \mapsto nm$ for all $n \in \mathbb{Z}$ and $m \in M$. Clearly, f is an R-isomorphism. Therefore by [4, 4.1.1], V_T is quasi-projective. On the other hand, M is not quasi-projective.

Theorem 3.7. Let V_1 and V_2 be two right T-modules with $(X_1, Y_1)_{f_1}$, $(X_2, Y_2)_{f_2}$ the corresponding triples. If V_2 is V_1-projective, then Y_2 is Y_1-projective and $X_2/f_2(Y_2 \otimes M)$ is $X_1/f_1(Y_1 \otimes M)$-projective.

Proof. Let $\eta_1 : Y_1 \longrightarrow Y_1/K_1$ be the natural epimorphism and $\alpha_1 : Y_2 \longrightarrow Y_1/K_1$ be any B-homomorphism, where $K_1 \leq Y_1$. Then we can construct the quotient $(0 \oplus Y_1/K_1)_T$ of $(X_1 \oplus Y_1)_T$ with the following commutative diagram:

$$\begin{tikzcd}
K_1 \otimes M \arrow[r, "j_1 \otimes 1_M"] \arrow[d, "f_1'"] & Y_1 \otimes M \arrow[r, "\eta_1 \otimes 1_M"] \arrow[d, "f_1"] & Y_1/K_1 \otimes M \arrow[r] \arrow[d, "0"] & 0 \\
X_1 \arrow[r, "1"] & X_1 \arrow[r, "0"] & 0 \arrow[r] & 0
\end{tikzcd}$$

Now we can construct those morphisms in Ω:

$$(0, \alpha_1) : (X_2, Y_2)_{f_2} \longrightarrow (0, Y_1/K_1)_0$$

and

$$(0, \eta_1) : (X_1, Y_1)_{f_1} \longrightarrow (0, Y_1/K_1)_0.$$

Thus we have the T-homomorphisms

$$\alpha : (X_2 \oplus Y_2)_T \longrightarrow (0 \oplus Y_1/K_1)_T \text{ with } \alpha(x_2, y_2) = (0, \alpha_1(y_2))$$

and

$$\eta : (X_1 \oplus Y_1)_T \longrightarrow (0 \oplus Y_1/K_1)_T \text{ with } \eta(x_1, y_1) = (0, \eta_1(y_1)).$$

Note that η is the natural epimorphism from $(X_1 \oplus Y_1)_T$ to its quotient $(0 \oplus Y_1/K_1)_T$. Since V_2 is V_1-projective, there is a T-homomorphism $\beta : V_2 \longrightarrow V_1$ such that $\eta\beta = \alpha$. Namely, there exists a B-homomorphism $\beta_2 : Y_2 \longrightarrow Y_1$ and an A-homomorphism $\beta_1 : X_2 \longrightarrow X_1$ such that $\beta_1 f_2 = f_1(\beta_2 \otimes 1_M)$ and $\beta(x_2, y_2) = (\beta_1(x_2), \beta_2(y_2))$. Thus $\eta_1\beta_2 = \alpha_1$. Hence Y_2 is Y_1-projective.

Now consider the following diagram:

$$\begin{tikzcd}
 & X_2/f_2(Y_2 \otimes M) \arrow[d, "\mu"] & \\
X_1/f_1(Y_1 \otimes M) \arrow[r, "\nu"'] & \frac{X_1/f_1(Y_1 \otimes M)}{X_1'/f_1(Y_1 \otimes M)} \arrow[r] & 0
\end{tikzcd}$$

where ν is the natural epimorphism, μ is any A-homomorphism and $X_1'/f_1(Y_1 \otimes M)$ is a submodule of $X_1/f_1(Y_1 \otimes M)$. Let γ be the isomorphism from $(X_1/f_1(Y_1 \otimes M))/(X_1'/f_1(Y_1 \otimes M))$ to X_1/X_1', $\pi_1 : X_1 \longrightarrow X_1/f_1(Y_1 \otimes M)$ and $\pi_2 : X_2 \longrightarrow X_2/f_2(Y_2 \otimes M)$ be the natural epimorphisms. It is clear that $(X_1' \oplus Y_1)_T$ is a submodule of V_1 with $f_1' = f_1$ and $((X_1/X_1') \oplus 0)_T$ is a factor module of V_1 with $f_1'' = 0$, namely we have the following commutative diagram:

$$\begin{tikzcd}
Y_1 \otimes M \arrow[r, "1_{Y_1} \otimes 1_M"] \arrow[d, "f_1' = f_1"] & Y_1 \otimes M \arrow[r, "0"] \arrow[d, "f_1"] & 0 \otimes M \arrow[r] \arrow[d, "f_1'' = 0"] & 0 \\
X_1' \arrow[r, "j_1"] & X_1 \arrow[r, "\eta_1"] & X_1/X_1' \arrow[r] & 0
\end{tikzcd}$$

Now $(\gamma\mu\pi_2, 0) : (X_2, Y_2)_{f_2} \longrightarrow (X_1/X_1', 0)_0$ is a T-homomorphism and $(\gamma\nu\pi_1, 0) : (X_1, Y_1)_{f_1} \longrightarrow (X_1/X_1', 0)_0$ is a T-epimorphism. Since V_2 is V_1-projective, we have a T-homomorphism with the pair $(\mu_1, \mu_2) : (X_2, Y_2)_{f_2} \longrightarrow (X_1, Y_1)_{f_1}$ which makes the following diagram commutative:

$$\begin{tikzcd}
 & (X_2, Y_2)_{f_2} \arrow[dl, dashed, "(\mu_1, \mu_2)"'] \arrow[d, "(\gamma\mu\pi_2, 0)"] & \\
(X_1, Y_1)_{f_1} \arrow[r, "(\gamma\nu\pi_1, 0)"'] & (X_1/X_1', 0)_0 \arrow[r] & 0
\end{tikzcd}$$

Note that we have the compositions $\mu_1 f_2 = f_1(\mu_2 \otimes 1_M)$ and $\nu\pi_1\mu_1 = \mu\pi_2$. Let us define the A-homomorphism $\overline{\mu} : X_2/f_2(Y_2 \otimes M) \longrightarrow X_1/f_1(Y_1 \otimes M)$ by $x_2 + f_2(Y_2 \otimes M) \mapsto \mu_1(x_2) + f_1(Y_1 \otimes M)$. Since $\mu_1 f_2 = f_1(\mu_2 \otimes 1_M)$, $\overline{\mu}$ is well-defined and since $\nu\pi_1\mu_1 = \mu\pi_2$, $\nu\overline{\mu} = \mu$. Therefore the following diagram is commutative:

$$\begin{tikzcd} & X_2/f_2(Y_2 \otimes M) \arrow[dl, dashed, "\overline{\mu}"'] \arrow[d, "\mu"] & \\ X_1/f_1(Y_1 \otimes M) \arrow[r, "\nu"'] & \frac{X_1/f_1(Y_1\otimes M)}{X_1'/f_1(Y_1\otimes M)} \arrow[r] & 0 \end{tikzcd}$$

Therefore $X_2/f_2(Y_2 \otimes M)$ is $X_1/f_1(Y_1 \otimes M)$-projective. $\square$

Let V_1 and V_2 be two right T-modules with $(X_1, Y_1)_{f_1}$ and $(X_2, Y_2)_{f_2}$ the corresponding triples, respectively. If V_1 is V_2-projective, the relative projectivity of Y_1 with respect to Y_2 is also proven in [4, 4.1.3] and under the assumption that $f_1(Y_1 \otimes M)$ is a direct summand of X_1, the relative projectivity of $X_1/f_1(Y_1 \otimes M)$ with respect to $X_2/f_2(Y_2 \otimes M)$ is proven in [4, 4.1.4].

Corollary 3.8. If $(X \oplus Y)_T$ is quasi-projective, then $(X/YM)_A$ and Y_B are quasi-projective.

Example 3.9. Let R be a ring and M be a right R-module. Consider the ring $T = \begin{bmatrix} R & 0 \\ M & \mathbb{Z} \end{bmatrix}$. Let K be a nonzero submodule of $\mathbb{Q}_{\mathbb{Z}}$ with $K \ncong \mathbb{Z}$ and $K \ncong \mathbb{Q}_{\mathbb{Z}}$. By [18, Corollary 4.4], $K \oplus \mathbb{Z}$ is not semi-projective hence not quasi-projective over $\mathbb{Z}$. Then by Corollary 3.8, none of the right T-modules in the form $(X \oplus (K \oplus \mathbb{Z}))_T$ is quasi-projective, where X is any right R-module.

Corollary 3.10. If $(X \oplus Y)_T$ has a quasi-projective cover, then $(X/YM)_A$ and Y_B have semi-projective covers.

Proof. Let $\varphi : (U \oplus V)_T \longrightarrow (X \oplus Y)_T$ be a quasi-projective cover of $(X \oplus Y)_T$. Assume that the objects $(U, V)_g$ and $(X, Y)_f$ in Ω determine the right T-modules $(U \oplus V)_T$ and $(X \oplus Y)_T$, respectively. Then there exist homomorphisms $\varphi_1 : U_A \longrightarrow X_A$, $\varphi_2 : V_B \longrightarrow Y_B$ such that $(\varphi_1, \varphi_2) : (U, V)_g \longrightarrow (X, Y)_f$ is a morphism in Ω with $\varphi_1 g = f(\varphi_2 \otimes 1_M)$ and $(\varphi_1(u), \varphi_2(v)) = \varphi(u, v)$. By [3, Theorem 2.4], the epimorphism $\varphi_2 : V_B \longrightarrow Y_B$ has small kernel and we have the epimorphism $\overline{\varphi_1} : U/VM \longrightarrow X/YM$ with small kernel. Thus $(X/YM)_A$ and Y_B have semi-projective covers with the epimorphisms $\overline{\varphi_1}$ and φ_2, respectively by Corollary 3.8. $\square$

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Received: February 10, 2014.

Accepted: April 22, 2014.