

GAUSS LEGENDRE MULTIPLICATION FORMULA FOR p -ADIC BETA FUNCTION

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Abstract In the present work we study a p -adic analogue of the classical beta function by using Y. Morita’s p -adic gamma function. We obtain Gauss Legendre multiplication type formulas for the p -adic beta function.

1 Introduction

Let p be a odd prime number and let \mathbb{Z}_p and \mathbb{Q}_p denote the ring of p -adic integers and the field of p -adic numbers, respectively. The p -adic analogue of the classical gamma function Γ is defined by Y. Morita (1975) [10] as the continuous extension to \mathbb{Z}_p of the function $n \rightarrow (-1)^n \prod_{\substack{1 \leq j < n \\ (p,j)=1}} j$,

i.e., $\Gamma_p(x)$ is defined by the formula

$$\Gamma_p(x) = \lim_{n \rightarrow x} (-1)^n \prod_{\substack{j < n \\ (p,j)=1}} j$$

for $x \in \mathbb{Z}_p$, where n approaches x through positive integers. The p -adic gamma function Γ_p had been studied by J. Diamond (1977) [5], D. Barsky (1979) [2], T. K. Kim (1997) [8] and others. The relationship between some special functions and the p -adic gamma function $\Gamma_p(x)$ were investigated by B. Gross and N. Koblitz (1979) [7], H. Cohen and E. Friedman (2008) [4] and I. Shapiro (2012) [12].

We use the following properties of the p -adic gamma function Γ_p (for details see [6] and [11])

(i) For all $x \in \mathbb{Z}_p$

$$\Gamma_p(x + 1) = h_p(x)\Gamma_p(x) \tag{1.1}$$

where

$$h_p(x) := \begin{cases} -x, & \text{if } |x|_p = 1 \\ -1, & \text{if } |x|_p < 1 \end{cases}.$$

(ii) $\Gamma_p(0) = 1, \Gamma_p(1) = -1, \Gamma_p(2) = 1$ and $|\Gamma_p(x)|_p = 1$ for all $x \in \mathbb{Z}_p$.

(iii) For all $x \in \mathbb{Z}_p$

$$\Gamma_p(x)\Gamma_p(1 - x) = (-1)^{\ell(x)} \quad (x \in \mathbb{Z}_p) \tag{1.2}$$

where $\ell : \mathbb{Z}_p \rightarrow \{1, 2, \dots, p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, \dots, p\}$ modulo $p\mathbb{Z}_p$. Hence,

$$\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\ell\left(\frac{1}{2}\right)} \tag{1.3}$$

and

$$\Gamma_p\left(\frac{1}{2}\right)^2 = \begin{cases} 1 & \text{if } p \equiv 3 \pmod{4} \\ -1 & \text{if } p \equiv 1 \pmod{4}. \end{cases}$$

Assume that $m, n \in \mathbb{N}$ and $m \geq 2$ is not divisible by p . The first version of the p -adic Gauss Legendre multiplication formula ([11]) states that

$$\Gamma_p(n)\Gamma_p\left(n + \frac{1}{m}\right)\dots\Gamma_p\left(n + \frac{m-1}{m}\right) = \left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right) \Gamma_p(mn)m^{-\mu(mn)} \quad (n \in \mathbb{N}) \tag{1.4}$$

where $\mu(k)$ is the number of elements of $\{1, 2, \dots, k - 1\}$ that are not divisible by p , i.e.

$$\mu(k) = k - 1 - \left\lfloor \frac{k - 1}{p} \right\rfloor.$$

In the case $m = 2$, the first version of the p -adic Gauss Legendre formula says

$$\Gamma_p(n)\Gamma_p\left(n + \frac{1}{2}\right) = \Gamma_p\left(\frac{1}{2}\right)\Gamma_p(2n)2^{-(2n-1)}2^{\left\lfloor \frac{2n-1}{p} \right\rfloor} \tag{1.5}$$

so that

$$\Gamma_p(2n) = 2^{2n-1}\Gamma_p\left(\frac{1}{2}\right)^{-1}\Gamma_p(n)\Gamma_p\left(n + \frac{1}{2}\right)2^{-\left\lfloor \frac{2n-1}{p} \right\rfloor}. \tag{1.6}$$

The Gauss Legendre multiplication formula for p -adic gamma function as follows:

For each $x \in \mathbb{Z}_p$, let $\ell(x) \in \{1, 2, \dots, p\}$ be such that $|x - \ell(x)|_p < 1$. Further, let $\ell_1(x) = p^{-1}(x - \ell(x))$ ($x \in \mathbb{Z}_p$). Then for $m > 1$, m not divisible by p

$$\prod_{j=0}^{m-1} \Gamma_p\left(x + \frac{j}{m}\right) = \left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right) m^{1-\ell(mx)} (m^{p-1})^{-\ell_1(mx)} \Gamma_p(mx). \tag{1.7}$$

A p -adic analogue of classical beta function by using Morita's p -adic gamma function can be defined as follows

Definition 1.1. The p -adic beta function $B_p : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ is defined by the formula

$$B_p(x, y) := \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p(x+y)}, \quad x, y \in \mathbb{Z}_p$$

In 1980 M. Boyarsky [3] used the p -adic beta function in Dwork cohomology and gave an cohomological interpretation of the p -adic beta function. In 2006 F. Baldassarri [1] considered two constructions of the p -adic beta functions as the p -adic etale and p -adic crystalline beta functions, and gave some comparisons between them with relations via Fontaine's periods. In [9] the basic properties of the p -adic beta function are given.

In the present work we consider the p -adic beta function and we derive Gauss Legendre multiplication type formulas for the p -adic beta function.

2 Main Results

For the p -adic beta function, as a first version of Gauss Legendre multiplication formula we obtain the following result.

Theorem 2.1. Let $m, n \in \mathbb{N}$ and $m \geq 2$ is not divisible by p .

(i) If m is odd, then

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \frac{(-1)^{\frac{m-1}{2}} B_p(mn, mn) \left(\prod_{j=1}^{\frac{m-1}{2}} B_p\left(\frac{j}{m}, \frac{m-j}{m}\right)\right)}{\prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right) m^{2\mu(mn) - \mu(2mn)}} \tag{2.1}$$

(ii) If m is even, then

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \frac{(-1)^{\frac{m}{2}} \left(\prod_{j=0}^{\frac{m}{2}-1} B_p\left(\frac{2j+1}{m}, \frac{m-(2j+1)}{m}\right)\right) 2^{-2\mu(mn)}}{\prod_{j=0}^{\frac{m}{2}-1} h_p\left(2n + \frac{2j}{m}\right)} \tag{2.2}$$

where

$$\mu(k) = k - 1 - \left\lfloor \frac{k - 1}{p} \right\rfloor$$

Proof. From the first version of Gauss Legendre multiplication formula for p -adic gamma function that

$$\Gamma_p(n)\Gamma_p\left(n + \frac{1}{m}\right)\dots\Gamma_p\left(n + \frac{m-1}{m}\right) = \left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right) \Gamma_p(mn)m^{-\mu(mn)} \quad (n \in \mathbb{N}).$$

Taking $2n$ in place of n , the relation becomes

$$\prod_{j=0}^{m-1} \Gamma_p\left(2n + \frac{j}{m}\right) = \prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)m^{-\mu(2mn)}\Gamma_p(2mn). \tag{2.3}$$

From Definition 1.1 we get

$$\begin{aligned} & B_p(n, n)B_p\left(n + \frac{1}{m}, n + \frac{1}{m}\right)\dots B_p\left(n + \frac{m-1}{m}, n + \frac{m-1}{m}\right) = \\ &= \frac{\Gamma_p(n)\Gamma_p(n)\Gamma_p\left(n + \frac{1}{m}\right)\Gamma_p\left(n + \frac{1}{m}\right)}{\Gamma_p(2n)\Gamma_p\left(2n + \frac{2}{m}\right)} \dots \frac{\Gamma_p\left(n + \frac{m-1}{m}\right)\Gamma_p\left(n + \frac{m-1}{m}\right)}{\Gamma_p\left(2n + \frac{2m-2}{m}\right)} \\ &= \frac{\left(\prod_{j=0}^{m-1} \Gamma_p\left(n + \frac{j}{m}\right)\right)^2}{\prod_{j=0}^{m-1} \Gamma_p\left(2n + \frac{2j}{m}\right)}. \end{aligned} \tag{2.4}$$

Let

$$f(n) := \prod_{j=0}^{m-1} \Gamma_p\left(2n + \frac{2j}{m}\right). \tag{2.5}$$

Assume that m is odd. Thus, we can write

$$\begin{aligned} f(n) &= \Gamma_p(2n)\Gamma_p\left(2n + \frac{2}{m}\right)\dots\Gamma_p\left(2n + \frac{m-1}{m}\right)\Gamma_p\left(2n + \frac{m+1}{m}\right)\dots\Gamma_p\left(2n + \frac{2m-2}{m}\right) \\ &= \Gamma_p(2n)\dots\Gamma_p\left(2n + \frac{m-1}{m}\right)\Gamma_p\left(2n + \frac{1}{m} + 1\right)\dots\Gamma_p\left(2n + \frac{m-2}{m} + 1\right). \end{aligned}$$

By the equality (1.1) we obtain

$$f(n) = \Gamma_p(2n)\Gamma_p\left(2n + \frac{1}{m}\right)\Gamma_p\left(2n + \frac{2}{m}\right)\dots\Gamma_p\left(2n + \frac{m-1}{m}\right)h_p\left(2n + \frac{1}{m}\right)\dots h_p\left(2n + \frac{m-2}{m}\right)$$

such that

$$\begin{aligned} \Gamma_p\left(2n + \frac{m+1}{m}\right) &= \Gamma_p\left(2n + \frac{1}{m} + 1\right) = \Gamma_p\left(2n + \frac{1}{m}\right)h_p\left(2n + \frac{1}{m}\right) \\ \Gamma_p\left(2n + \frac{m+3}{m}\right) &= \Gamma_p\left(2n + \frac{3}{m} + 1\right) = \Gamma_p\left(2n + \frac{3}{m}\right)h_p\left(2n + \frac{3}{m}\right) \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

$$\Gamma_p\left(2n + \frac{2m-2}{m}\right) = \Gamma_p\left(2n + \frac{m-2}{m} + 1\right) = \Gamma_p\left(2n + \frac{m-2}{m}\right)h_p\left(2n + \frac{m-2}{m}\right).$$

Then, we can write

$$f(n) = \prod_{j=0}^{m-1} \Gamma_p\left(2n + \frac{j}{m}\right) \prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right).$$

Hence, using $f(n)$ in (2.4) we get the formula

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \frac{\left(\prod_{j=0}^{m-1} \Gamma_p\left(n + \frac{j}{m}\right)\right)^2}{\prod_{j=0}^{m-1} \Gamma_p\left(2n + \frac{j}{m}\right) \prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right)}$$

By using (1.4) and (2.3) we have

$$\begin{aligned} \prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) &= \frac{\left[\left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right) \Gamma_p(mn) m^{-\mu(mn)}\right]^2}{\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right) m^{-\mu(2mn)} \Gamma_p(2mn) \left(\prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right)\right)} \\ &= \frac{\left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right) \Gamma_p(mn) \Gamma_p(mn) m^{-2\mu(mn)}}{m^{-\mu(2mn)} \Gamma_p(2mn) \left(\prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right)\right)}. \end{aligned}$$

According to Definition 1.1 we obtain the formula

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \frac{B_p(mn, mn) m^{-2\mu(mn) + \mu(2mn)} \left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right)}{\prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2n + \frac{2j+1}{m}\right)}. \tag{2.6}$$

Now we must arrange $\left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right)\right)$. First, notice that $\Gamma_p(1) = -1$ and $\Gamma_p(0) = 1$. By using Definition 1.1 it follows that

$$\begin{aligned} \prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right) &= \Gamma_p(0) \Gamma_p\left(\frac{1}{m}\right) \Gamma_p\left(\frac{2}{m}\right) \dots \Gamma_p\left(\frac{m-2}{m}\right) \Gamma_p\left(\frac{m-1}{m}\right) \\ &= \frac{\Gamma_p\left(\frac{1}{m}\right) \Gamma_p\left(\frac{m-1}{m}\right)}{\Gamma_p(1)} \Gamma_p(1) \dots \frac{\Gamma_p\left(\frac{\frac{m-1}{2}}{m}\right) \Gamma_p\left(\frac{m - \left(\frac{m-1}{2}\right)}{m}\right)}{\Gamma_p(1)} \Gamma_p(1) \\ &= (-1)^{\frac{m-1}{2}} B_p\left(\frac{1}{m}, \frac{m-1}{m}\right) \dots B_p\left(\frac{\frac{m-1}{2}}{m}, \frac{m - \left(\frac{m-1}{2}\right)}{m}\right) \\ &= (-1)^{\frac{m-1}{2}} \prod_{j=1}^{\frac{m-1}{2}} B_p\left(\frac{j}{m}, \frac{m-j}{m}\right). \end{aligned}$$

Using this formula in (2.6) we prove the formula (2.1) for odd number m .

Assume that m be even. By (2.5) we have

$$f(n) = \Gamma_p(2n) \dots \Gamma_p\left(2n + \frac{\frac{m}{2} - 1 + \frac{m}{2} - 1}{m}\right) \Gamma_p\left(2n + \frac{m}{m}\right) \dots \Gamma_p\left(2n + \frac{m-1 + m-1}{m}\right).$$

Now, we arrange $f(n)$. By using the equality (1.1) we get

$$\begin{aligned} f(n) &= \Gamma_p(2n) \Gamma_p\left(2n + \frac{2}{m}\right) \dots \Gamma_p\left(2n + \frac{m-2}{m}\right) \Gamma_p(2n+1) \dots \Gamma_p\left(2n + \frac{m-2}{m} + 1\right) \\ &= \Gamma_p(2n) \dots \Gamma_p(2n) h_p(2n) \dots \Gamma_p\left(2n + \frac{m-2}{m}\right) h_p\left(2n + \frac{m-2}{m}\right) \end{aligned}$$

or

$$f(n) = \left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(2n + \frac{2j}{m}\right)\right)^2 \left(\prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right)\right). \tag{2.7}$$

Let $g(n) := \prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(2n + \frac{2j}{m}\right)$. Taking $m = 2k$, we have

$$g(n) = \prod_{j=0}^{k-1} \Gamma_p\left(2n + \frac{j}{k}\right)$$

Then, we can arrange $g(n)$. From the equality (2.3) we have

$$g(n) = \prod_{j=0}^{k-1} \Gamma_p\left(\frac{j}{k}\right) k^{-\mu(2kn)} \Gamma_p(2kn).$$

Taking $k = \frac{m}{2}$ we get

$$g(n) = \prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j}{m}\right) m^{-\mu(mn)} 2^{\mu(mn)} \Gamma_p(mn)$$

and using $g(n)$ in (2.7) we obtain

$$f(n) = \left(\prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j}{m}\right) m^{-\mu(mn)} 2^{\mu(mn)} \Gamma_p(mn) \right)^2 \left(\prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right) \right).$$

Thus, using $f(n)$ in (2.4) we get

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \frac{\left(\prod_{j=0}^{m-1} \Gamma_p\left(n + \frac{j}{m}\right) \right)^2}{\left(\prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j}{m}\right) m^{-\mu(mn)} 2^{\mu(mn)} \Gamma_p(mn) \right)^2 \left(\prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right) \right)}.$$

Using the equality (1.4) we have that

$$\begin{aligned} \prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) &= \frac{\left(\prod_{j=0}^{m-1} \Gamma_p\left(\frac{j}{m}\right) \right)^2 m^{-2\mu(mn)} (\Gamma_p(mn))^2}{\left(\prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j}{m}\right) \right)^2 m^{-2\mu(mn)} 2^{2\mu(mn)} (\Gamma_p(mn))^2 \prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right)} \\ &= \frac{\left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(\frac{2j}{m}\right) \right)^2 \left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(\frac{2j+1}{m}\right) \right)^2}{\left(\prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j}{m}\right) \right)^2 2^{2\mu(mn)} \left(\prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right) \right)} \end{aligned}$$

or

$$\prod_{j=0}^{m-1} B_p\left(n + \frac{j}{m}, n + \frac{j}{m}\right) = \left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(\frac{2j+1}{m}\right) \right)^2 2^{-2\mu(mn)} \left(\prod_{j=0}^{\frac{m-2}{2}} h_p\left(2n + \frac{2j}{m}\right) \right)^{-1} \quad (2.8)$$

Now we must arrange $\left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(\frac{2j+1}{m}\right) \right)^2$. Let $s(n) := \left(\prod_{j=0}^{\frac{m-2}{2}} \Gamma_p\left(\frac{2j+1}{m}\right) \right)^2$. Using $\Gamma_p(1) = -1$ and Definition 1.1 we get that

$$\begin{aligned} s(n) &= \left(\prod_{j=0}^{\frac{m}{2}-1} \Gamma_p\left(\frac{2j+1}{m}\right) \right)^2 \\ &= \Gamma_p\left(\frac{1}{m}\right)^2 \Gamma_p\left(\frac{3}{m}\right)^2 \dots \Gamma_p\left(\frac{m-3}{m}\right)^2 \Gamma_p\left(\frac{m-1}{m}\right)^2 \\ &= \Gamma_p(1) \frac{\Gamma_p\left(\frac{1}{m}\right) \Gamma_p\left(\frac{m-1}{m}\right)}{\Gamma_p(1)} \dots \Gamma_p(1) \frac{\Gamma_p\left(\frac{3}{m}\right) \Gamma_p\left(\frac{m-3}{m}\right)}{\Gamma_p(1)} \Gamma_p(1) \frac{\Gamma_p\left(\frac{1}{m}\right) \Gamma_p\left(\frac{m-1}{m}\right)}{\Gamma_p(1)} \\ &= (-1)^{\frac{m}{2}-1} \prod_{j=0}^{\frac{m}{2}-1} B_p\left(\frac{2j+1}{m}, \frac{m-(2j+1)}{m}\right). \end{aligned}$$

Using $s(n)$ in (2.8) we prove the formula (2.2). \square

In similar way, by using the equality (1.1) and the property (1.7) we can prove the Gauss Legendre multiplication formula for p -adic beta function:

Theorem 2.2. For each $x \in \mathbb{Z}_p$, let $\ell(x) \in \{1, 2, \dots, p\}$ be such that $|x - \ell(x)|_p < 1$. Further, let $\ell_1(x) = p^{-1}(x - \ell(x))$ ($x \in \mathbb{Z}_p$). Then for $m > 1$, m not divisible by p

(i) If m is odd, then

$$\prod_{j=0}^{m-1} B_p\left(x + \frac{j}{m}, x + \frac{j}{m}\right) = \frac{B_p(mx, mx) \left(\prod_{j=1}^{\frac{m-1}{2}} B_p\left(\frac{j}{m}, \frac{m-j}{m}\right) \right) (m^{p-1})^{\ell_1(2mx) - 2\ell_1(mx)}}{(-1)^{\frac{m-1}{2}} \left[\prod_{j=0}^{\frac{m-1}{2}-1} h_p\left(2x + \frac{2j+1}{m}\right) \right] m^{-1 + \ell(mx) - \ell(2mx)}}$$

(ii) If m is even, then

$$\prod_{j=0}^{m-1} B_p\left(x + \frac{j}{m}, x + \frac{j}{m}\right) = \frac{(-1)^{\frac{m}{2}} \left(\prod_{j=0}^{\frac{m}{2}-1} B_p\left(\frac{2j+1}{m}, \frac{m-(2j+1)}{m}\right) \right)}{\left[\prod_{j=0}^{\frac{m}{2}-1} h_p\left(2x + \frac{2j}{m}\right) \right] 2^{2\ell(mx) - 2(2^{p-1})2\ell_1(mx)}}$$

Corollary 2.3. The equality

$$B_p(n, n) = 2^{\lfloor \frac{2n-1}{p} \rfloor - (2n-1)} B_p\left(n, \frac{1}{2}\right)$$

holds for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. From Definition 1.1 we know that

$$B_p(n, n) = \frac{\Gamma_p(n)\Gamma_p(n)}{\Gamma_p(2n)}.$$

Then, by using (1.6) we get

$$\begin{aligned} B_p(n, n) &= \frac{\Gamma_p(n)\Gamma_p(n)}{2^{2n-1}\Gamma_p(\frac{1}{2})^{-1}\Gamma_p(n)\Gamma_p(n+\frac{1}{2})2^{-\lfloor \frac{2n-1}{p} \rfloor}} \\ &= \frac{\Gamma_p(n)\Gamma_p(\frac{1}{2})}{\Gamma_p(n+\frac{1}{2})} 2^{\lfloor \frac{2n-1}{p} \rfloor - (2n-1)}. \end{aligned}$$

Using Definition 1.1 we obtain the formula

$$B_p(n, n) = 2^{\lfloor \frac{2n-1}{p} \rfloor - (2n-1)} B_p\left(n, \frac{1}{2}\right).$$

□

Corollary 2.4. For all $n \in \mathbb{N}$, the equality

$$B_p(n, n)B_p\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = (-1)^{\ell(\frac{1}{2})} 2^{-2(2n-1) + 2\lfloor \frac{2n-1}{p} \rfloor} (h_p(2n))^{-1}$$

holds, where $\ell : \mathbb{Z}_p \rightarrow \{1, 2, \dots, p\}$ assigns to $x \in \mathbb{Z}_p$ its residue $\in \{1, 2, \dots, p\}$ modulo $p\mathbb{Z}_p$

Proof. Let $n \in \mathbb{N}$. Using Definition 1.1 we have

$$\begin{aligned} B_p(n, n)B_p\left(n + \frac{1}{2}, n + \frac{1}{2}\right) &= \frac{\Gamma_p(n)\Gamma_p(n)}{\Gamma_p(2n)} \frac{\Gamma_p(n+\frac{1}{2})\Gamma_p(n+\frac{1}{2})}{\Gamma_p(2n+1)} \\ &= \frac{(\Gamma_p(n)\Gamma_p(n+\frac{1}{2}))^2}{\Gamma_p(2n)\Gamma_p(2n+1)}. \end{aligned}$$

From the equalities (1.1) and (1.5) we get that

$$\begin{aligned} B_p(n, n)B_p\left(n + \frac{1}{2}, n + \frac{1}{2}\right) &= \frac{(\Gamma_p(\frac{1}{2})\Gamma_p(2n)2^{-(2n-1)}2^{\lfloor \frac{2n-1}{p} \rfloor})^2}{\Gamma_p(2n)\Gamma_p(2n)h_p(2n)} \\ &= \frac{(\Gamma_p(\frac{1}{2}))^2(\Gamma_p(2n))^2 2^{-2(2n-1) + 2\lfloor \frac{2n-1}{p} \rfloor}}{(\Gamma_p(2n))^2 h_p(2n)} \\ &= \frac{(\Gamma_p(\frac{1}{2}))^2 2^{-2(2n-1) + 2\lfloor \frac{2n-1}{p} \rfloor}}{h_p(2n)}. \end{aligned}$$

Finally, by using the equality (1.3) we obtain

$$B_p(n, n)B_p\left(n + \frac{1}{2}, n + \frac{1}{2}\right) = (-1)^{\ell(\frac{1}{2})} 2^{-2(2n-1)+2\left[\frac{2n-1}{p}\right]} (h_p(2n))^{-1}.$$

□

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