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Composition of fuzzy sequential operators with special emphasis on FS-connectors

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Abstract. FS-closure and FS-interior operators both induce fuzzy sequential topologies on the underlying set. Do the composition of FS-closure and that of FS-interior operators provide any topological structure? If so, is there any relation among the topologies induced by the composition and that induced by the participants to the composition? We consider these questions in this article and also study relative FS-closure operators and FS-connectors.

1 Introduction

In 1968 C. L. Chang [6] introduced the concept of fuzzy topology after the initiation of fuzzy sets by L. A. Zadeh [18]. Towards the development of fuzzy set theory, fuzzy closure operators and fuzzy interior operators have been studied by Mashour and Ghanim [10], G. Gerla [8], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]. Notions of fuzzy sequential topological spaces (FSTS) and notions of FS-closure and FS-interior operators were introduced in [13] and [17] respectively.

Our purpose is to introduce FS-connectors connecting two fuzzy topologies on a set and to study the composition of FS-closure and that of FS-interior operators.

Section 2 deals with the composition of FS-closure operators, composition of FS-interior operators and the relation between collections of FS-closure and FS-interior operators. Section 3 deals with the relative FS-closure operators and the functions connecting two fuzzy topologies on a set, so called FS-connectors. The basic ideas behind the present work have been taken from the books ([5], [7] [9], [11]) and the articles ([4], [12], [14], [15], [16]).

In this paper, X will denote a non-empty set, $I=[0,1]$, the closed unit interval in the real line. Before entering into our work we recall the following definitions and results.

Definition 1.1. [13] A family $\delta(s)$ of fuzzy sequential sets on a set X satisfying the properties
(i) $X^r_f(s) \in \delta(s)$ for $r=0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}$, $\underset{j\in J}{\vee} A_{fj}(s) \in \delta(s)$
is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2. [13] If $(X,\delta(s))$ is an FSTS, then (X,δ_n) is a fuzzy topological space (FTS), where δ_n=$\{A^n_f;\ A^n_f(s)=\{A^n_f\}_n \in \delta(s)\}$, $n \in \mathbb{N}$. (X,δ_n), where $n \in \mathbb{N}$, is called the n^{th} component FTS of the FSTS $(X,\delta(s))$.

Proposition 1.3. [13] *Let* $A_f(s)$=$\{A^n_f\}_n$ *be an open* $(closed)$ *fuzzy sequential set in the FSTS* $(X,\delta(s))$*, then for each* $n \in \mathbb{N}$*,* A^n_f *is an open* $(closed)$ *fuzzy set in* (X,δ_n)*.*

Proposition 1.4. [13] *If* δ *be a fuzzy topology* (FT) *on a set* X*, then* $\delta^{\mathbb{N}}$ *forms an FST on* X*.*

Definition 1.5. [13] Let $A_f(s)$ be a fuzzy sequential set (fs-set) in an FSTS $(X,\delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overset{o}{A}_f(s)$ of $A_f(s)$ are defined as

$$\overline{A_f(s)} = \wedge\{C_f(s);\, A_f(s) \le C_f(s),\, (C_f(s))^c \in \delta(s)\},$$

$$\overset{o}{A}_f(s) = \vee\{O_f(s);\, O_f(s) \le A_f(s),\, O_f(s) \in \delta(s)\}.$$

Definition 1.6. [17] An operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-closure operator on X if it satisfies the following conditions:
$(FSC1)$ $\mathbf{Cl}(X^0_f(s)) = X^0_f(s)$.
$(FSC2)$ $A_f(s) \leq \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC3)$ $\mathbf{Cl}(\mathbf{Cl}(A_f(s))) = \mathbf{Cl}(A_f(s))$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSC4)$ $\mathbf{Cl}(A_f(s) \vee B_f(s)) = \mathbf{Cl}(A_f(s)) \vee \mathbf{Cl}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Definition 1.7. [17] An operator **I** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is said to be an FS-interior operator on X if it satisfies the following conditions:
$(FSI1)$ $\mathbf{I}(X^1_f(s)) = X^1_f(s)$.
$(FSI2)$ $\mathbf{I}(A_f(s)) \leq A_f(s)$ for all $A_f(s) \in (I^X)^{\mathbb{N}}$.
$(FSI3)$ $\mathbf{I}(\mathbf{I}(A_f(s))) = \mathbf{I}(A_f(s)))$ for all $A_f(s)) \in (I^X)^{\mathbb{N}}$.
$(FSI4)$ $\mathbf{I}(A_f(s) \wedge B_f(s)) = \mathbf{I}(A_f(s)) \wedge \mathbf{I}(B_f(s))$ for all $A_f(s), B_f(s) \in (I^X)^{\mathbb{N}}$.

Theorem 1.8. [17] *If* ***Cl*** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-closure operator on* X*, then the operator* $\boldsymbol{I_{Cl}} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{I_{Cl}}(A_f(s)) \;\; = \;\; X^1_f(s) - \boldsymbol{Cl}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-interior operator on X*. Again, if* $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *be an FS-interior operator on* X*, then the operator* $\boldsymbol{Cl_I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ *defined by*

$$\boldsymbol{Cl_I}(A_f(s)) \;\; = \;\; X^1_f(s) - \boldsymbol{I}((A_f(s))^c) \; \forall A_f(s) \in (I^X)^{\mathbb{N}},$$

is an FS-closure operator on X.

Theorem 1.9. [17] *The map* $t : \mathcal{C}_X \to \mathcal{I}_X$ *defined by*

$$t(\boldsymbol{Cl}) \;\; = \;\; \boldsymbol{I_{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X$$

is a bijection, where $\mathcal{C}_X$ *and* $\mathcal{I}_X$ *respectively, denote the collections of all FS-closure operators and all FS-interior operators on* X.

2 Composition of FS-closure and FS-interior operators

Definition 2.1. If $\mathbf{C}_1$, $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be two FS-closure operators on X, then the mapping $\mathbf{C}_2 \circ \mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{C}_2 \circ \mathbf{C}_1)(A_f(s)) \;\; = \;\; \mathbf{C}_2(\mathbf{C}_1(A_f(s))) \; \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

is called the composition of the FS-closure operators $\mathbf{C}_1$ and $\mathbf{C}_2$.

It is easy to see that composition of FS-closure operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.2.

Example 2.2. Let us consider the FS-closure operator $\mathbf{C}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_1(A_f(s)) = A_f(s) \vee D_f(s)$ whenever $A_f(s) \neq X^0_f(s)$ and $\mathbf{C}_1(X^0_f(s)) = X^0_f(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-closure operator $\mathbf{C}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{C}_2(A_f(s)) = \{A^n_f \vee A^{n+1}_f\}^{\infty}_{n=1} \; \forall A_f(s) = \{A^n_f\}^{\infty}_{n=1} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{C}_2 \circ \mathbf{C}_1 \neq \mathbf{C}_1 \circ \mathbf{C}_2$. and $(\mathbf{C}_2 \circ \mathbf{C}_1) \circ (\mathbf{C}_2 \circ \mathbf{C}_1) \neq (\mathbf{C}_2 \circ \mathbf{C}_1)$.

Theorem 2.3. *If* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*, then* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *satisfies* $FSC1$*,* $FSC2$ *and* $FSC4$*. Further, it satisfies* $FSC3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$ *forms an FS-closure operator.*
Proof: *Proof is omitted.*

Theorem 2.4. *Let* $\boldsymbol{C}_1$ *and* $\boldsymbol{C}_2$ *be two FS-closure operators on* X*. Under commutative composition,* $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s) = \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$*, where* $\delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$*,* $\delta_{\boldsymbol{C}_2}(s)$ *and* $\delta_{\boldsymbol{C}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{C}_2 \circ \boldsymbol{C}_1$*,* $\boldsymbol{C}_2$ *and* $\boldsymbol{C}_1$.
Proof: *Let* $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$*, then*

$$(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) \;\; = \;\; (A_f(s))^c$$

Now,

$$\begin{aligned}
\boldsymbol{C}_1((A_f(s))^c) &= \boldsymbol{C}_1((\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c)) \\
&= \boldsymbol{C}_1((\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c)) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c))) \\
&= \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\
&= (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\
&= (A_f(s))^c.
\end{aligned}$$

Similarly, $\boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$. *Hence* $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$.
Again, let $A_f(s) \in \delta_{\boldsymbol{C}_2}(s) \wedge \delta_{\boldsymbol{C}_1}(s)$, *then*

$$\boldsymbol{C}_1((A_f(s))^c) = (A_f(s))^c \textit{ and } \boldsymbol{C}_2((A_f(s))^c) = (A_f(s))^c$$

Now,

$$\begin{aligned}
(\boldsymbol{C}_2 \circ \boldsymbol{C}_1)((A_f(s))^c) &= \boldsymbol{C}_2(\boldsymbol{C}_1((A_f(s))^c)) \\
&= \boldsymbol{C}_2((A_f(s))^c) \\
&= (A_f(s))^c
\end{aligned}$$

Thus $A_f(s) \in \delta_{\boldsymbol{C}_2 \circ \boldsymbol{C}_1}(s)$ *and hence the theorem.*

Definition 2.5. If $\mathbf{I}_1$, $\mathbf{I}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be two FS-interior operators on X, then the mapping $\mathbf{I}_2 \circ \mathbf{I}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by

$$(\mathbf{I}_2 \circ \mathbf{I}_1)(A_f(s)) = \mathbf{I}_2(\mathbf{I}_1(A_f(s))) \; \forall A_f(s) \in (I^X)^{\mathbb{N}}$$

is called the composition of the FS-interior operators $\mathbf{I}_1$ and $\mathbf{I}_2$.

It is easy to see that composition of FS-interior operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example** 2.6.

Example 2.6. Let us consider the FS-interior operator $\mathbf{I}_1 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_1(A_f(s)) = A_f(s) \wedge D_f(s)$ whenever $A_f(s) \neq X_f^0(s)$ and $\mathbf{I}_1(X_f^1(s)) = X_f^1(s)$, where $D_f(s)$ is a fixed fuzzy sequential set in X. Also consider FS-interior operator $\mathbf{I}_2 : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X, defined by $\mathbf{I}_2(A_f(s)) = \{A_f^n \wedge A_f^{n+1}\}_{n=1}^{\infty} \; \forall A_f(s) = \{A_f^n\}_{n=1}^{\infty} \in (I^X)^{\mathbb{N}}$. Then $\mathbf{I}_2 \circ \mathbf{I}_1 \neq \mathbf{I}_1 \circ \mathbf{I}_2$ and $(\mathbf{I}_2 \circ \mathbf{I}_1) \circ (\mathbf{I}_2 \circ \mathbf{I}_1) \neq (\mathbf{I}_2 \circ \mathbf{I}_1)$.

Theorem 2.7. *If* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X, *then* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ *satisfies* $FSI1$, $FSI2$ *and* $FSI4$. *Further, it satisfies* $FSI3$ *if the composition is commutative, that is, under commutative composition,* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$ *forms an FS-interior operator.*
Proof: *Proof is omitted.*

Theorem 2.8. *Let* $\boldsymbol{I}_1$ *and* $\boldsymbol{I}_2$ *be two FS-interior operators on* X. *Under commutative composition,* $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s) = \delta_{\boldsymbol{I}_2}(s) \wedge \delta_{\boldsymbol{I}_1}(s)$, *where* $\delta_{\boldsymbol{I}_2 \circ \boldsymbol{I}_1}(s)$, $\delta_{\boldsymbol{I}_2}(s)$ *and* $\delta_{\boldsymbol{I}_1}(s)$ *respectively denote the FST's induced by* $\boldsymbol{I}_2 \circ \boldsymbol{I}_1$, $\boldsymbol{I}_2$ *and* $\boldsymbol{I}_1$.
Proof: *The proof is similar to that in case of FS-closure operators.*

Theorem 2.9. *Under commutative composition,* $(\mathcal{I}_X, \circ)$ *and* $(\mathcal{C}_X, \circ)$ *both form semigroups with identity. Further, there exists a semigroup isomorphism between them.*
Proof: *First part is easy to check. For the second part, define* $t : \mathcal{C}_X \to \mathcal{I}_X$ *by*

$$t(\boldsymbol{Cl}) = \boldsymbol{I}_{\boldsymbol{Cl}} \; \forall \boldsymbol{Cl} \in \mathcal{C}_X$$

From ***Theorem*** 1.9, t *is a bijection. Also for* $\boldsymbol{C}_1$, $\boldsymbol{C}_2 \in \mathcal{C}_X$ *and* $A_f(s) \in (I^X)^{\mathbb{N}}$

$$\begin{aligned}
(\boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2})(A_f(s)) &= \boldsymbol{I}_{\boldsymbol{C}_1}(X_f^1(s) - \boldsymbol{C}_2((A_f(s))^c)) \\
&= X_f^1(s) - \boldsymbol{C}_1(\boldsymbol{C}_2((A_f(s))^c)) \\
&= X_f^1(s) - (\boldsymbol{C}_1 \circ \boldsymbol{C}_2)((A_f(s))^c) \\
&= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2}(A_f(s)).
\end{aligned}$$

Therefore

$$\begin{aligned}
t(\boldsymbol{C}_1 \circ \boldsymbol{C}_2) &= \boldsymbol{I}_{\boldsymbol{C}_1 \circ \boldsymbol{C}_2} \\
&= \boldsymbol{I}_{\boldsymbol{C}_1} \circ \boldsymbol{I}_{\boldsymbol{C}_2} \\
&= t(\boldsymbol{C}_1) \circ t(\boldsymbol{C}_2)
\end{aligned}$$

Hence t *is an isomorphism.*

3 Relative FS-closure Operators and FS-connectors

Definition 3.1. Let $A_f(s)$ be an fs-set in X and **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. A function $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ defined by $(\mathbf{Cl})^n_{A_f(s)}(B) = n^{th}$ term of $\mathbf{Cl}({}_{nB}A_f(s))$, where ${}_{nB}A_f(s)$ is the fs-set in X obtained from $A_f(s)$ replacing n^{th} term of it by B, is called n^{th} relative FS-closure operator of **Cl** with respect to $A_f(s)$.

If **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X, then it is obvious that $(\mathbf{Cl})^n_{X^0_f(s)} = (\mathbf{Cl})^n_f$ and consequently $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}} = \delta_{(\mathbf{Cl})^n_f}$, $\delta_{(\mathbf{Cl})^n_{X^0_f(s)}}$ and $\delta_{(\mathbf{Cl})^n_f}$ being the fuzzy topologies induced by $(\mathbf{Cl})^n_{X^0_f(s)}$ and $(\mathbf{Cl})^n_f$ respectively. It is also true that the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)}$ of an FS-closure operator **Cl** with respect to an fs-set $A_f(s)$ satisfies $FSC2$, $FSC3$ and $FSC4$ but it may not satisfy $FSC1$ shown by **Example** 3.2. Hence $(\mathbf{Cl})^n_{A_f(s)}$ may not be a fuzzy operator.

Example 3.2. Define a function **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ by

$$\begin{aligned} \mathbf{Cl}(B_f(s)) &= X^1_f(s) \text{ if } B_f(s) \neq X^0_f(s), \\ &= X^0_f(s) \text{ if } B_f(s) = X^0_f(s) \end{aligned}$$

Then for any fs-set $A_f(s)$ in X, having at least two non zero components, $(\mathbf{Cl})^n_{A_f(s)}(\overline{0}) = \overline{1}$ for all $n \in \mathbb{N}$.

Theorem 3.3. *Let $(\boldsymbol{Cl})^n_{A_f(s)} : I^X \to I^X$ be the n^{th} relative FS-closure operator of an FS-closure operator $\boldsymbol{Cl}$: $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ on X with respect to an fs-set $A_f(s)$. Then $\delta_{(\boldsymbol{Cl})^n_{A_f(s)}} = \{\overline{1}, B;$ $B \in I^X$ and $(\boldsymbol{Cl})^n_{A_f(s)}(B^c) = B^c\}$ forms a fuzzy topology on X. Further, the closure in the FTS $(X, \delta_{(\boldsymbol{Cl})^n_{A_f(s)}})$ and $(\boldsymbol{Cl})^n_{A_f(s)}$ are identical on $I^X - \{\overline{0}\}$.*
Proof: *Proof is omitted.*

Definition 3.4. The fuzzy topology $\delta_{(\mathbf{Cl})^n_{A_f(s)}} = \{\overline{1}, B; B \in I^X$ and $(\mathbf{Cl})^n_{A_f(s)}(B^c) = B^c\}$ induced by the n^{th} relative FS-closure operator $(\mathbf{Cl})^n_{A_f(s)} : I^X \to I^X$ is called the n^{th} relative fuzzy topology induced by the FS-closure operator **Cl** : $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ with respect to the fs-set $A_f(s)$.

Theorem 3.5. *Let $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ be an fs-set in a set X and $\boldsymbol{Cl}$: $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Let $(\boldsymbol{Cl})^n_f$, $n \in \mathbb{N}$ be the n^{th} component of $\boldsymbol{Cl}$. Then*
(1) $\boldsymbol{Cl}(A_f(s)) \geq \{(\boldsymbol{Cl})^n_f(A^n_f)\}$ and the equality holds if $A_f(s)$ is a closed fs-set in $(X, \delta_{\boldsymbol{Cl}}(s))$.
(2) If $\boldsymbol{Cl}(A_f(s)) = \{(\boldsymbol{Cl})^n_f(A^n_f)\}$ and A_n is closed in $(X, \delta_{(\boldsymbol{Cl})^n_f})$ for each $n \in \mathbb{N}$, then $A_f(s)$ is closed in $(X, \delta_{\boldsymbol{Cl}}(s))$.
(3) $\boldsymbol{Cl}(A_f(s)) = \{(\boldsymbol{Cl})^n_{A_f(s)}(A^n_f)\}$.
Proof: *Proof is omitted.*

In an FSTS $(X, \delta(s))$ if $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ is closed, then A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$ but the converse is not true [13]. **Corollary** 3.6 provides a pair of if and only if conditions for an fs-set $A_f(s)$ to be closed in an FSTS.

Corollary 3.6. *In an FSTS $(X, \delta(s))$, an fs-set $A_f(s) = \{A^n_f\}^{\infty}_{n=1}$ is closed:*
(1) if and only if $\overline{A_f(s)} = \{B^n_f\}$ and A^n_f is closed in (X, δ_n) for each $n \in \mathbb{N}$, where $B^n_f = n^{th}$ component of $\overline{{}_{nA^n_f}X^0_f(s)}$.
(2) if and only if A^n_f is closed in $(X, \delta_{R^n_{A_f(s)}})$ for each $n \in \mathbb{N}$, where $R^n_{A_f(s)}$ is the n^{th} relative FS-closure operator of the closure operator in $(X, \delta(s))$ with respect to $A_f(s)$.

Theorem 3.7. *If $\{A_{\lambda f}(s); \lambda \in \Lambda\}$ be a chain of fs-sets in $((I^X)^{\mathbb{N}}, \leq)$, then $\{\delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}, \lambda \in \Lambda\}$ is a chain of fuzzy topologies on X for each $n \in \mathbb{N}$, where $\boldsymbol{Cl}$: $(I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ is an FS-closure*

operator on X.
Proof: *Let $A_{\lambda f}(s) \leq A_{\mu f}(s)$, λ, $\mu \in \Lambda$. It suffices to show that $\delta_{(\boldsymbol{Cl})^n_{A_{\mu f}(s)}} \leq \delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}$.*

$$\text{Let } B \in \delta_{(\boldsymbol{Cl})^n_{A_{\mu f}(s)}} \Rightarrow (\boldsymbol{Cl})^n_{A_{\mu f}(s)}(\overline{1} - B) = \overline{1} - B$$
$$\Rightarrow n^{th} \text{ term of } \boldsymbol{Cl}({}_{n(\overline{1}-B)}A_{\mu f}(s)) = \overline{1} - B$$

Therefore n^{th} term of $\boldsymbol{Cl}({}_{n(\overline{1}-B)}A_{\lambda f}(s)) \leq \overline{1} - B$
$\Rightarrow (\boldsymbol{Cl})^n_{A_{\lambda f}(s)}(\overline{1} - B) \leq \overline{1} - B$.

Hence $B \in \delta_{(\boldsymbol{Cl})^n_{A_{\lambda f}(s)}}$.

Definition 3.8. Each member except possibly $\overline{1}$ of $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is contained in $\overline{1} - (\mathbf{Cl})^n_{A_f(s)}(\overline{0})$ and so $\delta_{(\mathbf{Cl})^n_{A_f(s)}}$ is called $(\overline{1} - (\mathbf{Cl})^n_{A_f(s)}(\overline{0}))$-cut of $\delta_{(\mathbf{Cl})^n_f}$.

Theorem 3.9. *Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. Then the operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}^{\infty}_{n=1} \in (I^X)^{\mathbb{N}}$ is an FS-closure operator on X.*
Proof: *The proof is omitted.*

Definition 3.10. Let $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators on X. The operator $C : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ defined by $C(A_f(s)) = \{C_n(A^n_f)\}$ for all $A_f(s) = \{A^n_f\}^{\infty}_{n=1} \in (I^X)^{\mathbb{N}}$ is called an FS-closure operator induced by a sequence $\{C_n : I^X \to I^X\}$ of fuzzy closure operators on X.

Definition 3.11. Let δ and δ' be two fuzzy topologies on a set X. A subset K_f of δ'^{δ} is called an FS-connector of δ to δ' if it satisfies the following conditions:
(1) $A_\lambda \in \delta$ and $f_\lambda \in K_f$, $\lambda \in \Lambda \Rightarrow$ there exist $f \in K_f$ so that $f(\vee_{\lambda\in\Lambda}A_\lambda) = \vee_{\lambda\in\Lambda}f_\lambda(A_\lambda)$,
(2) $A_i \in \delta$ and $f_i \in K_f$, $i = 1(1)n \Rightarrow$ there exist $f \in K_f$ so that $f(\wedge^n_{i=1}A_i) = \wedge^n_{i=1}f_i(A_i)$ and
(3) $\delta' = \vee_{f\in K_f}f(\delta)$.

Example 3.12. Let δ and δ' be two fuzzy topologies on a set X. A function $f : \delta \to \delta'$ defined by $f(A) = O$ for all $A \in \delta$, where O is a fixed element of δ', is called a constant function from δ into δ'. If K_f be the collection of all such constant functions from δ into δ', then K_f forms an FS-connector from δ to δ'.

Definition 3.13. Let δ and δ' be two fuzzy topologies on a set X. Then the collection of all constant functions from δ into δ' forms an FS-connector of δ to δ'. This is called the discrete FS-connector of δ to δ'.

If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X, then any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, provides a unique FST on X (**Theorem** 3.14) which is denoted by $\delta(s) < \{\delta_n\}, \{K_n\} >$ such that the n^{th} components $(\delta < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$ and it is called the FST generated by $\{\delta_n\}$ and $\{K_n\}$. If further each K_n is the discrete FS-connector of δ_n to δ_{n+1}, then the FST is said to be generated by $\{\delta_n\}$ and is denoted by $\delta < \{\delta_n\} >$.

Theorem 3.14. *Let $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_n\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$, $n \in \mathbb{N}$. Also for any FSTS $(X,\delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects δ_n to δ_{n+1} and $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.*
Proof: *Let $K = \prod^{\infty}_{n=1} K_n$, $g = \{g_n\} \in K$ and $A \in \delta_1$. Define $H_1 = A$ and $H_n = g_{n-1}g_{n-2}.....g_2g_1A$, $n > 1$. Let $H^g_A(s) = \{H_n\} \in (I^X)^{\mathbb{N}}$ and consider $\delta(s) < \{\delta_n\}, \{K_n\} >= \{X^1_f(s), X^0_f(s)\} \vee \{H^g_A(s); g \in K \text{ and } A \in \delta_1\}$. Consider*

$$H_\lambda(s) = H^{g_\lambda}_{A_\lambda}(s) \in \delta(s),\ \lambda \in \Lambda$$

where Λ is an index set and

$$A = \vee_{\lambda\in\Lambda}A_\lambda \in \delta_1.$$

For $g_{\lambda 1} \in K_1$ and $A \in \delta_1$ there exist $g_1 \in K_1$ such that

$$g_1 A = \vee_{\lambda \in \Lambda} g_{\lambda 1} A_\lambda;\ g_{\lambda n} \in K_n$$

and for $g_{n-1} g_{n-2} \ldots\ldots g_2 g_1 A \in \delta_n$ there exist $g_n \in K_n$ such that

$$g_n g_{n-1} \ldots\ldots g_2 g_1 A = \vee_{\lambda \in \Lambda} g_{\lambda n} g_{\lambda(n-1)} \ldots . g_{\lambda 2} g_{\lambda 1} A_\lambda .$$

Obviously,

$$\vee_{\lambda \in \Lambda} H_\lambda(s) = \vee_{\lambda \in \Lambda} H^{g_\lambda}_{A_\lambda}(s) = H^g_A(s) \in \delta(s) < \{\delta_n\}, \{K_n\} >$$

where $g = g_n$. Arguing in the same way it can be shown that $\delta(s) < \{\delta_n\}, \{K_n\} >$ is closed under finite intersection. Therefore, $(X,\ \delta(s) < \{\delta_n\}, \{K_n\} >)$ is a fuzzy sequential topological space. The third condition to be an FS-connector ensures that $(\delta(s) < \{\delta_n\}, \{K_n\} >)_n = \delta_n$ for all $n \in \mathbb{N}$. For the next part, for each $n \in \mathbb{N}$ define a relation $R^{n,n+1}$ on $\delta(s)$ by $A_f(s) = \{A^n_f\} R^{n,n+1} B_f(s) = \{B^n_f\}$ if and only if $A^n_f = B^n_f$. Then $R^{n,n+1}$ defines a partition of $\delta(s)$ say

$$\{Cls(A_f(s));\ A_f(s) \in \delta^{n,n+1}(s) \subset \delta(s)\}$$

where $\delta^{n,n+1}(s)$ is a family of open fs-sets taking exactly one from each class of the partition of $\delta(s)$ by $R^{n,n+1}$ and $Cls(A_f(s))$ represents the class of $A_f(s)$. Let

$$K^{n,n+1} = \prod_{A_f(s) \in \delta^{n,n+1}(s)} Cls(A_f(s))$$

Then each $t \in K^{n,n+1}$ defines a function $g_t : \delta_n \to \delta_{n+1}$ and $K_n = \{g_t;\ t \in K^{n,n+1}\}$ is an FS-connector connecting δ_n to δ_{n+1} and properties of FS-connectors ensures that $\delta(s) = \delta(s) < \{\delta_n\}, \{K_n\} >$.

Corollary 3.15. *Let $\boldsymbol{Cl} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-closure operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{Cl})^n_f}$ and the components of the closure operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{Cl})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X, \delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{Cl})^n_f}$ to $\delta_{(\boldsymbol{Cl})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{Cl})^n_f}\}, \{K_n\} >$.*

Corollary 3.16. *Let $\boldsymbol{I} : (I^X)^{\mathbb{N}} \to (I^X)^{\mathbb{N}}$ be an FS-interior operator on X. Then for any sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$ on X such that $(\delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)_n = \delta_{(\boldsymbol{I})^n_f}$ and the components of the interior operator on $(X,\ \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >)$ are $(\boldsymbol{I})^n_f$, $n \in \mathbb{N}$. Also for any FSTS $(X, \delta(s))$, there is a sequence $\{K_n\}$ of FS-connectors such that K_n connects $\delta_{(\boldsymbol{I})^n_f}$ to $\delta_{(\boldsymbol{I})^{n+1}_f}$ and $\delta(s) = \delta(s) < \{\delta_{(\boldsymbol{I})^n_f}\}, \{K_n\} >$.*

Corollary 3.17. *If $\{\delta_n\}$ be a sequence of fuzzy topologies on a set X such that $\delta_n = \delta$ for all $n \in \mathbb{N}$, then $\delta(s) < \{\delta_n\} >= \delta^{\mathbb{N}}$.*

Corollary 3.18. *If $\{C_n : I^X \to I^X\}$ be a sequence of fuzzy closure operators and $\boldsymbol{C}$ be an FS-closure operator induced by $\{C_n\}$, then $\delta_{\boldsymbol{C}}(s) = \delta(s) < \{\delta_n\} >$ where δ_n is the fuzzy topology on X induced by C_n, $n \in \mathbb{N}$.*

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