Palestine Journal of Mathematics

Vol. 5(1) (2016) , 35–42

The Lomax-Gumbel Distribution

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Communicated by S. P. Goyal

MSC 2010 Classifications: Primary 60E

Keywords and phrases: Lomax-Gumbel distribution , characteristic function, estimation, hazard rate function, moments, survivor function.

Abstract
In this paper, we introduce a new statistical distribution constructed by combining the cumulative density functions (cdf) of Lomax and Gumbel distributions and name it as the Lomax-Gumbel distribution. We derive expressions for its characteristic function, moments, hazard rate function and survivor function. We plot some graphs for its probability density function (pdf) and hazard rate function using the software 'Mathematica'. We also discuss estimation by the method of maximum-likelihood.

1 Introduction

The Gumbel distribution is an extreme value distribution of type I, widely applied for problems in engineering. Some of its major application areas include flood frequency analysis, network engineering, nuclear engineering, offshore engineering, risk based engineering, space engineering, software reliability engineering, structural engineering and wind engineering. However, in many applied areas like reliability, lifetime analysis, finance, and insurance, there is a clear need for extended forms of classical distributions, that is, new distributions which are more flexible to model real data in these areas. The data in these areas can present a high degree of skewness and kurtosis and we can have additional control over them by adding new parameters to the existing distributions.

Recent developments focus on new techniques for building meaningful distributions. These include the two-piece approach introduced by Hansen (1994), the perturbation approach of Azzalini and Capitanio (2003), and the generator approach pioneered by Eugene *et al.* (2002) and Jones (2004). Many researchers have worked using such technique, a few to mention are Gupta and Kundu (2001), Nadarajah and Kotz (2004, 2005), Akinsete et al. (2008), Zografos and Balakrishnan (2009), Codeiro et al. (2010), Barreto-Souza et al. (2010), Zhu and Galbraith (2010), Cordeiro et al.(2011, 2012, 2012, 2013) and Nadarajah (2013).

In the present paper, we introduce a new approach of combining the cdf's of two known distributions to define the Lomax-Gumbel distribution. The rest of the paper is organized as follows. In Section 2, we give some basic definitions which will be required in subsequent sections. In Section 3, we introduce the Lomax-Gumbel distribution and plot graphs of its pdf. We also obtain the expressions for characteristic function, moments, hazard rate function and the survivor function for this distribution. In Section 4, we discuss estimation of the four parameters $\alpha, \beta, \mu, \sigma$ by the method of maximum-likelihood. In Section 5, we give some conclusion.

2 Preliminaries

2.1 Lomax Distribution

The Lomax distribution is a heavy tail probability distribution, often used in business, economics and actuarial modelling. Its applications in modelling and analysing the lifetime data in medical and biological sciences and engineering are widely studied by Lomax (1954) and Moghadam et al. (2012).

The probability density function (pdf) of Lomax distribution is given by

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-(1+\alpha)} , \qquad \alpha,\beta>0,\ x\geq 0 \tag{2.1}$$

And the cdf of Lomax distribution is given by

$$F(x;\alpha,\beta) = 1-\left(1+\frac{x}{\beta}\right)^{-\alpha}, \qquad \alpha,\beta>0,\ x\geq 0 \tag{2.2}$$

2.2 Gumbel Distribution

The Gumbel distribution (Kotz & Nadarazah, 2000) is an important extreme value distribution of type I, which is used to model the distribution of the maximum of a number of samples of various distributions. The pdf of Gumbel distribution is given by

$$g(x) = \frac{1}{\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}+\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)\right\}\right)\right\} -\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma>0 \tag{2.3}$$

and the cdf of the Gumbel distribution is given by

$$G(x) = \frac{1}{\sigma}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}, \quad -\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma>0 \tag{2.4}$$

2.3 The composition of cdf's

Let F(x) and f(x) be the cdf and pdf of a statistical distribution referred as the first and having the support $[0,a]$, $where\ 1\leq a<\infty$ and G(x) and g(x) be the cdf and pdf of the second distribution with the support as $[b,c]$, where b, c $\in R$or $[0,\infty)$ or $(-\infty,\infty)$. We define the new distribution with cdf as given by

$$F_G(x) = \frac{F(G(x))}{F(1)} \tag{2.5}$$

and the corresponding pdf as

$$f_G(x) = \frac{F'(G(x))}{F(1)}g(x) \tag{2.6}$$

Clearly the support of the new distribution is same as that of the second distribution. Also for different choices of first and second distributions we can construct a large number of new distributions. The additional advantage of the new distribution is that it has more parameters to have a better control.

3 The Lomax-Gumbel distribution

In equations (2.5) and (2.6), taking F(x) as Lomax cdf given by (2.2) and G(x) as Gumbel cdf given by (2.4) we define the cdf of Lomax-Gumbel distribution as follows

$$F_{LG}(x;\alpha,\beta,\mu,\sigma) = K\left[1-\left\{1+\frac{1}{\beta}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right\}^{-\alpha}\right] \tag{3.1}$$

and the pdf is then given by

$$f_{LG}(x;\alpha,\beta,\mu,\sigma) = \frac{K\alpha}{\beta\sigma}\exp\left\{-\left(\frac{x-\mu}{\sigma}+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}\left[1+\frac{1}{\beta}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{-(\alpha+1)} \tag{3.2}$$

where $K^{-1} = 1-\left(1+\beta^{-1}\right)^{-\alpha}$, $\alpha,\ \beta,\ \sigma>0,\ -\infty<\mu<\infty, -\infty<x<\infty.$

Remarks

(i) On taking ţ=0 and σ=1, in (3.1) and (3.2), the cdf and pdf are reduced to the following forms respectively

$$F(x;\alpha,\beta,0,1) = K\left[1-\left\{1+\frac{1}{\beta}\exp\left(-e^{-x}\right)\right\}^{-\alpha}\right] \tag{3.3}$$

and

$$f(x;\alpha,\beta,0,1) = \frac{K\alpha}{\beta}\exp\left\{-\left(x+e^{-x}\right)\right\}\left[1+\frac{1}{\beta}\exp\left(-e^{-x}\right)\right]^{-(\alpha+1)} \tag{3.4}$$

where $K^{-1} = 1 - \left(1+\beta^{-1}\right)^{-\alpha}$, $\alpha,\, \beta > 0, -\infty < x < \infty.$

The above distribution may be referred as the standard Lomax-Gumbel distribution.

(ii) We observe that $\underset{x\to\infty}{Lim}\, f_{LG}\,(x) \;=\; 0$ and $\underset{x\to -\infty}{Lim}\, f_{LG}\,(x) \;=\; 0.$

3.1 Graphs

Some graphs illustrating the effects of various parameters on the shape of the pdf have been drawn in support to the study.

Here, we have drawn four graphs using the software 'Mathematica'. In Fig 1, we fix the parameters μ =1.5, σ=3, β=0.5 and plot the graph of Lomax-Gumbel pdf for varying values of α. In Fig. 2, we fix the parameters μ =1.5, σ=3, α =1 and plot the graph for varying values of β. In Fig. 3, we fix the parameters α=1, β=2 and μ =1.5 and plot the graph for varying values of σ. In Fig. 4, we fix the parameters α=1, β=2, σ=3 and plot the graph for varying values of μ .

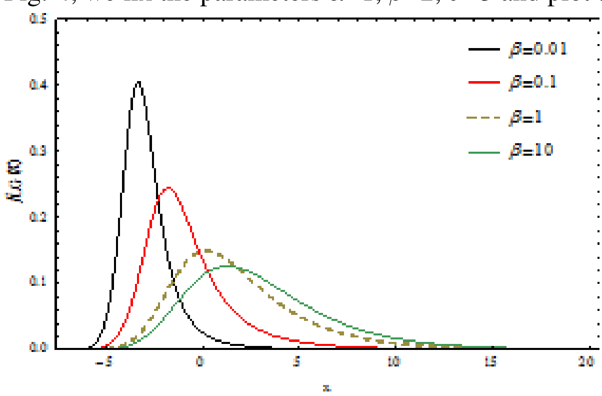

Fig 1 The Lomax-Gumbel pdf f(x)
for μ=1.5, σ=3, β=0.5

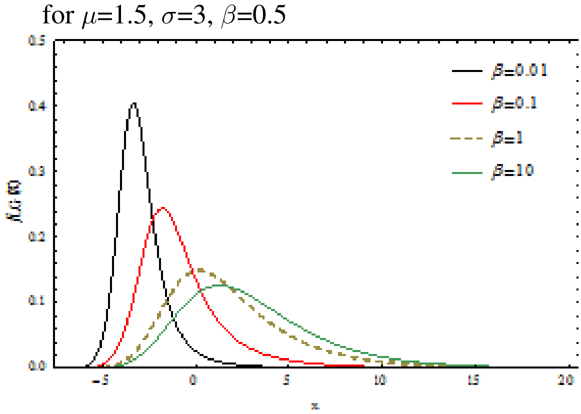

Fig 2 The Lomax-Gumbel pdf f(x)
for μ=1.5, σ=3, α =1

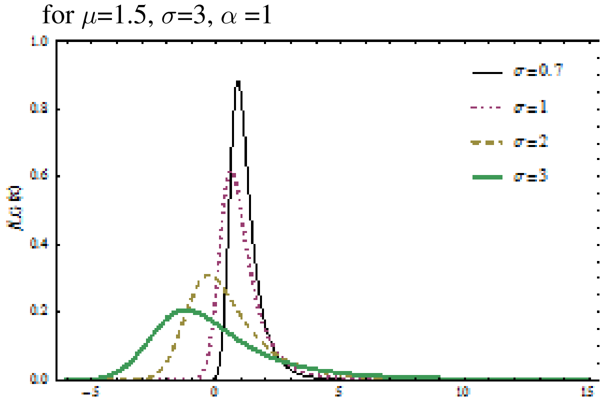

Fig 3 The Lomax-Gumbel pdf f(x)
for α=1, β=2, μ=1.5

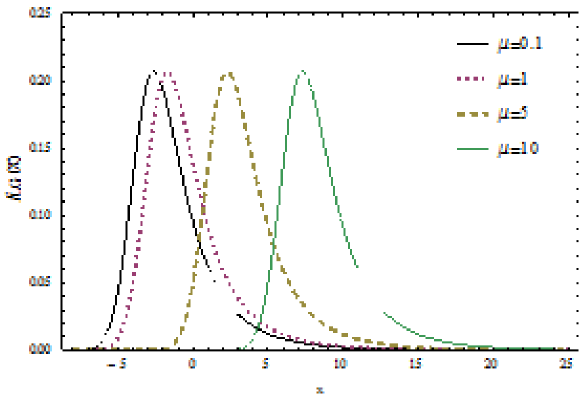

Fig 4 The Lomax-Gumbel pdf f(x)
for α=1, β=2 and σ=3

3.2 The characteristic function

The characteristic function of pdf f(x) is given by

$$\phi(t) = E\left(\exp\left(itx\right)\right) = \int_{-\infty}^{\infty} \exp\left(itx\right)\ f(x)\,dx, i = \upsilon - 1 \tag{3.5}$$

Substituting the value of f(x) from (3.2) in (3.5), using binomial expansion and then evaluating the integral using a known result (Gradshteyn et al., 1994, p.353, eq.(3.312.3)), we get

$$\phi(t) = \frac{K\alpha}{\beta} e^{itx} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha+1)_k}{k!} \left(\frac{1}{\beta}\right)^k \frac{\Gamma(it\sigma+1)}{(k+1)^{it\sigma+1}};\ \alpha,\ \beta,\ \sigma > 0,\ -\infty < \mu < \infty$$

Which can further be written as

$$\phi(t) = \frac{K\alpha}{\beta} e^{itx} \Gamma\left(it\sigma+1\right) \Phi^*_{\alpha+1}\left(-\frac{1}{\beta},\ it\sigma+1, 1\right), |\beta| > 1 \tag{3.6}$$

where $K^{-1} = 1 - \left(1+\beta^{-1}\right)^{-\alpha}$ and $\Phi^*_\alpha\left(z, s, a\right)$denotes the generalized Riemann Zeta function defined by Goyal and Laddha (1997).

3.3 The moments

The rth moment of the pdf f(x) about the origin is given by

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r\, f(x)\, dx\ ,\quad r = 1, 2, 3, \ldots \tag{3.7}$$

Substituting the value of f(x) from (3.2) and writing the second term using binomial expansion, we get

$$\begin{aligned} \mu'_r &= \tfrac{K\alpha}{\beta\sigma} \textstyle\sum_{n=0}^{\infty} \tfrac{(-1)^n (\alpha+1)_n}{n!} \left(\tfrac{1}{\beta}\right)^n \\ &\quad \times \textstyle\int_{-\infty}^{\infty} x^r\ \exp\left\{-\left(\tfrac{x-\mu}{\sigma} + \exp\left(-\tfrac{x-\mu}{\sigma}\right)\right)\right\}\ \left(\exp\left\{-\exp\left(-\tfrac{x-\mu}{\sigma}\right)\right\}\right)^n dx \end{aligned} \tag{3.8}$$

$\alpha,\ \beta,\ \sigma > 0,\ -\infty < \mu < \infty$, r=1,2,3,. . .
on substituting $\exp\left(-\frac{x-\mu}{\sigma}\right) = t$, the r.h.s. of eq.(3.8) reduces to

$$\frac{K\alpha}{\beta} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+1)_n}{n!} \left(\frac{1}{\beta}\right)^n \int_0^{\infty} \left(\mu - \sigma\log t\right)^r\ e^{-(1+n)t} dt \tag{3.9}$$

Next, writing the binomial expression in the integrand in series form and evaluating the integral using a known result in Prudnikov et al. (1986, 2.6.21.1), the rth moment of X can be expressed as

$$\mu'_r = \frac{K\alpha\mu^r}{\beta} \sum_{m=0}^{r} \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha+1)_n (-r)_m}{n!\, m!} \left(\frac{1}{\beta}\right)^n \left(\frac{\sigma}{\mu}\right)^m \left\{\left(\frac{\partial}{\partial\kappa}\right)^m \left((1+n)^{-\kappa}\,\Gamma(\kappa)\right)\right\}\Bigg|_{\kappa=1} \tag{3.10}$$

In particular, for r = 1, 2, 3..., we get the values of various moments as follows

$$\mu_1^{'} = \frac{K\alpha\mu}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n\,(1+\alpha)_n}{\beta^n\,\,(n+1)!}\left\{1-\gamma-\log\left(n+1\right)\right\}, \tag{3.11}$$

$$\begin{aligned}\mu_2^{'} = \tfrac{K\alpha}{2\beta}\textstyle\sum_{n=0}^{\infty}\tfrac{(-1)^n(1+\alpha)_n}{\beta^n\,(n+1)!}&\left\{2\mu^2-4\mu\gamma\sigma+\left(\pi^2+6\gamma^2\right)\sigma^2\right.\\ &\left.+4\sigma\left(3\gamma\sigma-\mu\right)\log\left(n+1\right)+6\sigma^2\log^2\left(n+1\right)\right\},\end{aligned} \tag{3.12}$$

$$\begin{aligned}\mu_3^{'} = \tfrac{K\alpha}{\beta}\textstyle\sum_{n=0}^{\infty}\tfrac{(-1)^n(1+\alpha)_n}{\beta^n\,(n+1)!}&\left\{\mu^3-3\mu^2\gamma\sigma+\mu\left(\pi^2+6\gamma^2\right)\sigma^2-5\gamma\left(2\gamma^2-\pi^2\right)\sigma^3\right.\\ &+\left(12\mu\gamma\sigma^2-3\mu^2\sigma-30\gamma^2\sigma^3-5\sigma^3\pi^2\right)\log\left(n+1\right)\\ &\left.+6\sigma^2\left(\mu-5\gamma\sigma\right)\log^2\left(n+1\right)-10\sigma^3\log^3\left(n+1\right)+\psi^{(2)}\left(n\right)\right\},\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\mu_4^{'} = \tfrac{K\alpha}{12\beta}\textstyle\sum_{n=0}^{\infty}\tfrac{(-1)^n(1+\alpha)_n}{\beta^n\,(n+1)!}&\\ \times\left\{12\mu^4-48\mu^3\gamma\sigma+20\mu^2\left(\pi^2+6\gamma^2\right)\sigma^2-240\mu\left(\gamma^3+\gamma\pi^2-\psi^{(2)}\left(n\right)\right)\sigma^3\right.&\\ +\left(20\gamma^4+20\gamma^2\pi^2+3\pi^2-80\gamma\psi^{(2)}\left(n\right)\right)\sigma^4-12\left(4\mu^3\sigma-20\mu^2\gamma\sigma^2\right.&\\ \left.-10\mu\left(6\gamma^2+\pi^2\right)\sigma^3+70\sigma^4\left(2\gamma^3+\gamma\pi^2-2-\psi^{(2)}\left(n\right)\right)\right)\log\left(n+1\right)&\\ +12\left(10\mu^4\sigma^2-6\gamma\mu\sigma^3+35\left(\pi^2+6\gamma^2\right)\sigma^4\right)\log^2\left(n+1\right)&\\ \left.-240\sigma^3\left(\mu+7\sigma\right)\log^3\left(n+1\right)+420\log^4\left(n+1\right)\right\},&\end{aligned} \tag{3.14}$$

where $K^{-1} = 1-\left(1+\beta^{-1}\right)^{-\alpha}$, γ denotes Euler's constant and $\psi^{(2)}\left(n\right)$denotes the polygamma function(Abramowitz et al., 1972, §6.4, p. 260].
The variance, skewness, and kurtosis measures can now be calculated using the following relations

$$Var\left(x\right) = {\mu_2}' - {\mu_1}'^2 \tag{3.15}$$

$$Skewness(x) = \frac{{\mu_3}' - 3{\mu_1}'\,{\mu_2}' + 2\,\mu_1'^3}{Var^{3/2}\left(x\right)} \tag{3.16}$$

$$Kurtosis\left(x\right) = \frac{\mu_4^{'}-4\mu_1^{'}\mu_3^{'}+6\mu_2^{'}\mu_1^{'2}-3\,\mu_1^{'4}}{Var^2\left(x\right)} \tag{3.17}$$

3.4 Hazard rate function

The hazard rate function h(x) is given by $h(x) = \frac{f(x)}{1-F(x)}$.
Substituting the values of f(x) and F(x) from (3.2) and (3.1) respectively, and simplifying we get

$$h\left(x\right) = \frac{\frac{\alpha}{\beta\sigma}\,\exp\left\{-\left(\frac{x-\mu}{\sigma}+\exp\left(-\frac{x-\mu}{\sigma}\right)\right)\right\}\left[1+\frac{1}{\beta}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right]^{-\alpha-1}}{\left\{1+\frac{1}{\beta}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right\}^{-\alpha}-\left(1+\beta^{-1}\right)^{-\alpha}} \tag{3.18}$$

for $\alpha, \beta, \sigma > 0$, $-\infty < \mu < \infty$.
Fig. 3.4.1 illustrates some possible shapes of h(x) for different values of parameter β. It is interesting to see, decreasing hazard rate for $\beta < 0.6$while increasing hazard rate for $\beta > 0.6$. This could be an attractive property of Lomax-Gumbel distribution in reliability and life-testing experiments.

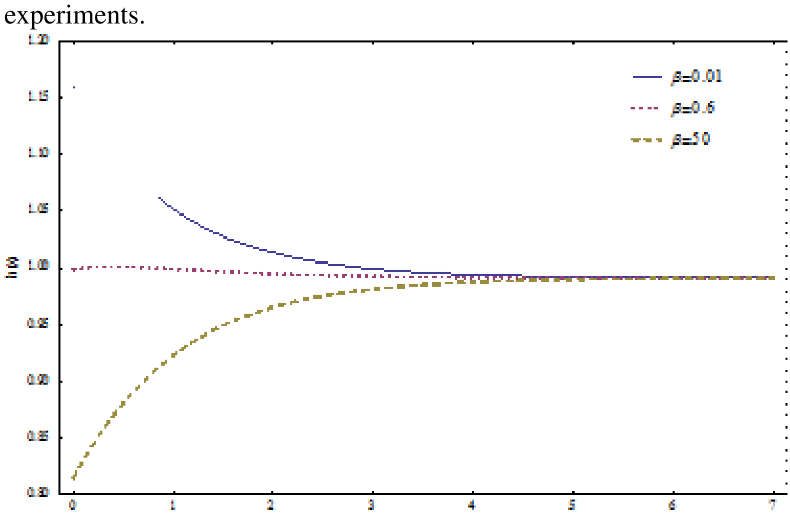

Fig. 3.4.1 Hazard rate function of Lomax-Gumbel distribution for different values of β with $\alpha = 0.85$, $\mu = -0.95$, $\sigma = 1.009$.

3.5 Survivor function

The survivor function S(x) is given by

$$S(x) = P(X > x) = \int_x^{\infty} f(u)du = 1 - F(x)$$

Substituting the value of F(x) from (3.1), we get

$$S(x) = \left(1+\beta^{-1}\right)^{-\alpha}\left[1+\left\{1+\frac{1}{\beta}\exp\left\{-\exp\left(-\frac{x-\mu}{\sigma}\right)\right\}\right\}^{-\alpha}\right] \quad (3.19)$$

$\alpha, \beta, \sigma > 0, -\infty < \mu < \infty$

4 Estimation

We estimate the parameters by the method of maximum likelihood. The log-likelihood for a random sample $x_1, \ldots, x_n$ from pdf (3.2) is

$$\log L(\alpha,\beta,\mu,\sigma) = \log(\alpha) - nlog(\beta) - nlog(\sigma) - nlog\left\{1-\left(1+\tfrac{1}{\beta}\right)^{-a}\right\} - \sum_{i=1}^{n}\tfrac{x_i-\mu}{\sigma}$$
$$+\sum_{i=1}^{n}\exp\left\{-\left(\tfrac{x_i-\mu}{\sigma}\right)\right\} - (\alpha+1)\sum_{i=1}^{n}\log\left[1+\tfrac{1}{\beta}\exp\left\{-\exp\left(-\left(\tfrac{x_i-\mu}{\sigma}\right)\right)\right\}\right] \quad (4.1)$$

The first order partial derivatives of (4.1) with respect to the four parameters are

$$\frac{\partial \log L}{\partial \alpha} = \frac{1}{\alpha} - \frac{n\log\left(1+\beta^{-1}\right)}{\left(1+\beta^{-1}\right)^{\alpha}-1} - \sum_{i=1}^{n}\log\left[1+\frac{1}{\beta}\exp\left\{-\exp\left(-\left(\frac{x_i-\mu}{\sigma}\right)\right)\right\}\right] \quad (4.2)$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{n}{\beta} + \frac{\alpha n\left(1+\beta^{-1}\right)^{-1}}{\beta^2\left[\left(1+\beta^{-1}\right)^{\alpha}-1\right]} - \frac{(\alpha+1)}{\beta^2}\sum_{i=1}^{n}\frac{\exp\{-\exp\left(-(x_i-\mu)/\sigma\right)\}}{\left[1+\beta^{-1}\exp\{-\exp\left(-(x_i-\mu)/\sigma\right)\}\right]} \quad (4.3)$$

$$\frac{\partial \log L}{\partial \mu} = \frac{n}{\sigma} + \frac{1}{\sigma}\sum_{i=1}^{n}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)\right\} + (\alpha+1)\sum_{i=1}^{n}\frac{\exp\{-\exp\left(-(x_i-\mu)/\sigma\right)-(x_i-\mu)/\sigma\}}{\beta\sigma\left[1+\beta^{-1}\exp\{-\exp\left(-(x_i-\mu)/\sigma\right)\}\right]} \quad (4.4)$$

$$\frac{\partial \log L}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2} + \sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2}\exp\left\{-\left(\frac{x_i-\mu}{\sigma}\right)\right\}$$
$$+(\alpha+1)\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma^2}\right)\frac{\exp\{-\exp(-(x_i-\mu)/\sigma)-(x_i-\mu)/\sigma\}}{\beta[1+\beta^{-1}\exp\{-\exp(-(x_i-\mu)/\sigma)\}]} \quad (4.5)$$

Setting these expressions to zero and solving them simultaneously yields the maximum-likelihood estimates of the four parameters.

5 Conclusion

In this paper we have developed a new technique for constructing a family of new statistical distributions by combining the cdf of two known statistical distributions. We have thus defined and studied a new distribution by combining the cdf's of Lomax and Gumbel distributions and named it as Lomax-Gumbel distribution. This distribution contains four parameters, two from Lomax and two from Gumbel and thus has more flexibility as compared to its constituent distributions. We have derived explicit expressions for the moments, characteristic function, hazard rate function, and survivor function for the Lomax-Gumbel distribution. Using the software 'Mathematica' we plot some graphs for its pdf which show the effect of variation of different parameters occurring in the definition and investigate the variation of the hazard rate function. We also discuss estimation by the method of maximum-likelihood.

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Received: December 11, 2014.

Accepted: January 12, 2015