Composition of fuzzy sequential operators with special emphasis on FS-connectors

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Abstract. FS-closure and FS-interior operators both induce fuzzy sequential topologies on the underlying set. Do the composition of FS-closure and that of FS-interior operators provide any topological structure? If so, is there any relation among the topologies induced by the composition and that induced by the participants to the composition? We consider these questions in this article and also study relative FS-closure operators and FS-connectors.

1 Introduction

In 1968 C. L. Chang [6] introduced the concept of fuzzy topology after the initiation of fuzzy sets by L. A. Zadeh [18]. Towards the development of fuzzy set theory, fuzzy closure operators and fuzzy interior operators have been studied by Mashour and Ghanim [10], G. Gerla [8], Bandler and Kohout [1], R. Belohlavek [2], R. Belohlavek and T. Funiokova [3]. Notions of fuzzy sequential topological spaces (FSTS) and notions of FS-closure and FS-interior operators were introduced in [13] and [17] respectively.

Our purpose is to introduce FS-connectors connecting two fuzzy topologies on a set and to study the composition of FS-closure and that of FS-interior operators.

Section 2 deals with the composition of FS-closure operators, composition of FS-interior operators and the relation between collections of FS-closure and FS-interior operators. Section 3 deals with the relative FS-closure operators and the functions connecting two fuzzy topologies on a set, so called FS-connectors. The basic ideas behind the present work have been taken from the books ([5], [7], [9], [11]) and the articles ([4], [12], [14], [15], [16]).

In this paper, X will denote a non-empty set, I = [0, 1], the closed unit interval in the real line. Before entering into our work we recall the following definitions and results.

Definition 1.1. [13] A family δ(s) of fuzzy sequential sets on a set X satisfying the properties

(i) $X^j_s \in \delta(s)$ for $r = 0$ and 1,
(ii) $A_f(s), B_f(s) \in \delta(s) \Rightarrow A_f(s) \wedge B_f(s) \in \delta(s)$ and
(iii) for any family $\{A_{fj}(s) \in \delta(s), j \in J\}, \forall j A_{fj}(s) \in \delta(s)$

is called a fuzzy sequential topology (FST) on X and the ordered pair $(X, \delta(s))$ is called fuzzy sequential topological space (FSTS). The members of $\delta(s)$ are called open fuzzy sequential sets in X. Complement of an open fuzzy sequential set in X is called closed fuzzy sequential set in X.

Definition 1.2. [13] If $(X, \delta(s))$ is an FSTS, then $(X, \delta_n)$ is a fuzzy topological space (FTS), where $\delta_n = \{A^n_f; A^n_f(s) = \{A^n_f\}_{n} \in \delta(s)\}, n \in \mathbb{N}$. $(X, \delta_n)$, where $n \in \mathbb{N}$, is called the $n^{th}$ component FTS of the FSTS $(X, \delta(s))$.

Proposition 1.3. [13] Let $A_f(s) = \{A^n_f\}_{n} be an open (closed) fuzzy sequential set in the FSTS $(X, \delta(s))$, then for each $n \in \mathbb{N}, A^n_f is an open (closed) fuzzy set in $(X, \delta_n)$.

Proposition 1.4. [13] If $\delta$ be a fuzzy topology (FT) on a set X, then $\delta^\mathbb{N}$ forms an FST on X.

Definition 1.5. [13] Let $A_f(s)$ be a fuzzy sequential set (fs-set) in an FSTS $(X, \delta(s))$. The closure $\overline{A_f(s)}$ and interior $\overline{A_f(s)}$ of $A_f(s)$ are defined as

$\overline{A_f(s)} = \wedge \{C_f(s); A_f(s) \leq C_f(s), (C_f(s))^c \in \delta(s)\}$,
$\overline{A_f(s)} = \vee \{O_f(s); O_f(s) \leq A_f(s), O_f(s) \in \delta(s)\}$. 


Example 2.2. An operator \( C : (I^X)^N \rightarrow (I^X)^N \) is called the composition of the FS-closure operators \( A_f(s) \) if it satisfies the following conditions:

\[
\text{(FSC1)} \quad Cl(X^0_f(s)) = X^0_f(s).
\]

\[
\text{(FSC2)} \quad A_f(s) \leq Cl(A_f(s)) \quad \forall A_f(s) \in (I^X)^N.
\]

\[
\text{(FSC3)} \quad Cl(Cl(A_f(s))) = Cl(A_f(s)) \quad \forall A_f(s) \in (I^X)^N.
\]

\[
\text{(FSC4)} \quad Cl(A_f(s) \lor B_f(s)) = Cl(A_f(s)) \lor Cl(B_f(s)) \quad \forall A_f(s), B_f(s) \in (I^X)^N.
\]

Definition 2.1. [17] An operator \( I : (I^X)^N \rightarrow (I^X)^N \) is said to be an FS-interior operator on \( X \) if it satisfies the following conditions:

\[
\text{(FSI1)} \quad I(X^0_f(s)) = X^0_f(s).
\]

\[
\text{(FSI2)} \quad I(A_f(s)) \leq A_f(s) \quad \forall A_f(s) \in (I^X)^N.
\]

\[
\text{(FSI3)} \quad I(I(A_f(s))) = I(A_f(s)) \quad \forall A_f(s) \in (I^X)^N.
\]

\[
\text{(FSI4)} \quad I(A_f(s) \land B_f(s)) = I(A_f(s)) \land I(B_f(s)) \quad \forall A_f(s), B_f(s) \in (I^X)^N.
\]

Theorem 1.8. [17] If \( Cl : (I^X)^N \rightarrow (I^X)^N \) be an FS-closure operator on \( X \), then the operator \( I_{Cl} : (I^X)^N \rightarrow (I^X)^N \) defined by

\[
I_{Cl}(A_f(s)) = X^0_f(s) - Cl((A_f(s))^c) \quad \forall A_f(s) \in (I^X)^N,
\]

is an FS-interior operator on \( X \). Again, if \( I : (I^X)^N \rightarrow (I^X)^N \) be an FS-interior operator on \( X \), then the operator \( Cl_{I} : (I^X)^N \rightarrow (I^X)^N \) defined by

\[
Cl_{I}(A_f(s)) = X^0_f(s) - I((A_f(s))^c) \quad \forall A_f(s) \in (I^X)^N,
\]

is an FS-closure operator on \( X \).

Theorem 1.9. [17] The map \( t : C_X \rightarrow I_X \) defined by

\[
t(Cl) = I_{Cl} \forall Cl \in C_X
\]

is a bijection, where \( C_X \) and \( I_X \) respectively denote the collections of all FS-closure operators and all FS-interior operators on \( X \).

2 Composition of FS-closure and FS-interior operators

Definition 2.1. If \( C_1, C_2 : (I^X)^N \rightarrow (I^X)^N \) be two FS-closure operators on \( X \), then the mapping \( C_2 \circ C_1 : (I^X)^N \rightarrow (I^X)^N \) defined by

\[
(C_2 \circ C_1)(A_f(s)) = C_2(C_1(A_f(s))) \quad \forall A_f(s) \in (I^X)^N
\]

called the composition of the FS-closure operators \( C_1 \) and \( C_2 \).

It is easy to see that composition of FS-closure operators is associative but it may not be commutative and it may not be idempotent, as shown by Example 2.2.

Example 2.2. Let us consider the FS-closure operator \( C_1 : (I^X)^N \rightarrow (I^X)^N \) on \( X \), defined by \( C_1(A_f(s)) = A_f(s) \lor D_f(s) \) whenever \( A_f(s) \neq X^0_f(s) \) and \( C_1(X^0_f(s)) = X^0_f(s) \), where \( D_f(s) \) is a fixed fuzzy sequential set in \( X \). Also consider FS-closure operator \( C_2 : (I^X)^N \rightarrow (I^X)^N \) on \( X \), defined by \( C_2(A_f(s)) = \{ A^1_f \lor A^1_{f+1} \}_{n=1} \forall A_f(s) = \{ A^1_f \}_{n=1} \in (I^X)^N \). Then \( C_2 \circ C_1 \neq C_1 \circ C_2 \) and \( (C_2 \circ C_1) \circ (C_2 \circ C_1) \neq (C_2 \circ C_1) \).

Theorem 2.3. If \( C_1 \) and \( C_2 \) be two FS-closure operators on \( X \), then \( C_2 \circ C_1 \) satisfies FSC1, FSC2 and FSC4. Further, it satisfies FSC3 if the composition is commutative, that is, under commutative composition, \( C_2 \circ C_1 \) forms an FS-closure operator.

Proof: Proof is omitted.

Theorem 2.4. Let \( C_1 \) and \( C_2 \) be two FS-closure operators on \( X \). Under commutative composition, \( \delta_{C_2 \circ C_1}(s) = \delta_{C_2}(s) \land \delta_{C_1}(s) \), where \( \delta_{C_2 \circ C_1}(s), \delta_{C_2}(s) \) and \( \delta_{C_1}(s) \) respectively denote the FST’s induced by \( C_2 \circ C_1, C_2 \) and \( C_1 \).

Proof: Let \( A_f(s) \in \delta_{C_2 \circ C_1}(s) \), then

\[
(C_2 \circ C_1)((A_f(s))^c) = (A_f(s))^c
\]
Now,
\[ C_1((A_f(s))^c) = C_1(C_2 \circ C_1((A_f(s))^c)) = C_1((C_1 \circ C_2)((A_f(s))^c)) = C_1(C_1(C_2((A_f(s))^c))) = C_1(C_2((A_f(s))^c)) = (C_1 \circ C_2)((A_f(s))^c) = (A_f(s))^c. \]

Similarly, \( C_2((A_f(s))^c) = (A_f(s))^c \). Hence \( A_f(s) \in \delta_{C_1}(s) \land \delta_{C_2}(s) \).

Again, let \( A_f(s) \in \delta_{C_1}(s) \land \delta_{C_2}(s) \), then
\[ C_1((A_f(s))^c) = (A_f(s))^c \text{ and } C_2((A_f(s))^c) = (A_f(s))^c \]

Now,
\[ (C_2 \circ C_1)((A_f(s))^c) = C_2(C_1((A_f(s))^c)) \]
\[ = C_2((A_f(s))^c) \]
\[ = (A_f(s))^c \]

Thus \( A_f(s) \in \delta_{C_1 \circ C_2}(s) \) and hence the theorem.

**Definition 2.5.** If \( I_1, I_2 : (I^X)^N \rightarrow (I^X)^N \) be two FS-interior operators on \( X \), then the mapping \( I_2 \circ I_1 : (I^X)^N \rightarrow (I^X)^N \) defined by
\[ (I_2 \circ I_1)(A_f(s)) = I_2(I_1(A_f(s))) \forall A_f(s) \in (I^X)^N \]
is called the composition of the FS-interior operators \( I_1 \) and \( I_2 \).

It is easy to see that composition of FS-interior operators is associative but it may not be commutative and it may not be idempotent, as shown by **Example 2.6**.

**Example 2.6.** Let us consider the FS-interior operator \( I_1 : (I^X)^N \rightarrow (I^X)^N \) on \( X \), defined by
\[ I_1(A_f(s)) = A_f(s) \land D_f(s) \text{ whenever } A_f(s) \neq X_N^0(\ast) \text{ and } I_1(X_N^1(\ast)) = X_N^1(\ast), \]
where \( D_f(s) \) is a fixed fuzzy sequential set in \( X \). Also consider FS-interior operator \( I_2 : (I^X)^N \rightarrow (I^X)^N \) on \( X \), defined by
\[ I_2(A_f(s)) = \{ A_f^n \land A_f^{n+1} \}_{n=1}^\infty \forall A_f(s) = \{ A_f^n \}_{n=1}^\infty \in (I^X)^N. \] Then \( I_2 \circ I_1 \neq I_1 \circ I_2 \) and \( (I_2 \circ I_1) \circ (I_2 \circ I_1) \neq (I_2 \circ I_1) \).

**Theorem 2.7.** If \( I_1 \) and \( I_2 \) be two FS-interior operators on \( X \), then \( I_2 \circ I_1 \) satisfies FS11, FS12 and FS14. Further, it satisfies FS13 if the composition is commutative, that is, under commutative composition, \( I_2 \circ I_1 \) forms an FS-interior operator.

**Proof:** Proof is omitted.

**Theorem 2.8.** Let \( I_1 \) and \( I_2 \) be two FS-interior operators on \( X \). Under commutative composition, \( \delta_{I_2 \circ I_1}(s) = \delta_{I_1}(s) \lor \delta_{I_2}(s) \), where \( \delta_{I_1}(s) \) and \( \delta_{I_2}(s) \) respectively denote the FST’s induced by \( I_2 \circ I_1, I_2 \) and \( I_1 \).

**Proof:** The proof is similar to that in case of FS-closure operators.

**Theorem 2.9.** Under commutative composition, \( (I_X, \circ) \) and \( (C_X, \circ) \) both form semigroups with identity. Further, there exists a semigroup isomorphism between them.

**Proof:** First part is easy to check. For the second part, define \( t : C_X \rightarrow I_X \) by
\[ t(\mathcal{C}I) = \mathcal{I}_t \forall \mathcal{C}I \in C_X \]

From **Theorem 1.9**, \( t \) is a bijection. Also for \( C_1, C_2 \in C_X \) and \( A_f(s) \in (I^X)^N \)
\[ (I_{C_1} \circ I_{C_2})(A_f(s)) = I_{C_1}(X_1^1(s) - C_2((A_f(s))^c)) = X_1^1(s) - C_1(C_2((A_f(s))^c)) = X_1^1(s) - (C_1 \circ C_2)((A_f(s))^c) = I_{C_1 \circ C_2}(A_f(s)). \]

Therefore
\[ t(C_1 \circ C_2) = t(C_1) \circ t(C_2) = I_{C_1 \circ C_2} \]

Hence \( t \) is an isomorphism.
3 Relative FS-closure Operators and FS-connectors

**Definition 3.1.** Let $A_f(s)$ be an fs-set in $X$ and $Cl : (I^X)^N \rightarrow (I^X)^N$ be an FS-closure operator on $X$. A function $(Cl)^n_{A_f(s)} : I^X \rightarrow I^X$ defined by $(Cl)^n_{A_f(s)}(B) = n^{th}$ term of $Cl(n_B A_f(s))$, where $n_B A_f(s)$ is the fs-set in $X$ obtained from $A_f(s)$ replacing $n^{th}$ term of it by $B$, is called $n^{th}$ relative FS-closure operator of $Cl$ with respect to $A_f(s)$.

If $Cl : (I^X)^N \rightarrow (I^X)^N$ be an FS-closure operator on $X$, then it is obvious that $(Cl)^n_{X^0(s)} = (Cl)^0_n$ and consequently $\delta(Cl)^n_{X^0(s)} = \delta(Cl)^n_{X^0(s)}$ and $\delta(Cl)^n_{X^0(s)}$ being the fuzzy topologies induced by $(Cl)^n_{X^0(s)}$ and $(Cl)^n_{X^0(s)}$ respectively. It is also true that the $n^{th}$ relative FS-closure operator $(Cl)^n_{A_f(s)}$ of an FS-closure operator $Cl$ with respect to an fs-set $A_f(s)$ satisfies FSC2, FSC3 and FSC4 but it may not satisfy FSC1 shown by **Example 3.2.** Hence $(Cl)^n_{A_f(s)}$ may not be a fuzzy operator.

**Example 3.2.** Define a function $Cl : (I^X)^N \rightarrow (I^X)^N$ by

$$Cl(B_f(s)) = \begin{cases} X_f^1(s) & \text{if } B_f(s) \neq X_f^0(s), \\ X_f^0(s) & \text{if } B_f(s) = X_f^0(s) \end{cases}$$

Then for any fs-set $A_f(s)$ in $X$, having at least two non zero components, $(Cl)^n_{A_f(s)}(\emptyset) = \overline{\emptyset}$ for all $n \in N$.

**Theorem 3.3.** Let $(Cl)^n_{A_f(s)} : I^X \rightarrow I^X$ be the $n^{th}$ relative FS-closure operator of an FS-closure operator $Cl : (I^X)^N \rightarrow (I^X)^N$ on $X$ with respect to an fs-set $A_f(s)$. Then $\delta(Cl)^n_{A_f(s)} = (\overline{\emptyset}, B; B \in I^X$ and $(Cl)^n_{A_f(s)}(B^c) = B^c)$ forms a fuzzy topology on $X$. Further, the closure in the FTS $(X, \delta(Cl)^n_{A_f(s)})$ and $(Cl)^n_{A_f(s)}$ are identical on $I^X - \{\emptyset\}$.

**Proof:** Proof is omitted.

**Definition 3.4.** The fuzzy topology $\delta(Cl)^n_{A_f(s)} = (\overline{\emptyset}, B; B \in I^X$ and $(Cl)^n_{A_f(s)}(B^c) = B^c)$ induced by the $n^{th}$ relative FS-closure operator $(Cl)^n_{A_f(s)} : I^X \rightarrow I^X$ is called the $n^{th}$ relative fuzzy topology induced by the FS-closure operator $Cl : (I^X)^N \rightarrow (I^X)^N$ with respect to the fs-set $A_f(s)$.

**Theorem 3.5.** Let $A_f(s) = \{A_f^n\}_{n=1}^\infty$ be an fs-set in a set $X$ and $Cl : (I^X)^N \rightarrow (I^X)^N$ be an FS-closure operator on $X$. Let $(Cl)^n_s$, $n \in N$ be the $n^{th}$ component of $Cl$. Then

1. $Cl(A_f(s)) \geq \{[Cl]^n(A_f^n)\}$ and the equality holds if $A_f(s)$ is a closed fs-set in $(X, \delta(Cl))$.
2. If $Cl(A_f(s)) = \{[Cl]^n(A_f^n)\}$ and $A_n$ is closed in $(X, \delta(Cl))$ for each $n \in N$, then $A_f(s)$ is closed in $(X, \delta(Cl))$.
3. $Cl(A_f(s)) = \{[Cl]^n_{A_f(s)}(A_f^n)\}$.

**Proof:** Proof is omitted.

In an FSTS $(X, \delta(s))$ if $A_f(s) = \{A_f^n\}_{n=1}^\infty$ is closed, then $A_f^n$ is closed in $(X, \delta)$ for each $n \in N$ but the converse is not true [13]. **Corollary 3.6** provides a pair of if and only if conditions for an fs-set $A_f(s)$ to be closed in an FSTS.

**Corollary 3.6.** In an FSTS $(X, \delta(s))$, an fs-set $A_f(s) = \{A_f^n\}_{n=1}^\infty$ is closed if and only if $A_f^n = \{B_f^n\}$ and $A_f^n$ is closed in $(X, \delta_n)$ for each $n \in N$, where $B_f^n = n^{th}$ component of $n_B A_f^n(X_f^n(s))$.

2. If and only if $A_f^n$ is closed in $(X, \delta(R^n_{A_f(s)}))$ for each $n \in N$, where $R^n_{A_f(s)}$ is the $n^{th}$ relative FS-closure operator of the closure operator in $(X, \delta(s))$ with respect to $A_f(s)$.

**Theorem 3.7.** If $\{A_f^\lambda; \lambda \in \Lambda\}$ be a chain of fs-sets in $((I^X)^N, \leq)$, then $\delta(Cl)^n_{A_f^\lambda, s}$, $\lambda \in \Lambda$ is a chain of fuzzy topologies on $X$ for each $n \in N$, where $Cl : (I^X)^N \rightarrow (I^X)^N$ is an FS-closure
operator on $X$.

**Proof:** Let $A_{\lambda f}(s) \leq A_{\mu f}(s)$, $\lambda, \mu \in \Lambda$. It suffices to show that $\delta\left(C_{n}^{(A_{\lambda f}(s))}\right) \leq \delta\left(C_{n}^{(A_{\mu f}(s))}\right)$.

Let $B \in \delta\left(C_{n}^{(A_{\mu f}(s))}\right)$.

Then $\left(C_{n}^{(A_{\lambda f}(s))}(\top - B) = \top - B\right)$.

Therefore $n^{th}$ term of $\left(C_{n}^{(A_{\lambda f}(s))}(\top - B)\right)$ is contained in $\top = (\delta\left(C_{n}^{(A_{\lambda f}(s))}\right)) - \delta\left(C_{n}^{(A_{\lambda f}(s))}\right)$.

Hence $B \in \delta\left(C_{n}^{(A_{\lambda f}(s))}\right)$.

**Definition 3.8.** Each member except possibly $\top$ of $\delta\left(C_{n}^{(A_{\lambda f}(s))}\right)$ is contained in $\top = (\delta\left(C_{n}^{(A_{\lambda f}(s))}\right)) - \delta\left(C_{n}^{(A_{\lambda f}(s))}\right)$.

**Theorem 3.9.** Let $\{C_{n} : I^{X} \rightarrow I^{X}\}$ be a sequence of fuzzy closure operators on $X$. Then the operator $C : (I^{X})^{\mathbb{N}} \rightarrow (I^{X})^{\mathbb{N}}$ defined by $C(A^{f}(s)) = \{C_{n}(A_{n})\}$ for all $A^{f}(s) = \{A^{n}_{f}\}_{n=1}^{\infty} \subseteq (I^{X})^{\mathbb{N}}$ is an FS-closure operator on $X$.

**Proof:** The proof is omitted.

**Definition 3.10.** Let $\{C_{n} : I^{X} \rightarrow I^{X}\}$ be a sequence of fuzzy closure operators on $X$. The operator $C : (I^{X})^{\mathbb{N}} \rightarrow (I^{X})^{\mathbb{N}}$ defined by $C(A^{f}(s)) = \{C_{n}(A_{n})\}$ for all $A^{f}(s) = \{A^{n}_{f}\}_{n=1}^{\infty} \subseteq (I^{X})^{\mathbb{N}}$ is called an FS-closure operator induced by a sequence $\{C_{n} : I^{X} \rightarrow I^{X}\}$ of fuzzy closure operators on $X$.

**Definition 3.11.** Let $\delta$ and $\delta'$ be two fuzzy topologies on a set $X$. A subset $K_{f}$ of $\delta^{\mathbb{N}}$ is called an FS-connector of $\delta$ to $\delta'$ if it satisfies the following conditions:

1. $A_{\lambda} \in \delta$ and $f_{\lambda} \in K_{f}, \lambda \in \Lambda \Rightarrow$ there exist $f \in K_{f}$ such that $f(\forall_{\lambda \in \Lambda} A_{\lambda}) = \forall_{\lambda \in \Lambda} f(A_{\lambda})$.
2. $A_{\lambda} \in \delta$ and $f_{\lambda} \in K_{f}, i = 1(1)n \Rightarrow$ there exist $f \in K_{f}$ such that $f(\forall_{n=1}^{\infty} A_{i}) = \forall_{n=1}^{\infty} f(A_{i})$.
3. $\delta' = \forall_{f \in K_{f}} f(\delta)$.

**Example 3.12.** Let $\delta$ and $\delta'$ be two fuzzy topologies on a set $X$. A function $f : \delta \rightarrow \delta'$ is defined by $f(A) = O$ for all $A \in \delta$, where $O$ is a fixed element of $\delta'$, is called a constant function from $\delta$ into $\delta'$. If $K_{f}$ be the collection of all such constant functions from $\delta$ into $\delta'$, then $K_{f}$ forms an FS-connector from $\delta$ to $\delta'$.

**Definition 3.13.** Let $\delta$ and $\delta'$ be two fuzzy topologies on a set $X$. Then the collection of all constant functions from $\delta$ into $\delta'$ forms an FS-connector of $\delta$ to $\delta'$. This is called the discrete FS-connector of $\delta$ to $\delta'$.

If $\{\delta_{n}\}$ be a sequence of fuzzy topologies on a set $X$, then any sequence $\{K_{n}\}$ of FS-connector such that $K_{n}$ connects $\delta_{n}$ to $\delta_{n+1}$ for all $n \in \mathbb{N}$, provides a unique FST on $X$ (Theorem 3.14) which is denoted by $\delta(s) < \{\delta_{n}\}, \{K_{n}\} >$ such that the $n^{th}$ components $\delta < \{\delta_{n}\}, \{K_{n}\} >_{n} = \delta_{n}$ for all $n \in \mathbb{N}$ and it is called the FST generated by $\{\delta_{n}\}$ and $\{K_{n}\}$. If further each $K_{n}$ is the discrete FS-connector of $\delta_{n}$ to $\delta_{n+1}$, then the FST is said to be generated by $\{\delta_{n}\}$ and is denoted by $\delta < \{\delta_{n}\} >$.

**Theorem 3.14.** Let $\{\delta_{n}\}$ be a sequence of fuzzy topologies on a set $X$. Then for any sequence $\{K_{n}\}$ of FS-connector such that $K_{n}$ connects $\delta_{n}$ to $\delta_{n+1}$ for all $n \in \mathbb{N}$, there is a unique FST $\delta(s) < \{\delta_{n}\}, \{K_{n}\} >$ on $X$ such that $\delta(s) < \{\delta_{n}\}, \{K_{n}\} >_{n} = \delta_{n}$. Also for any FSTS $(X, \delta(s))$, there is a sequence $\{K_{n}\}$ of FS-connector such that $K_{n}$ connects $\delta_{n}$ to $\delta_{n+1}$ and $\delta(s) < \{\delta_{n}\}, \{K_{n}\} >$.

**Proof:** Let $K = \prod_{n=1}^{\infty} K_{n}, g = \{g_{n}\} \in K$ and $A \in \delta_{1}$. Define $H_{1} = A$ and $H_{n} = g_{n-1}g_{n-2}...g_{1}A, n > 1$. Let $H^{\lambda}_{n}(s) = \{H_{n}\} \in (I^{X})^{\mathbb{N}}$ and consider $\delta(s) < \{\delta_{n}\}, \{K_{n}\} > = \{X^{\lambda}_{1}(s), X^{\lambda}_{2}(s)\} \cup \{H^{\lambda}_{n}(s) ; g \in K \text{ and } A \in \delta_{1}\}$. Consider

$$H_{\lambda}(s) = H^{\lambda}_{n}(s) \in \delta(s), \lambda \in \Lambda$$

where $\Lambda$ is an index set and

$$A = \forall_{\lambda \in \Lambda} A_{\lambda} \in \delta_{1}.$$
For \( g_1 \in K_1 \) and \( A \in \delta_1 \) there exist \( g_1 \in K_1 \) such that
\[
g_1 A = \lor_{\lambda \in A} g_1 A_{\lambda}; \; g_{\lambda n} \in K_n
\]
and for \( g_{n-1}g_{n-2} \ldots g_2 g_1 A \in \delta_n \) there exist \( g_n \in K_n \) such that
\[
g_{n-1}g_{n-2} \ldots g_2 g_1 A = \lor_{\lambda \in A} g_{n-1} g_{n-2} \ldots g_2 g_1 A_{\lambda}.
\]

Obviously,
\[
\lor_{\lambda \in A} H_\lambda(s) = \lor_{\lambda \in A} H^{g_\lambda}_\lambda(s) = H^g_\lambda(s) \in \delta(s) \not\subseteq \{ \delta_n \}, \{ K_n \}
\]
where \( g = g_n \). Arguing in the same way it can be shown that \( \delta(s) \not\subseteq \{ \delta_n \}, \{ K_n \} \) is closed under finite intersection. Therefore, \( (X, \delta(s) \not\subseteq \{ \delta_n \}, \{ K_n \} ) \) is a fuzzy sequential topological space. The third condition to be an FS-connector ensures that \( \delta(s) \not\subseteq \{ \delta_n \}, \{ K_n \} > n = \delta_n \)
for all \( n \in \mathbb{N} \). For the next part, for each \( n \in \mathbb{N} \) define a relation \( R^{n,n+1} \) on \( \delta(s) \) by \( A_f(s) = \{ A^n_f \} R^{n,n+1} B^n_f = \{ B^n_f \} \) if and only if \( A^n_f \supseteq B^n_f \). Then \( R^{n,n+1} \) defines a partition of \( \delta(s) \) say
\[
\{ \text{Cls}(A_f(s)) \}; \; A_f(s) \in \delta^{n,n+1}(s) \not\subseteq \delta(s)
\]
where \( \delta^{n,n+1}(s) \) is a family of open fs-sets taking exactly one from each class of the partition of \( \delta(s) \) by \( R^{n,n+1} \) and \( \text{Cls}(A_f(s)) \) represents the class of \( A_f(s) \). Let
\[
K^{n,n+1} = \prod_{A_f(s) \in \delta^{n,n+1}(s)} \text{Cls}(A_f(s))
\]
Then each \( t \in K^{n,n+1} \) defines a function \( g_t : \delta_n \to \delta_{n+1} \) and \( K_n = \{ g_t ; t \in K^{n,n+1} \} \) is an FS-connector connecting \( \delta_n \) to \( \delta_{n+1} \) and properties of FS-connectors ensures that \( \delta(s) \not\subseteq \{ \delta_n \}, \{ K_n \} \).

Corollary 3.15. Let \( CI : (I^X)^\mathbb{N} \to (I^X)^\mathbb{N} \) be an FS-closure operator on \( X \). Then for any sequence \( \{ \delta_n \} \) of FS-connectors such that \( K_n \) connects \( \delta(s) \) to \( \delta(s) \) for all \( n \in \mathbb{N} \), there is a unique FST \( \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) on \( X \) such that \( \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) and the components of the closure operator on \( (X, \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) are \( (CI) \), \( n \in \mathbb{N} \).

Corollary 3.16. Let \( I : (I^X)^\mathbb{N} \to (I^X)^\mathbb{N} \) be an FS-interior operator on \( X \). Then for any sequence \( \{ \delta_n \} \) of FS-connectors such that \( K_n \) connects \( \delta(s) \) to \( \delta(s) \) for all \( n \in \mathbb{N} \), there is a unique FST \( \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) on \( X \) such that \( \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) and the components of the interior operator on \( (X, \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \) are \( (I) \), \( n \in \mathbb{N} \). Also for any FSTS \( (X, \delta(s)) \), there is a sequence \( \{ K_n \} \) of FS-connectors such that \( K_n \) connects \( \delta(s) \) to \( \delta(s) \) and \( \delta(s) \) to \( \delta(s) \) such that \( \delta(s) \not\subseteq \{ \delta(s) \}, \{ K_n \} \).

Corollary 3.17. If \( \{ \delta_n \} \) be a sequence of fuzzy topologies on a set \( X \) such that \( \delta_n = \delta \) for all \( n \in \mathbb{N} \), then \( \delta(s) \not\subseteq \{ \delta_n \} \Rightarrow \delta(s) \not\subseteq \{ \delta_n \} \).

Corollary 3.18. If \( \{ C_n : I^X \to I^X \} \) be a sequence of fuzzy closure operators and \( C \) be an FS-closure operator induced by \( C_n \), then \( C(s) = \delta(s) \not\subseteq \{ \delta_n \} \) where \( \delta_n \) is the fuzzy topology on \( X \) induced by \( C_n \), \( n \in \mathbb{N} \).

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