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SPHERICAL CHAINS INSIDE A SPHERICAL SEGMENT

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Abstract We consider a spherical chain whose centers are on a vertical plane that can be drawn inside a spherical fragment and we show some geometric properties related to the chain itself. We also give recursive and non recursive formulas for calculating the coordinates of the centers and the radii of the spheres.

1 Introduction

Let us consider a sphere "$PNQMP$" with diameter PQ and center R. If we cut this sphere by a plane, parallel to the coordinate plane then we get a circle. Now we construct the coordinate system considering C as origin on the diameter PQ (see Figure 1). It is possible to construct an infinite chain of spheres inside a spherical fragment where the centers of all sphere of the chain lie on a vertical plane, parallel to the YZ plane or may be YZ plane and each tangent to the plane $Y = 0$ and spherical fragment MQN and to its two immediate neighbors.

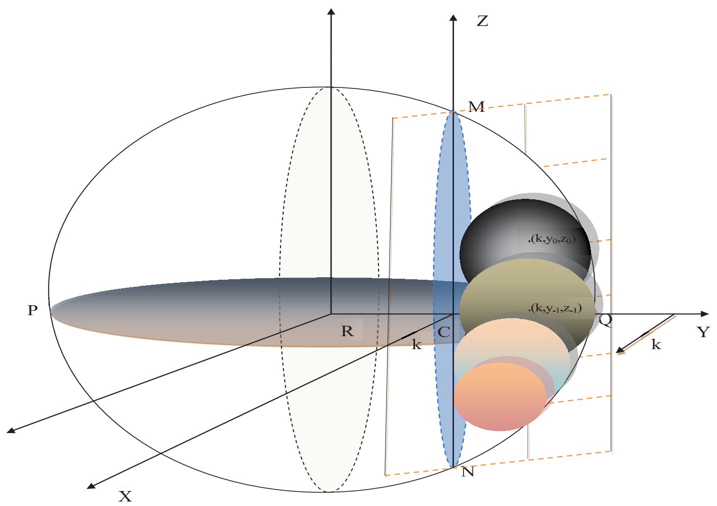

Figure 1. Spherical Chains inside a spherical segment

Let $2(a+b)$ be the diameter of the sphere and $2b$ be the length of the segment CQ. Here we have set up a Cartesian coordinate system with origin at C and let us consider spheres, tangent to the spherical fragment MQN and the plane $Y = 0$, with centers (k, y_i, z_i) lying on a vertical plane parallel to the YZ plane or on YZ plane depending upon the value of k and radii r_i for integer values of i, positive and negative and k is fixed but the values of k may be positive, negative or zero. That means for different values of k we get vertical planes parallel to YZ plane.

Here we have seen that the locus of the centers of the spherical chain mentioned above is a particular type of curve. Also we have shown that locus of the point of centers of the spheres of the chain lie on a sphere. We have also deduced recursive and non recursive formulae to find coordinates of centers and radii of the spheres of a spherical chain.

2 Some Geometric Properties of a Spherical Chain Touching $Y = 0$ Plane and Spherical Fragment MQN

Here we have discussed some basic properties of the infinite chain of spheres as mentioned above.

Theorem 2.1. *The locus of the centers lying on a vertical plane of the spheres touching the plane $Y = 0$ and spherical fragment MQN of a spherical chain is a parabola with axis is parallel to, Y-axis, focus is at a distance k from YZ plane and the vertex is at* $(k, b - \frac{k^2}{4a}, 0)$.

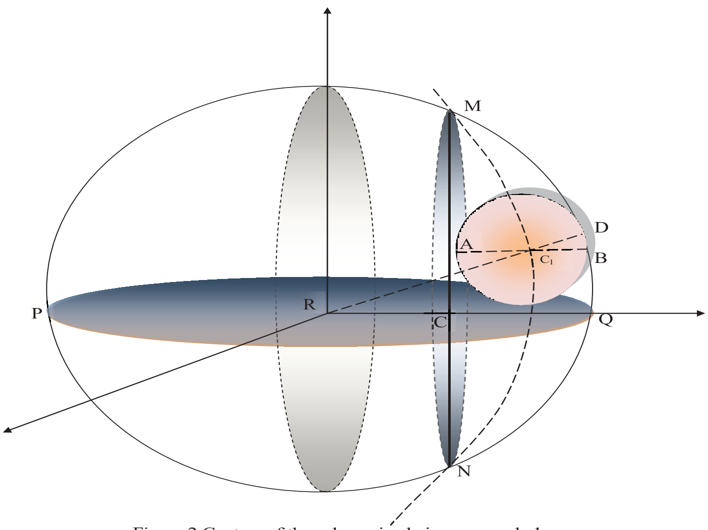

Figure 2 Centers of the spheres in chain on a parabola

Proof. Let us consider a sphere of the chain with center $C_1(k, y, z)$, lie on a vertical plane which is parallel to the coordinate plane YZ, diameter AB, radius r, tangent to $Y = 0$ at A and to the spherical fragment MQN at D. Since RD contains C_1 (see Figure 2), we have by taking into account that R, where R is the center of the sphere which contains the spherical chains, has coordinate $(0, b - a, 0)$ and

$$RD = a + b,$$

$$RC_1 = \sqrt{k^2 + (y - b + a)^2 + z^2},$$

$$C_1D = AC_1 = r = y.$$

Now, it is clear that

$$RC_1 = RD - C_1D.$$

From these, we have

$$\sqrt{k^2 + (y - b + a)^2 + z^2} = a + b - y,$$

which simplifies into

$$z^2 = -4a(y - (b - \frac{k^2}{4a})). \tag{2.1}$$

This represents a parabola symmetric with respect to the axis parallel to Y - axis with vertex $(k, b - \frac{k^2}{4a}, 0)$ and focus $(k, b - a - \frac{k^2}{4a}, 0)$.

Theorem 2.2. *The points of tangency between consecutive spheres of the chain lie on a sphere.*

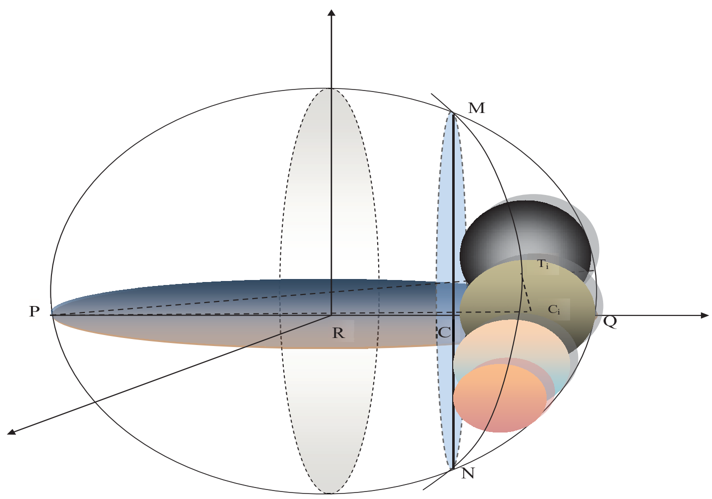

Figure 3. Points of tangency on a spherical arc

Proof. Consider two neighboring spheres with centers $C_i(k, y_i, z_i)$, $T_i(k, y_{i+1}, z_{i+1})$, radii r_i, r_{i+1} respectively, tangent to each other at T_i (see Figure 3) and touching the spherical fragment MQN and the plane $Y = 0$. By using Theorem 2.1 and noting that P has coordinate $(0, -2a, 0)$, we have

$$PC_i^2 = k^2 + (y_i + 2a)^2 + z_i^2 = k^2 + (-\frac{z^2}{4a} + b - \frac{k^2}{4a} + 2a)^2 + z_i^2,$$

$$r_i^2 = y_i^2 = (-\frac{z^2}{4a} + b - \frac{k^2}{4a})^2.$$

Applying the Pythagorean theorem to the right triangle PC_iT_i, we have

$$PT_i^2 = PC_i^2 - r_i^2 = 4a(a + b) = PC.PQ = PM^2.$$

Thus it follows that T_i lie on the sphere with center at P and radius PM.

Theorem 2.3. *If a sphere of the chain touches the plane $Y = 0$ at A and the spherical fragment, that containing MQN at D, then the points P (end point of the diameter opposite to plane $Y = 0$), A, D are collinear.*

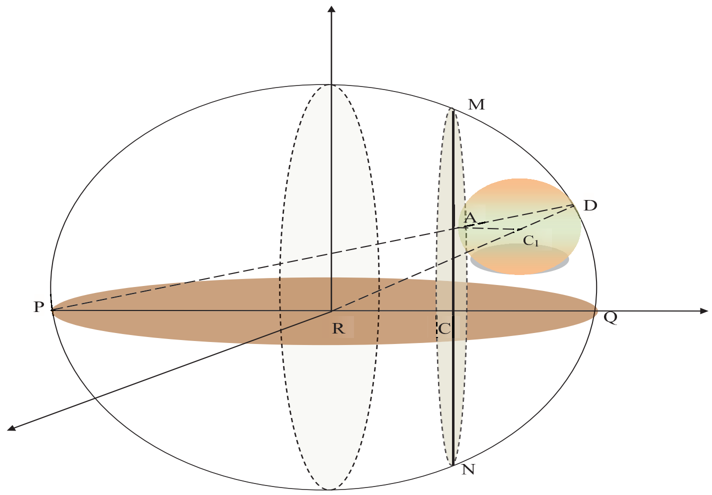

Figure 4. Line joining points of tangency

Proof. Suppose a sphere has center C_1 of a spherical chain which touches the plane $Y = 0$ at A and the spherical fragment MQN at D (see Figure 4). Note that triangles RPD and C_1AD are isosceles triangles where $\angle RDP = \angle RPD = \angle C_1DA = \angle C_1AD$. Thus P, A, D must be collinear as the triangles RPD and C_1AD are similar.

3 Recursive and Non Recursive Formulae to Find Coordinates of Centers and Radii of the Spheres of a Spherical Chain

From Figure 5, the triangle $C_iC_{i-1}P_i$ is right angle triangle (as C_iP_i is perpendicular drawn on r_{i-1}) with the centers C_{i-1} and C_i of two neighboring spheres of the chain. Since these spheres have radii $r_{i-1} = y_{i-1}$ and $r_i = y_i$ respectively, we have

$$(k-k)^2 + (y_i - y_{i-1})^2 + (z_i - z_{i-1})^2 = (r_i + r_{i-1})^2 = (y_i + y_{i-1})^2$$

$$(z_i - z_{i-1})^2 = 4y_iy_{i-1}.$$

Using (2.1), we can write

$$(z_i - z_{i-1})^2 = 4(b - \frac{k^2}{4a} - \frac{z_i^2}{4a})(b - \frac{k^2}{4a} - \frac{z_{i-1}^2}{4a}),$$

or

$$\frac{4a(a+b-k^2/4a) - z_{i-1}^2}{4a^2}.z_i^2 - 2z_{i-1}z_i + \frac{(a+b-k^2/4a)z_{i-1}^2 - 4a(b-k^2/4a)^2}{a} = 0 \quad (3.1)$$

If we index the spheres in the chain in such a way that the coordinate z_i increases with the index i, then from (3.1) we have

$$z_i = \frac{2z_{i-1} - (z_{i-1}^2/a - 4(b-k^2/4a))\sqrt{1 + \frac{(b-k^2/4a)}{a}}}{2(1 + \frac{(b-k^2/4a)}{a} - \frac{z_{i-1}^2}{4a^2})} \quad (3.2)$$

This is a recursive formula that can be applied provided that z_0 of the first circle is known. Note that z_0 must be chosen in the interval $(-2\sqrt{a(b-k^2/4a)}, 2\sqrt{a(b-k^2/4a)})$.

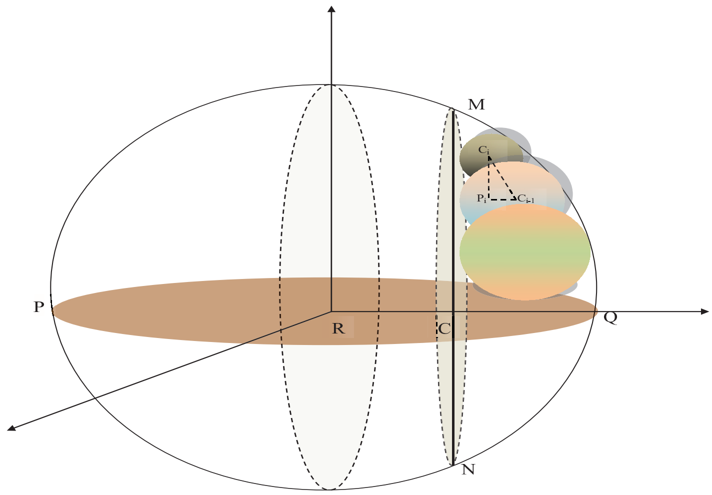

Figure 5. Construction for determination of recursive formula

Now the x coordinate is k and y_i are radii derived from (2.1), by

$$y_i = r_i = b - \frac{k^2}{4a} - \frac{z_i^2}{4a} \tag{3.3}$$

Now, it is possible to transform the recursion formula into a continued fraction and after some calculations, we get

$$z_i = 2a\left(\sqrt{1 + \frac{(b - k^2/4a)}{a}} - \frac{1}{\frac{z_{i-1}}{2a} + \sqrt{1 + \frac{(b-k^2/4a)}{a}}}\right). \tag{3.4}$$

Let

$$\ell = 2\sqrt{1 + \frac{(b - k^2/4a)}{a}}, \quad and \quad \zeta_i = \frac{z_i}{2a} - \sqrt{1 + \frac{(b - k^2/4a)}{a}}, \quad i = 1, 2, ..., \tag{3.5}$$

then, we have

$$\zeta_i = -\frac{1}{\ell + \zeta_{i-1}}$$

Thus, for positive integral values of i,

$$\zeta_i = -\frac{1}{\ell - \frac{1}{\ell - \frac{1}{\frac{\cdots}{-\frac{1}{\ell + \zeta_{0+}}}}}}$$

here we have used ζ_{0+} in place of ζ_0 and

$$\zeta_{0+} = \frac{z_0}{2a} - \sqrt{1 + \frac{(b - k^2/4a)}{a}}.$$

Now, if we solve equation (3.1) for z_{i-1} then we get

$$z_{i-1} = \frac{2z_i + (z_i^2/a - 4(b - k^2/4a))\sqrt{1 + \frac{(b-k^2/4a)}{a}}}{2(1 + \frac{(b-k^2/4a)}{a} - \frac{z_i^2}{4a^2})} \tag{3.6}$$

Thus, for negative integral values of i, with

$$\zeta_{-i} = \frac{z_{-i}}{2a} + \sqrt{1 + \frac{(b - k^2/4a)}{a}},$$

we have

$$\zeta_{-i} = -\cfrac{1}{-\ell - \cfrac{1}{-\ell - \cfrac{1}{\cfrac{\cdots}{-\cfrac{1}{-\ell+\zeta_{0+}}}}}},$$

where

$$\zeta_{0-} = \frac{z_0}{2a} + \sqrt{1 + \frac{(b - k^2/4a)}{a}}.$$

Therefore it is possible to give nonrecursive formulae for calculating z_i and z_{-i}. In the following, here we shall consider only z_i for positive integer indices because, as far as z_{-i} is concerned, it is enough to change, in all the formulae involved, ℓ into $-\ell$, z_i into z_{i-1}. Starting from (3.4), and by considering its particular structure, one can write, for $i = 1, 2, 3, ...$

$$\zeta_i = -\frac{\phi_{i-1}(\ell)}{\phi_i(\ell)}$$

where $\phi_i(\ell)$ are polynomials with integer coefficients. Here are the first five of them.

$\phi_0(\ell)$	1
$\phi_1(\ell)$	$\ell + \zeta_{0+}$
$\phi_2(\ell)$	$(\ell^2 - 1) + \ell\zeta_{0+}$
$\phi_3(\ell)$	$(\ell^3 - 2\ell) + (\ell^2 - 1)\zeta_{0+}$
$\phi_4(\ell)$	$(\ell^4 - 3\ell^2 + 1) + (\ell^3 - 2\ell)\zeta_{0+}$
$\phi_5(\ell)$	$(\ell^5 - 4\ell^3 + 3\ell) + (\ell^4 - 3\ell^2 + 1)\zeta_{0+}$

According to a fundamental property of continued fraction [1], these polynomials satisfy the second order linear recurrence

$$\phi_i(\ell) = \ell\phi_{i-1}(\ell) - \phi_{i-2}(\ell). \tag{3.7}$$

We can further write

$$\phi_i(\ell) = \psi_i(\ell) + \psi_{i-1}(\ell)\zeta_{0+}, \tag{3.8}$$

for a sequence of simpler polynomials $\psi_i(\ell)$, each either odd or even. In fact, from (3.7) and (3.8), we have

$$\psi_{i+2}(\ell) = \ell\psi_{i+1}(\ell) - \psi_i(\ell).$$

Explicitly,

$$\psi_i(\ell) = \left\{ \begin{array}{ll} 1, & i = 0 \\ \sum_{n=0}^{\frac{1}{2}} (-1)^{\frac{1}{2}+n} \begin{pmatrix} \frac{1}{2} + n \\ 2n \end{pmatrix} \ell^{2n}, & i = 2, 4, 6, ... \\ \sum_{n=1}^{\frac{i+1}{2}} (-1)^{\frac{i+1}{2}+n} \begin{pmatrix} \frac{i-1}{2} + n \\ 2n - 1 \end{pmatrix} \ell^{2n-1}, & i = 1, 3, 5, ... \end{array} \right\}$$

From (3.5), we have

$$z_i = a(\ell - 2\frac{\phi_{i-1}(\ell)}{\phi_i(\ell)}), \tag{3.9}$$

for $i = 1, 2, ...$

References

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