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Uniform Decay Rates for the Energy in Nonlinear Fluid Structure Interaction with Monotone Viscous Damping

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Abstract. Uniform stability of *finite energy solutions* to a *nonlinear* fluid-structure interaction model in a bounded domain $\Omega \in \mathbb{R}^2$ is considered. Boundary interface and interior viscous feedback controls are selected to stabilize the *energy* of the overall system. It will be shown that the energy decays to zero at a uniform rate. If appropriate geometric condition is imposed on the interface, then the decay rate could be improved to an exponential rate.

1 Introduction

1.1 Description of the problem

We consider fluid-structure interaction (FSI) described by a coupled system of partial differential equations (PDEs) comprising of the nonlinear Navier-Stokes equation and a system of elasticity of wave equation. The coupling between two systems occurs on the boundary-interface between two environments: fluid and a solid. The FSI model is well established in the literature and has numerous engineering applications that range from naval and aerospace engineering to cell biology and biomedical engineering [36, 15, 21, 18, 17] and references therein.

However, due to mismatch of regularity between the particular hyperbolic component (dynamic system of elasticity) and parabolic component (fluid) the basic mathematical questions such as well-posedness of *finite energy* physical solutions had not been resolved until recently [10, 11, 19, 20, 16]. It is known by now that weak (finite energy) solutions corresponding to fluid structure interaction exist globally and they are *unique* when the dimension of the domain is equal to two. Thus, in the two dimensional case there exists a well defined semi-flow describing the associated dynamical system.

There are two common physical settings for this interaction model: solid submerged in the fluid and fluid flown in the solid. Examples of the former include a submarine submerged in the water; examples of the latter include membrane valves. The main aim of this paper is to establish uniform stability results for the FSI model with "fluid in the solid" setting under different scenarios. We shall consider the following configurations: (1) *nonlinear* interior feedback in a form of frictional damping affecting the solid; (2) linear boundary feedback affecting the interface. No strong assumptions are imposed on the interior damping when the magnitude of the solution is small (see Assumption 2.10). Geometry plays an important role dispensing with tangential derivatives of the displacement of the solid at the interface in a key estimate.

1.2 The model

The model is defined on a bounded domain $\Omega \in \mathbb{R}^2$, that describes an elastic body interacting with an interior incompressible viscous fluid. Ω is a bounded simply connected domain, consisting of two open sub-domains Ω_s and Ω_f where Ω_s is the exterior domain with non-overlapping boundaries $\partial\Omega$ and Γ_s, so that $\partial\Omega_s = \partial\Omega \cup \Gamma_s$; and Ω_f is the interior domain with boundary $\Gamma_s = \partial\Omega_f$. Ω_s is occupied by an elastic solid, while Ω_f is filled with fluid. The interaction between the fluid and the solid occurs at the interface Γ_s.

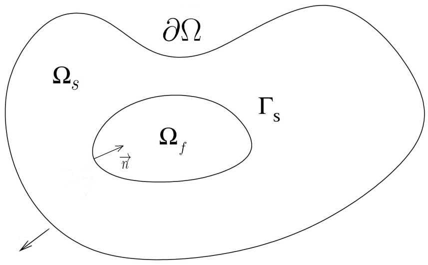

Figure 1. Geometry of Ω.

The dynamics of the fluid is described by the Navier-Stokes equation and the dynamics of the elastic body is described by an elasto-dynamic system of wave equations. $u(t,x) \in \mathbb{R}^2$ is a vector-valued function representing the velocity of the fluid and $p(t,x)$ is a scalar-valued function representing pressure. $w(t,x), w_t(t,x) \in \mathbb{R}^2$ denote the displacement and the velocity functions of the elastic solid Ω_s. $\overrightarrow{n}$ denotes the unit outward normal vector on Γ_s with respect to the region Ω_s. See Figure 1.

This leads to the following coupled PDE system defined for the state variables $[u, w, w_t, p]$ [34]:

$$\begin{cases} u_t - \triangle u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } \Omega_f \times (0,\infty) \\ \text{div } u = 0 & \text{in } \Omega_f \times (0,\infty) \\ w_{tt} = \triangle w - \rho(x) g(w_t) & \text{in } \Omega_s \times (0,\infty) \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial u}{\partial n} - p\overrightarrow{n} + \dfrac{1}{2}(u\cdot\overrightarrow{n})u & \text{on } \Gamma_s \times (0,\infty) \\ u = w_t + \beta(x)\dfrac{\partial w}{\partial n} & \text{on } \Gamma_s \times (0,\infty) \\ w = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(0,\cdot) = u_0 & \text{in } \Omega_f \\ w(0,\cdot) = w_0,\ w_t(0,\cdot) = w_1 & \text{in } \Omega_s \end{cases} \tag{1.1}$$

where $\beta(x), \rho(x) > 0$ are smooth functions of x.

In (1.1), $g(w_t)$ represents an interior viscous -frictional nonlinear feedback while $\dfrac{\partial w}{\partial n}$ represents porous damping on the interface between the solid and fluid. Both damping mechanism are typical for dynamics of oscillating structures governed by wave equation [24, 23, 22, 25, 13]. It will be assumed that $g(\cdot)$ is monotone, continuously differentiable, and of a polynomial growth with $g(0) = 0$.

Remark 1.1. We note that the functions $\beta(x)$ and $\rho(x)$ while positive, they are not required to be uniformly positive (the latter is a standard assumption in stabilization theory). One of the goals in this paper is to study the effects of potential degeneracy of one of the dampings.

The interaction between the fluid and the solid occurs at the interface Γ_s and is reflected by the velocity and stress matching boundary conditions. The model considered accounts for small but rapid oscillations of the elastic displacements [17]. This allows one to assume that the interface is static.

It is convenient at this stage to introduce the energy functional of the system

$$E(t) \equiv \frac{1}{2}\int_{\Omega_f} |u|^2 dx + \frac{1}{2}\int_{\Omega_s} (|\nabla w|^2 + |w_t|^2) dx \tag{1.2}$$

where the states $[u, w, w_t]^T$ belong to the finite energy space [10]

$$\mathcal{H} \equiv H \times [H^1_{\partial\Omega}(\Omega_s)]^2 \times [L_2(\Omega_s)]^2$$

with $H \equiv \{u \in [L_2(\Omega_f)]^2 : \text{div } u = 0\}$ and $H^1_{\partial\Omega}(\Omega_s)$ denotes the Sobolev space $H^1(\Omega_s)$ with zero boundary condition on $\partial\Omega$.

1.3 Mathematical Challenges

The major mathematical difficulty stems from the mismatch between the boundary regularity of the hyperbolic wave equation and the parabolic Navier-Stokes equation, which does not provide sufficient regularity for the boundary traces. In dealing with this particular difficulty, several strategies have been developed in earlier mathematical literatures where either a structural damping is added to the wave equation or a very smooth local-in-time solution were considered. Only recently the existence, uniqueness (in two dimension), of the solutions *in the natural energy level* were shown to hold [10]. This was accomplished by taking advantage of recently discovered hyperbolic trace theory [27] applied on the interface of the structure. Regularity of weak solutions was subsequently developed in [11], and also in [19, 20] for a slightly different topological setting. Smooth solutions with moving interface have been analyzed in [16].

As mentioned above, stability results are available for the *linearized* model with the presence of pressure: strong stability in [2, 3, 5] where geometric dependency is first discovered; exponential decay rate with additional boundary damping in [3, 4]. The main tool used to establish the strong stability results for linear models is spectral theory [1], which has no extension to nonlinear models. For stability results of the *nonlinear* model in "solid in the fluid" setting, one can consult [28] for strong stability with no damping but subject to partial flatness geometric conditions on Γ_s; [29] for exponential uniform stability with boundary damping and [30] for polynomial uniform stability with nonlinear interior damping. The main contribution of the present paper, however, is to establish uniform stability for nonlinear model in the "fluid in the solid" setting. This particular setting is motivated by numerous applications in biology and medicine. For instance, a classical fluid structure interaction model arises in a context of body fluid encapsulated in a human body (blood flowing through the arteries). On the other hand, mathematical analysis of such setup has several novel aspects. Notwithstanding is the fact that the damping coefficients are now variable. Moreover, the present setup allows to eliminate "partial flatness" condition used in [28, 29, 2] which is restrictive where it comes to applications to structures that are circular (like veins, arteries). Indeed, partial flatness eliminates perfect symmetries such as circles, spheres etc. The technical difficulty in the "solid in the fluid" setting is that the energy functional (1.2) does not control $|w|_{L_2(\Omega_s)}$. In the new setting, however, the energy functional does control $|w|_{L_2(\Omega_s)}$ by the virtue of $w|_{\partial\Omega} = 0$ and Poincarè's inequality. This lack of control of lower order term now transfers to the fluid component, which requires delicate treatments in our analysis. The dissipation from the frictional damping eliminates nontrivial solutions in the asymptotic dynamics, a point we will elaborate below.

2 Preliminaries and Main Results

In this section, we will review some preliminary definitions and known results and then introduce the main results.

2.1 Notations

The following (standard) notations will be used:

$$(u,v)_f = \int_{\Omega_f} uv\ d\Omega_f, (u,v)_s = \int_{\Omega_s} uv\ d\Omega_s, \langle u,v\rangle = \int_{\Gamma_s} uv\ d\Gamma_s;$$

$$|u|_{\alpha,D} = |u|_{H^\alpha(D)}, |u|_f = |u|_{0,\Omega_f}, (u,v)_{1,f} \equiv \int_{\Omega_f} \nabla u\cdot\nabla v\ d\Omega_f$$

$$Q_s \equiv (0,T]\times\Omega_s;\ Q_f \equiv (0,T]\times\Omega_f;\ \Sigma_s \equiv (0,T]\times\Gamma_s;\ \Sigma_f \equiv (0,T]\times\Gamma_f$$

2.2 Feedback control mechanisms

A natural interior nonlinear frictional damping to consider is the following function $g(s) = [g_i(s_i)]_{i=1,2}$ which is subject to the following assumptions:

Assumption 2.1. The function $g(s) = [g_i(s_i)]_{i=1,2}$, where $g_i(s_i), s_i \in \mathbb{R}$ are monotone increasing,continuous functions, zero at the origin and subject to the following conditions for $|s| \geq 1$

$$m|s_i|^2 \leq g_i(s_i)s_i,\ \ |g(s)| \leq M|s|^p, p > 0$$

for some positive constants $0 < m, M < \infty$.

We will specify the range of p in our main results.

Remark 2.2. (a). Note that no conditions are imposed on the damping function at the origin. This is one of the issues when dealing with questions of stability and decay rates [32].
(b). One could impose more general structure of monotone frictional damping allowing for mixing of the wave coordinates. However, the main challenges of the problem are present already in this special configuration. In order to focus reader's attention we shall consider the frictional damping in this form only. For more general structures of frictional damping acting on the wave vectors we refer the reader to [14].

2.3 Existence, uniqueness and regularity of finite energy solutions

Define the following key space

$$V \equiv H \cap [H^1(\Omega_f)]^2$$

Projecting the equations on H and utilizing the boundary conditions allows us to define weak solutions to our PDE system in the variational form for a.e. $t \in (0,T)$:

$$(u_t,\phi)_f + \langle\frac{\partial w}{\partial n},\phi\rangle + (\nabla u,\nabla\phi)_f + ((u\cdot\nabla)u,\phi)_f - \langle\frac{1}{2}(u\cdot\overrightarrow{n})u,\phi\rangle = 0,\ \ \forall\,\phi\in V \tag{2.1}$$

$$(w_{tt},\psi)_s - \langle\frac{\partial w}{\partial n},\psi\rangle + (\nabla w,\nabla\psi)_s + \rho(g(w_t),\psi)_s = 0,\ \ \forall\,\psi\in[H^1(\Omega_s)]^2 \tag{2.2}$$

We recall some results for wellposedness and regularity of finite energy solutions. Global-in-time existence of weak solutions is obtained in [10].

Theorem 2.3. *(Existence and uniqueness of weak solutions [10]) Given any initial condition $(u_0,w_0,w_1)\in\mathcal{H}$, and any $T>0$, there exists unique weak (finite energy) solution $(u,w,w_t)\in C_w([0,T],\mathcal{H})$ to the system (1.1) with the following additional properties:*

(i) $u \in L_2(0,T;V), u_t \in L_2(0,T;V')$, $w_{tt} \in L_2(0,T;[[H^1(\Omega_s)]^2]')$,
$u|_{\Gamma_s} = w_t|_{\Gamma_s} + \beta(x)\dfrac{\partial w}{\partial n}$

(ii) $\rho(x)g(w_t)w_t \in L_1(Q_s)$, $\quad w_t|_{\Gamma_s} \in L_2((0,T);[H^{1/2}(\Gamma_s)]^2)$

Moreover, the said solution depends continuously on the initial data (with respect to the topology induced by $\mathcal{H}$.

Finite energy solutions are constructed as limits of monotone approximations to Navier Stokes problem. More specifically, the nonlinear N-S term is truncated so that the resulting problem is maximally monotone. Weak solutions are shown to be strong limits of these approximations [10].

Remark 2.4. When $g(w_t) \in L_2(\Omega)$ one can also show [10] that weak solutions satisfy $\dfrac{\partial w}{\partial n} \in L_2((0,T);[H^{-1/2}(\Gamma_s)]^2)$. The latter happens when $p=1$ in Assumption 2.1.

Additional regularity including differentiability of weak solutions is asserted in [11] (see also [19] for different topological configuration) for solutions with more regular initial data.

Theorem 2.5. *(Regularity [11]) Let* $(u_0,w_0,w_1) \in \mathcal{H} \cap \{([H^2(\Omega_f)]^2 \cap V \times [H^2(\Omega_s)]^2 \times [H^1(\Omega_s)]^2)\}$ *satisfy the usual boundary compatibility conditions imposed on the boundary. Then, for any* $T>0$, (u,w,w_t) *satisfies the variational form (2.1), (2.2) and we have :*

(i) $(u,p) \in L_2((0,T);[H^2(\Omega_f)]^2 \times H^1(\Omega_f))$

(ii) $(u_t,w_t,w_{tt}) \in L_\infty((0,T);\mathcal{H})$, $w \in L_\infty((0,T);[H^2(\Omega_s)]^2)$.

Theorem 2.3 and Theorem 2.5 were proved in [10, 11] without the damping $F(w)$ and $g(w_t)$ and in the "solid in the fluid" setting. However, the same proof can be carried in the presence of frictional damping that is assumed *monotone* and subject to polynomial growth condition when dimension of Ω is equal to two for the "fluid in the solid" setting.

2.4 Energy identity

Let u,w be regular solutions obtained in Theorem 2.5. Choose as the test functions $\phi = u$ and $\psi = w_t$ in the formulation (2.1)-(2.2). Noticing cancelation occurring in the nonlinear term

$$((u\cdot\nabla)u,u)_f - \langle \frac{1}{2}(u\cdot\overrightarrow{n})u,u\rangle = 0$$

and utilizing the transmission condition $u = w_t + \beta\dfrac{\partial w}{\partial n}$ on Γ_s, one obtains the following energy identity for $0 \le s \le t$

$$E(t) + \int_s^t \left[|\nabla u|^2_{\Omega_f} + \left|\beta^{\frac{1}{2}}\frac{\partial w}{\partial n}\right|^2_{\Gamma_s} + (\rho g(w_t),w_t)_{\Omega_s} \right] d\tau = E(s),\ 0 \le s < t \tag{2.3}$$

where $E(t)$ is the energy functional defined in (1.2). Denote the dissipation terms in (2.3) as

$$D(t) = |\nabla u|^2_{\Omega_f} + \left|\beta^{\frac{1}{2}}\frac{\partial w}{\partial n}\right|^2_{\Gamma_s} + (\rho g(w_t),w_t)_{\Omega_s}$$

The energy identity can be rewritten as

$$E(t) + \int_s^t D(\tau)d\tau = E(s),\ 0 \le s < t \tag{2.4}$$

A few observations are in order:

Remark 2.6. The energy identity (2.3) reveals the potential sources of dissipation: one from the Navier-Stokes equation, one from the boundary damping and one from the interior damping $g(w_t)$. If no damping are imposed ($\beta, \rho = 0$), the only source of dissipation is the one propagated from the fluid dynamics. In this case, the dissipation of energy is so weak that it does not have any impact on the boundary normal displacement of the solid. The geometric optic conditions for the wave equation are violated. Uniform stability for the overall dynamics is thus impossible [25]. This is the case even for the linear model for which one can show that there are infinitely many eigenvalues with real parts on the imaginary axis [5, 2]. The best stability result one can hope for is strong stability for energy only. And to compensate the weakness of dissipation, geometric conditions such as partially flatness need to be place on the interface Γ_s, as shown for the linear model in [2] with initial data restricted to a closed subspace of $\mathcal{H}$, which eliminates a subspace corresponding to zero eigenvalue of the linear generator. We note that the above condition fails when Ω is a ball. Thus, the aforementioned condition is not compatible with a perfect symmetry of the domain. To improve strong stability to uniform stability, the interior frictional damping and the boundary dynamic damping play pivot roles.

2.5 Main Results

Our main results answer the question of decay rates of (1.1). With boundary and/or interior damping and suitable geometric conditions, one obtains uniform decay of energy with either an exponential rate or a polynomial rate determined by a solution of suitably constructed nonlinear ODE [32]:

Theorem 2.7. (Uniform Decay Rates for Energy).
Suppose $\rho(x) \geq \rho_0 > 0$ for all $x \in \Omega_s$ and $\beta = 0$. Assume $p \geq 1$ in Assumption 2.1. Moreover assume that the initial conditions satisfy compatibility condition: $\int_{\Gamma_s} w_0 \cdot \overrightarrow{n} d\Gamma_s = 0$. Then, there exist constant $T_0 > 0$, such that the energy satisfies

$$E(t) \leq S(t), \ for \ t > T_0$$

where $S(t)$ satisfies the following ODE:

$$\frac{d}{dt}S(t) + q(S(t)) = 0, S(0) = E(0) \tag{2.5}$$

with $q(s) \sim \hat{h}^{-1}(s)$ where $\hat{h}(s) = (meas\ Q_s)\ h\left(\frac{s}{meas\ Q_s}\right)$ with h monotone increasing, continuous, $h(0) = 0$, concave and determined from the inequality $s^2 \leq h(sg(s)), |s| \leq 1$.

Remark 2.8. Note that owing to monotonicity of $g(s)$, function $h(s)$ can be always constructed as a concave envelope [32]. Function $h(s)$ captures the behavior of the dissipation $g(s)$ at the origin. This is the most sensitive region with respect to the decay rates. Thus, the task of finding decay rates is reduced to solving an ODE equation (2.5) with a given function $h(s)$ (hence $q(s)$) . In fact, when $g(s) = as$ then $h(s) = a^{-1}s$ and the decay rates are exponential of the type e^{-at}. For polynomial $g(s)$ at the origin the decay rates are polynomial as well (algebraic) $t^{-\frac{2}{p-1}}$. These are optimal algebraic decay rates. See [32] and also [33] for many other examples.

Theorem 2.9. (Exponential Decay Rates for Energy).
Suppose $\Omega_1 \subset supp\ \rho(x)$ where $\overline{\Omega_1}$ has nonzero measure in a layer of Γ_s and $\beta(x) \geq \beta_0 > 0$ for all x in Γ_s. In addition we assume that $p \leq 1$ in Assumption 2.1 and the following geometric condition:

Assumption 2.10. $l \cdot \overrightarrow{n} > 0$, where $l(x) \equiv x - x^0$ is defined for $x \in \overline{\Omega_s}$ arbitrary with $x^0 \in \mathbb{R}^2$ fixed; and $\overrightarrow{n}$ is the unit outward normal vector of Γ_s with respect to Ω_s.

Then, there exist constants $M \geq 1$, $\delta > 0$ and a time $T_1 > 0$, such that

$$E(t) \leq Me^{-\delta t}E(0), \text{ for } t > 0 \tag{2.6}$$

Remark 2.11. Though our method does not depend on dimensionality, we restrict dim Ω to two for the reason that when $n = 3$, weak solutions are not known to be unique, thus the decay rates obtained for strong solutions only can not be extended to all weak solutions. In that case the result remains valid for smooth solutions which are global (e.g. corresponding to small initial data -as shown in [11]).

The main mechanism that drives the decay of energy of the overall dynamics are different under the conditions of Theorem 2.7 and 2.9: for the former, it is the frictional damping $g(w_t)$; while for the latter, it is the boundary damping. The difficulty arises from lack of control of $|u|_{0,\Omega_f}$ by $|\nabla u|_{0,\Omega_f}$. In light of this, the proofs of our results will strongly depart from previous treatments and will require serious modifications.

2.6 Strong decay of energy

Before we show the proof of main results of this paper, we should point out that with weaker assumptions on ρ as in Theorem 2.9 and $\beta > 0$, following similar argument as in [28], one can show that energy $E(t) \to 0$ as $t \to \infty$ without assuming *any* geometric conditions. We cover a few key points of the proof of this claim, which also validates an aforementioned observation that the frictional damping eliminates nontrivial steady state solutions. The key is the "uniqueness result" highlighted below.

Let $[\overline{u_0}, \overline{w_0}, \overline{w_1}]$ be an element in $\omega(u_0, w_0, w_1)$ for $[u_0, \xi_0, v_0] \in \mathcal{D}$ (see [28] for definitions). By definition, there exists a sequence $t_n \to \infty$ such that $[u(t_n), w(t_n)] \to [\overline{u_0}, \overline{w_0}]$ strongly in $L_2(\Omega_f) \times L_2(\Omega_s)$ and $w_t(t_n) \rightharpoonup \overline{w_1}$ weakly in $L^2(\Omega_s)$. Denote $X(t; X_0) := [u(t), w(t), w_t(t)]$ a solution with initial data $X_0 = [u_0, w_0, w_1]$. For this sequence t_n, consider the translate $X_n(t) := X(t + t_n; X_0)$. X_n is bounded in $L_\infty((0, \infty); \mathcal{H})$. Thus, X_n has a subsequence, which we still denote by X_n, such that it converges to $\overline{X} := [\overline{u}, \overline{w}, \overline{w_t}]$ weakly in $L_2((0, T); \mathcal{H})$ and weak* in $L_\infty((0, \infty); \mathcal{H})$. We will show that $\overline{X} \equiv 0$.

Choose $t > 0$ fixed, the energy identity (2.3) implies that

$$E(t + t_n) + \int_0^t \left[|\nabla u_n|^2_{\Omega_f} + \left| \beta^{\frac{1}{2}} \frac{\partial w_n}{\partial n} \right|^2_{\Gamma_s} + (\rho g(w_{t,n}), w_{t,n})_{\Omega_s} \right] d\tau = E(t_n)$$

Let $t_n \to \infty$. Since $E(t)$ is monotonically non-increasing and positive, $E(t + t_n), E(t_n)$ must converge to the same limit, which forces $\nabla u_n \to 0$ in $L_1(0, T; L_2(\Omega_f))$, $\dfrac{\partial w_n}{\partial n} \to 0$ in $L_1(0, T; L_2(\Gamma_s))$ and $\rho g(w_{t,n})w_{t,n} \to 0$ in $L_1(0, T, L_1(\Omega_s))$. Thus, $\nabla \overline{u} = 0$ in Ω_f, $(\rho g(\overline{w_t}), \overline{w_t})_{\Omega_s} = 0$ and $\dfrac{\partial \overline{w}}{\partial n} \equiv 0$ on Γ_s. Since the support of ρ contains a portion whose closure intersects with Γ_s, $\overline{w_t} \equiv 0$ on a nontrivial subset Γ_0 of Ω_s. The transmission condition on Γ_s in turn shows that $\overline{u} \equiv 0$ on Γ_0. Combining this observation with $\nabla \overline{u} = 0$ implies that $\overline{u} \equiv 0$ in Ω_s. Thus, $[\overline{u}, \overline{w}, \overline{w_t}]$ satisfies the following stationary problem:

$$\begin{cases} \Delta \overline{w} = 0 & \text{in } \Omega_s \times (0, T_1) \\ \dfrac{\partial \overline{w}}{\partial n} = 0 & \text{on } \Gamma_s \times (0, T_1) \\ \overline{w} = 0 & \text{on } \partial\Omega \times (0, T_1) \end{cases} \tag{2.7}$$

By Holmgren's uniqueness theorem, the above problem has only trivial solutions, thus $\overline{w} \equiv 0$ in Ω_s as well. To show how to improve from weak ω-limit set to strong ω-limit set, one can see [28] for details.

2.7 Further discussion

We conclude our introduction with the following open questions:

(i) Is it possible to obtain the result of Theorem 2.9 without assuming $p \leq 1$ in Assumption 2.1?

(ii) Is it possible to obtain a fully nonlinear counterpart of uniform decay results with nonlinear boundary damping. This is to say $\beta f\left(\frac{\partial w}{\partial n}\right)$ where $f(s)$ is -say-monotone.

The Assumption that $p \leq 1$ was used in Theorem 2.9 for purely technical reasons, due to the use of higher energy multipliers. From physical point of view it is plausible to think that such assumption may be eliminated. As to the second question, a related result is well known for a pure wave equation with boundary monotone damping. Whether one could also generalize porous damping to a nonlinear form, requires further studies.

The rest of the manuscript is devoted to the proofs of main results stated in Theorem 2.7 and Theorem 2.9 .

3 Uniform stability with damping

The proof of Theorem 2.7 and 2.9 is based on the multiplier's method. As usual, the critical step in proving Theorem 2.7 and 2.9 is the following estimate:

Theorem 3.1. ***(a).*** *Under the conditions of Theorem 2.7, there exists a time $T > 0$ and a constant $C_T > 0$, such that the energy at $t = T$ is dominated by the dissipation for all initial condition $[u_0, w_0, w_1] \in \mathcal{H}$:*

$$E(T) \leq H_T\left(\int_0^T D(t)dt\right) \tag{3.1}$$

where $H_T(s) : \mathbb{R}^+ \to \mathbb{R}^+$ is a concave, monotone increasing function and zero at the origin with asymptotic behavior dictated by $\hat{h}$.

(b). *Under the conditions of Theorem 2.9, $H_T(s)$ could be further identified as $H_T(s) = Cs$, where $C > 0$ is a constant determined by geometry of Ω_s and initial data.*

Once Theorem 3.1 is established, using the energy identity (2.3) and following the nonlinear version of an inductive argument in [32], one is able to show Theorem 2.7 and Theorem 2.9. Indeed, using the fact that the system is autonomous we reiterate the same estimate on the multiple T , which gives

$$E((m+1)T) \leq H_T\left(\int_{mT}^{(m+1)T} D(t)dt\right), m = 0, 1\ldots$$

By the energy identity (2.4)

$$E((m+1)T) \leq H_T(E(m(T)) - E((m+1)T)$$

$$H_T^{-1}(E((m+1)T)) \leq E(mT) - E(m+1)T$$

$$H_T^{-1}(E((m+1)T)) + E((m+1)T) \le E(mT)$$

The rest of the argument rests on ODE comparison theorem [32]. The existence of the function $H_T(s)$ will allow one to calculate the decay rates -as in [32]. Under the conditions of Theorem 2.9, solving the ODE (2.5) with $H_T(s) = Cs$ will yield an exponential decay rate for the energy functional. Thus, the main task is to establish the validity of Theorem 3.1.

4 Proof of Theorem 3.1 (a)

In this step, we will show *uniform stability* result for the model with the main damping mechanism being the interior frictional damping g with $\rho > \rho_0 > 0$ on the entire Ω_s. W.l.o.g, we take $\beta = 0$ throughout this section. Our goal is to establish the inequality

$$E(T) \le H_T\left(\int_0^T D(t)dt\right) \tag{4.1}$$

where $H_T(s)$ is a concave, continuous function, monotone and zero at the origin and such that asymptotically coincides with $\hat{h}$.

First of all, we can easily see that the kinetic energy of the wave is controlled by the dissipation. Indeed

$$\int_0^T |w_t|_s^2 dt = \int_{Q_s \cap |w_t| \ge 1} + \int_{Q_s \cap |w_t| \le 1} \le \frac{1}{m}\int_{Q_s} (g(w_t), w_t)_s dt + \int_{Q_s} h(g(w_t)w_t)dt$$

Hence by Jensen's inequality and monotonicity and concavity of h,

$$\int_0^T |w_t|_s^2 dt \le [\hat{h} + m^{-1}I]\left(\int_0^T (g(w_t), w_t)_s dt\right) \tag{4.2}$$

where $\hat{h} = \frac{1}{\text{meas } Q_s} h(\text{meas } Q_s \cdot)$.

The next step is the control of potential energy. Applying multiplier w to the wave equation $w_{tt} = \Delta w - \rho g(w_t)$ along with integration by parts yields

$$(w_t, w)_s\Big|_0^T - \int_0^T |w_t|_s^2 dt = \int_0^T \left[\langle \frac{\partial w}{\partial n}, w\rangle - |\nabla w|_s^2\right] dt - \int_0^T (\rho g(w_t), w)_s dt$$

Thus, the kinetic energy of the wave is given by

$$\int_0^T |\nabla w|_s^2 dt = \int_0^T |w_t|_s^2 dt + \int_0^T \langle \frac{\partial w}{\partial n}, w\rangle dt + \int_0^T (\rho g(w_t), w)_s dt - (w_t, w)_s\Big|_0^T \tag{4.3}$$

We need to estimate the last three terms on the right hand side of (4.3).

Step 1: Interior damping term:

$$|(\rho g(w_t), w)_s| \le I + II \tag{4.4}$$

where

$$I = \left|\int_{\Omega_s, |w_t| \le 1} \rho g(w_t) w \, d\Omega_s\right| \le C \int_{\Omega_s} |w_t||w| d\Omega_s \le C_\epsilon |w_t|_s^2 + \varepsilon |\nabla w|_s^2$$

$$II = \left| \int_{\Omega_s, |w_t| \geq 1} \rho g(w_t) w \, d\Omega_s \right| \leq \varepsilon |\nabla w|_s^2 + C_{\epsilon, E(0)} \int_{\Omega_s} g(w_t) w_t d\Omega_s \tag{4.5}$$

Detailed proof of (4.5): Take $r = \dfrac{1}{p} + 1$. Denote the conjugate of r as $\overline{r} = 1 + p$. Applying Hölder's inequality to II, taking into account that $\overline{r} \geq 2$ and $|w|_s \leq C|\nabla w|_s$ yields:

$$II \leq C|g(w_t)|_{r,s}|w|_{\overline{r},s} \leq C|g(w_t)|_{r,s}|w|_s \leq \varepsilon|\nabla w|_s^2 + C_\varepsilon \left(\int_{\Omega_s, |w_t| \geq 1} (g(w_t))^r d\Omega_s \right)^{\frac{2}{r}}$$

Since for $|s| \geq 1$, $|g(s)| \leq |s|^p$, continuing from above and using the fact that $p(r-1) = 1$, we have

$$\begin{aligned}
II &\leq \varepsilon|\nabla w|_s^2 + C_\varepsilon \left(\int_{\Omega_s, |w_t| \geq 1} (g(w_t))^{r-1} (g(w_t)) d\Omega_s \right)^{\frac{2}{r}} \\
&\leq \varepsilon|\nabla w|_s^2 + C_\varepsilon \left(\int_{\Omega_s, |w_t| \geq 1} (w_t)^{p(r-1)} (g(w_t)) d\Omega_s \right)^{\frac{2}{r}} \\
&\leq \varepsilon|\nabla w|_s^2 + C_\varepsilon \left(\int_{\Omega_s} w_t g(w_t) d\Omega_s \right)
\end{aligned} \tag{4.6}$$

The last is because $\dfrac{2}{r} = \dfrac{2p}{p+1} \geq 1$.

Step 2: Boundary term $\int_0^T \langle \frac{\partial w}{\partial n}, w \rangle dt$.

Define the following Stokes extension of the Dirichlet map D

$$z = Dg^* \;\Leftrightarrow\; \begin{cases} \triangle z = \nabla q, \quad \text{div } z = 0 & \text{in } \Omega_f \\ z|_{\Gamma_s} = g^* & \text{on } \Gamma_s \end{cases} \tag{4.7}$$

where we assume the compatibility $\int_{\Gamma_s} g^* \cdot \overrightarrow{n} d\Gamma_s = 0$. Stokes theory [38] gives that $D : H^\alpha(\Gamma_s) \to H^{\alpha+\frac{1}{2}}(\Omega_f)$ is well defined and continuous. In particular, D is continuous from $H^{\frac{1}{2}}(\Gamma_s)$ to V.

Choose the boundary term g^* as: $g^* = w|_{\Gamma_s}$ on Γ_s, where the boundary function satisfies the requisite compatibility condition. This is argued below. Since $u = w_t$, on Γ_s and $\int_{\Gamma_s} w_0 \cdot \overrightarrow{n} d\Gamma_s = 0$ as well as $\int_{\Gamma_s} u \cdot \overrightarrow{n} d\Gamma_s = 0$ we obtain that $\int_{\Gamma_s} w(t) \cdot \overrightarrow{n} d\Gamma_s = 0$ for all $t > 0$. Applying weak formulation of the fluid equation with the test function $\phi = Dw$ yields

$$(u_t, Dw)_f - \langle \frac{\partial w}{\partial n}, w \rangle + (\nabla u, \nabla Dw)_f + b(u, u, Dw) = 0 \tag{4.8}$$

where b is the trilinear form defined by

$$b(u, v, w) \equiv ((u \cdot \nabla)v, w)_f - \frac{1}{2}\langle (u \cdot \overrightarrow{n})v, w \rangle, \text{for } u \in V, v, w \in H^1(\Omega_f)$$

Integration by parts in time yields:

$$(u, Dw)_f\Big|_0^T - \int_0^T [(u, D(w_t))_f + \langle\frac{\partial w}{\partial n}, w\rangle + (\nabla u, \nabla Dw)_f]dt + \int_0^T b(u, u, Dw)dt = 0 \quad (4.9)$$

From here the estimate for the boundary term becomes:

$$\int_0^T \langle\frac{\partial w}{\partial n}, w\rangle dt \le \int_0^T (u, Dw_t)_f dt + \int_0^T (\nabla u, \nabla Dw)_f]dt + \int_0^T b(u, u, Dw)dt + (u, Dw)_f\Big|_0^T \quad (4.10)$$

We need to estimate each of the four terms on the right hand side of (4.10).

Step 2.1: Nonlinear term $\int_0^T b(u, u, Dw)dt$**.**
We apply a key estimate of the trilinear form on $[u, u, Dw]$ obtained by the virtue of Sobolev's embedding (as in [10]) when $\dim \Omega = 2$,

$$|b(u, u, Dw)| \le C\left[|u|_{\frac{1}{2},f}|u|_{1,f}|Dw|_{\frac{1}{2},f} + |u|_{\frac{1}{2},f}|u|_{\frac{3}{4},f}|Dw|_{\frac{3}{4},f}\right] \quad (4.11)$$

Elliptic theory [38], interpolation and Poincaré's Inequality thus imply

$$|Dw|_{1/2,f} \le |w|_{0,\Gamma_s} \le C|w|_s^{1/2}|w|_{1,s}^{1/2} \le C|\nabla w|_s \quad (4.12)$$

$$|Dw|_{3/4,f} \le |w|_{1/4,\Gamma_s} \le C|w|_s^{1/4}|w|_{1,s}^{3/4} \le C|\nabla w|_s \quad (4.13)$$

Combining (4.12), (4.13) with (4.11) and applying interpolation and Young's inequality then gives

$$\int_0^T |b(u, u, Dw)|dt \le \varepsilon\int_0^T |\nabla w|_s^2 dt + C_{\varepsilon,E(0)}\int_0^T \left[|\nabla u|_f^2 + |u|_f^2\right] dt \quad (4.14)$$

Step 2.2: Time derivatives. Elliptic theory, trace theory and transmission condition combined imply that $|Dw_t|_f^2 \le C|w_t|_{\frac{1}{2},\Gamma_s} = C\left|u\right|_{\Gamma_s} \le CD(t)$. Thus,

$$|(u, Dw_t)_f| \le C\left[D(t) + |u|_f^2\right] \quad (4.15)$$

Step 2.3 Gradient term. Applying elliptic theory, trace theory and Poincaré's inequality to the interior gradient term in (4.10) implies that

$$(\nabla u, \nabla Dw)_f \le C_\epsilon|\nabla u|_f^2 + \varepsilon|\nabla w|_s^2 \quad (4.16)$$

Step 2.4 The term: $(u, Dw)_f\Big|_0^T$**.** The boundedness of the operator D implies

$$(u, Dw)_f\Big|_0^T \le C[E(0) + E(T)] \quad (4.17)$$

Substituting inequalities (4.14), (4.15), (4.16), (4.17) into (4.10) gives the final bound for the boundary interface coupling term:

$$\int_0^T \langle\frac{\partial w}{\partial n}, w\rangle dt \le \varepsilon\int_0^T |\nabla w|_s^2 dt + C_{\epsilon,E(0)}\int_0^T D(t)dt + CE(0) + CE(T) \quad (4.18)$$

Step 3: Final estimation on (4.3). On the strength of Poincaré's inequality,

$$(w_t, w)_s\Big|_0^T \leq CE(0) + CE(T) \tag{4.19}$$

Taking suitably small ε and combining (4.4), (4.18), (4.19) with (4.3) yield

$$\int_0^T |\nabla w|_s^2 dt \leq C\int_0^T |w_t|_s^2 dt + C_{E(0)}\int_0^T D(t)dt + C\int_0^T |u|_f^2 dt + CE(0) + CE(T) \tag{4.20}$$

(4.2) ,(4.20) and energy identity then imply

$$\int_0^T E(t)dt \leq [\hat{h} + m^{-1}I]\left(\int_0^T D(t)dt\right) + C_{E(0)}\int_0^T D(t)dt + C\int_0^T |u|_f^2 dt + CE(T) \tag{4.21}$$

After obtaining the following inequality by energy identity

$$\frac{1}{2}E(T)T + \frac{1}{2}\int_0^T E(t)dt \leq C_{E(0)}[(1 + m^{-1})I + \hat{h}]\left(\int_0^T D(t)dt\right) + +C\int_0^T |u|_f^2 dt + CE(T)$$

and taking $T > 2C$, we arrive at

$$E(T) \leq C_{E(0)}H\left(\int_0^T D(t)dt\right) + C\int_0^T |u|_f^2 dt \tag{4.22}$$

where $H(s) = s + m^{-1}s + \hat{h}(s) \sim \hat{h}(s)$ for small s.

The final step is to absorb the lower order term by the dissipation in (4.22). We follow a standard nonlinear version of the compactness-uniqueness argument [32].

Lemma 4.1. *With reference to system (1.1), assume* $\rho(x) \geq \rho_0 > 0$ *and* $\beta \geq 0$*, then there exists a constant* $C_T(E(0)) > 0$ *such that the following holds*

$$\int_0^T |u|_f^2 dt \leq C_T(E(0))H\left(\int_0^T D(t)dt\right) \tag{4.23}$$

Proof. We prove by contradiction. Suppose such constant does not exist. Then, there exists a sequence of solutions $X_n(t) \equiv [u_n(t), w_n(t), w_{n,t}(t)]$ of (1.1) corresponding to initial data $X_n(0) \equiv [u_n(0), w_n(0), w_{n,t}(0)]$, which are uniformly bounded: $E_n(0) \leq M$, such that

$$\lim_{n\to\infty} \frac{\int_0^T |u_n|_f^2 dt}{H\left(\int_0^T D_n(t)dt\right)} = \infty \tag{4.24}$$

where $D_n(t)$ is the dissipation term in (2.3) for $X_n(t)$.

Denote by $c_n \equiv \dfrac{|u_n|_{L_2(0,T;L_2(\Omega_f))}}{M\sqrt{T}}$. Then, c_n is uniformly bounded by 1. We renormalize $X_n(t)$ and denote the normalized sequence by $\widehat{X_n}(t) \equiv \dfrac{X_n(t)}{c_n^2}$. By (4.24),

$$\lim_{n\to\infty} \frac{H\left(\int_0^T D_n(t)dt\right)}{c_n^2} = 0 \tag{4.25}$$

Applying (4.22) to $X_n(t)$ and dividing by c_n^2 yields for $0 \le t \le T$,

$$\widehat{E_n}(t) \le C \tag{4.26}$$

Thus, there exists a subsequence of $\widehat{X_n}(t)$ such that $\widehat{X_n}(t) \to \widehat{X}(t) \equiv [\hat{u}, \hat{w}, \hat{w}_t]$ weak-$*$ in $L_\infty(0,T;\mathcal{H})$. For this subsequence, since $c_n \le 1$, by concavity, continuity of H, $H(0) = 0$ and (4.25), we have

$$\int_0^T \widehat{D}_n(t)dt \to 0 \tag{4.27}$$

which in turn implies

$$\nabla \hat{u}_n \to 0, \text{ in } L_2(0,T;L_2(\Omega_f));\ \ \beta\frac{\partial \hat{w}_n}{\partial n} \to 0, \text{ in } L_2(0,T;L_2(\Gamma_s))$$

$$\frac{\rho g(w_{t,n})w_{t,n}}{c_n^2} \to 0, \text{ in } L_1(0,T;L_2(\Omega_s))$$

by the virtue of $g(s)s \ge m|s|^2$, the last of which implies

$$\hat{w}_{t,n} \to 0, \text{ in } L_2(0,T;L_2(\Omega_s))$$

Therefore, we conclude that $\nabla \hat{u} = 0$ in Ω_f, $\hat{w}_t = 0$ in Ω_s and $\dfrac{\partial \hat{w}}{\partial n} = 0$ on Γ_s. Combining the last two with the transmission condition then implies $\hat{u} \equiv 0$ on Γ_s, which allows to apply Poincarè's inequality on $\hat{u}$ yielding that $\hat{u} = 0$ in $H^1(\Omega_f)$. From here, the compactness of $H^1(\Omega_f)$ in $L_2(\Omega_f)$ gives that $\hat{u}_n \to 0$ in $L_2(0,T;L_2(\Omega_f))$, which contradicts with the fact that $|\hat{u}_n|_{L_2(0,T;L_2(\Omega_f))} \equiv$ const. □

Remark 4.2. We note that in the proof of the above lemma, β could be taken as 0. In this case, utilizing the transmission condition $u|_{\Gamma_s} = w_t|_{\Gamma_s}$, one can still conclude that $\hat{u} \equiv 0$ on Γ_s. The rest of the proof follows through.

Inserting (4.23) into (4.22) completes the proof for Theorem 3.1 (a). The function H determines the asymptotic behavior of the ODE (2.5).

5 Proof of Theorem 3.1 (b)

In this section, we prove part (b) of Theorem 3.1 under the geometric condition specified in Theorem 2.9: there exists l with $l \cdot \overrightarrow{n} > 0$, where $l(x) \equiv x - x^0$ is defined for $x \in \overline{\Omega_s}$ arbitrary with $x^0 \in \mathbb{R}^2$ fixed; and $\overrightarrow{n}$ is the unit outward normal vector of Γ_s with respect to Ω_s. W.l.o.g, throughout this section, we take $\beta = 1$.

Multiplying $w_{tt} = \Delta w - \rho g(w_t)$ with the classical $l \cdot \nabla w$ yields

$$
\begin{aligned}
(w_t, l\cdot\nabla w)_s\Big|_0^T - \int_0^T (w_t, l\cdot\nabla w_t)_s dt &= \int_0^T \langle \frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt - \int_0^T |\nabla w|_s^2 dt \\
-\frac{1}{2}\int_0^T \langle |\nabla w|^2, l\cdot\overrightarrow{n}\rangle dt &+ \frac{1}{2}\int_0^T (|\nabla w|_s^2, \text{div } l)_s dt - \int_0^T (\rho g(w_t), l\cdot\nabla w)_{\Omega_1} dt
\end{aligned} \tag{5.1}
$$

Integration by parts with respect to space then yields

$$
\int_0^T (w_t, l\cdot\nabla w_t)_s dt = \frac{1}{2}\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt - \frac{1}{2}\int_0^T (w_t^2, \text{div } l)_s dt \tag{5.2}
$$

Substituting (5.2) into (5.1) and noticing that div $l = n = 2$ gives

$$
\begin{aligned}
\int_0^T |w_t|_{\Omega_s}^2 dt = \frac{1}{2}\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt + \int_0^T \langle \frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt - \frac{1}{2}\int_0^T \langle |\nabla w|_{\Gamma_s}^2, l\cdot\overrightarrow{n}\rangle dt \\
- \int_0^T (\rho g(w_t), l\cdot\nabla w)_{\Omega_1} dt - (w_t, l\cdot\nabla w)_s\Big|_0^T
\end{aligned} \tag{5.3}
$$

Splitting $|w_t|_s^2 = \frac{1}{2}|w_t|_s^2 + \frac{1}{2}|w_t|_s^2$ and adding $\frac{1}{2}\int_0^T |\nabla w|_s^2 dt$ on both sides leads to

$$
\begin{aligned}
\frac{1}{2}\int_0^T \left[|w_t|_{\Omega_s}^2 + |\nabla w|_{\Omega_s}^2\right] dt = \frac{1}{2}\int_0^T \left[|\nabla w|_{\Omega_s}^2 - |w_t|_{\Omega_s}^2\right] dt + \int_0^T \langle \frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt \\
-\frac{1}{2}\int_0^T \langle |\nabla w|_{\Gamma_s}^2, l\cdot\overrightarrow{n}\rangle dt + \frac{1}{2}\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt \\
- \int_0^T (\rho g(w_t), l\cdot\nabla w)_{\Omega_1} dt - (w_t, l\cdot\nabla w)_s\Big|_0^T
\end{aligned} \tag{5.4}
$$

Applying a second multiplier w to $w_{tt} = \Delta w - \rho g(w_t)$ and rerunning the argument in the previous section before (4.3) gives

$$
\int_0^T \left[|\nabla w|_s^2 - |w_t|_s^2\right] dt = \int_0^T \langle \frac{\partial w}{\partial n}, w\rangle dt - \int_0^T (\rho g(w_t), w)_{\Omega_1} dt - (w_t, w)_s\Big|_0^T \tag{5.5}
$$

Substituting (5.5) into (5.4) then yields

$$
\begin{aligned}
\frac{1}{2}\int_0^T \left[|w_t|_{\Omega_s}^2 + |\nabla w|_{\Omega_s}^2\right] dt = \int_0^T \langle \frac{\partial w}{\partial n}, w\rangle dt + \int_0^T \langle \frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt \\
-\frac{1}{2}\int_0^T \langle |\nabla w|_{\Gamma_s}^2, l\cdot\overrightarrow{n}\rangle dt + \frac{1}{2}\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt \\
- \int_0^T \left[(\rho g(w_t), l\cdot\nabla w)_{\Omega_1} + (\rho g(w_t), w)_{\Omega_1}\right] dt \\
-(w_t, l\cdot\nabla w)_s\Big|_0^T - (w_t, w)_s\Big|_0^T
\end{aligned} \tag{5.6}
$$

We need to estimate each term on the right hand side of (5.6).

Step 1: Interior damping terms: Following a similar argument as the one to get (4.4), and applying growth condition $p \leq 1$, we have

$$(\rho g(w_t), l\cdot\nabla w)_{\Omega_1} + (\rho g(w_t), w)_{\Omega_1} \leq \varepsilon|\nabla w|^2_{\Omega_s} + C_\varepsilon|w_t|^2_{\Omega_1} + C_\varepsilon(g(w_t), w_t)_{\Omega_1} \tag{5.7}$$

Indeed, we can argue as in the previous section when $|w_t|_{\Omega_1} \leq 1$. If $|w_t|_{\Omega_1} \geq 1$, then for v in $L_2(\Omega_s)$,

$$\left|\int_{\Omega_1\cap|w_t|\geq 1} \rho g(w_t)v dx\right| \leq C\int_{\Omega_1\cap|w_t|\geq 1} |g(w_t)||v|dx \leq C\int_{\Omega_1\cap|w_t|\geq 1} |w_t|^p|v|dx$$
$$(\text{since } p\leq 1) \leq C\int_{\Omega_1\cap|w_t|\geq 1} |w_t||v|dx \leq \varepsilon|v|^2_{\Omega_s} + C_\varepsilon|w_t|^2_{\Omega_1}$$

Note that in (5.7), since $\Omega_1 \subset \Omega_s$ is small, C_ε could be chosen such that $C_\varepsilon|w_t|^2_{\Omega_1} < \frac{1}{2}|w_t|^2_{\Omega_s}$.

Step 2: Boundary terms: The key is to estimate the four boundary terms in (5.6).

Step 2.1: Estimate on $\int_0^T \langle\frac{\partial w}{\partial n}, w\rangle dt$:
With $\beta > 0$, we can now apply a more standard argument. Choosing $\varepsilon_1 > 0$ small, we can estimate this term simply as:

$$\int_0^T \langle\frac{\partial w}{\partial n}, w\rangle dt \leq C_{\varepsilon_1}\int_0^T \left|\frac{\partial w}{\partial n}\right|^2_{\Gamma_s} dt + \varepsilon_1\int_0^T |\nabla w|^2_{\Omega_s} dt \tag{5.8}$$

Step 2.2: Estimate on $\int_0^T \langle\frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt - \frac{1}{2}\int_0^T \langle|\nabla w|^2, l\cdot\overrightarrow{n}\rangle dt$:

Writing $\nabla w|_{\Gamma_s} = (\frac{\partial w}{\partial n})\overrightarrow{n} + (\frac{\partial w}{\partial \tau})\tau$ and using Young's inequality, for any $\varepsilon_2 > 0$, there exists a $C_{\varepsilon_2} > 0$ such that

$$\int_0^T \langle\frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt \leq C_{\varepsilon_2}\int_0^T \left|\frac{\partial w}{\partial n}\right|^2_{0,\Gamma_s} dt + \varepsilon_2 C_h\int_0^T \left|\frac{\partial w}{\partial \tau}\right|^2_{0,\Gamma_s} dt \tag{5.9}$$

Using the geometric condition $h\cdot\overrightarrow{n} > 0$ and choosing ε_2 small enough such that $\varepsilon_2 C_h < \frac{1}{2}\min h\cdot\overrightarrow{n}$ yields

$$\int_0^T \langle\frac{\partial w}{\partial n}, l\cdot\nabla w\rangle dt - \frac{1}{2}\int_0^T \langle|\nabla w|^2, l\cdot\overrightarrow{n}\rangle dt \leq C_{\varepsilon_2}\int_0^T \left|\frac{\partial w}{\partial n}\right|^2_{0,\Gamma_s} dt \tag{5.10}$$

Step 2.3: Estimate on $\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt$:

Using the transmission condition $u = w_t + \dfrac{\partial w}{\partial n}$ on Γ_s and majorizing $|u|_{0,\Gamma_s}$ by $|\nabla u|_{0,\Omega_s}$, we obtain

$$\frac{1}{2}\int_0^T \langle w_t^2, l\cdot\overrightarrow{n}\rangle dt \le C_h \int_0^T D(t)dt \tag{5.11}$$

Step 3: Final step on estimating (5.6):

By Young's and Poincaré's inequality,

$$-(w_t, l\cdot\nabla w)_s\Big|_0^T - (w_t, w)_s\Big|_0^T \le C[E(0)+E(T)] \tag{5.12}$$

Substituting (5.7) - (5.12) into (5.6) and taking small ε's yields

$$\int_0^T [|w_t|_{0,\Omega_s}^2 + |\nabla w|_{0,\Omega_s}^2]dt \le C_{h,\varepsilon_1,\varepsilon_2}\int_0^T D(t)dt + C[E(0)+E(T)] \tag{5.13}$$

□

Now proceeding as in the last step of previous section, we could recover the total energy and obtain:

$$E(T) \le C_{E(0)}\left(\int_0^T D(t)dt\right) + C\int_0^T |u|_f^2 dt \tag{5.14}$$

The final task now is again to absorb the lower order term into the dissipation:

$$\int_0^T |u|_f^2 dt \le C_{E(0)}\left(\int_0^T D(t)dt\right), \text{ for some } C_{E(0)} > 0 \tag{5.15}$$

To this aim, we revoke a key assumption in Theorem 2.9 that ρ does not vanish on a full measure near Γ_s and following the uniqueness-compactness argument as in Lemma 4.1 with $H(s) = Cs$, we could show that $\hat{u} \equiv 0$ on a nontrivial part of Γ_s. Poincarè's inequality then completes the proof of (5.15), which in turn implies (3.1), thus fulfills the proof of Theorem 3.1 (b).

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